Problems in formulating the consecution calculus of contraction–less relevant logics

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The contraction–less logic $RW^{\circ t}$ is the best known relevant system R ([1], p. 341) with co–tenability $\circ$ and t, but without the contraction axiom:

$$(W) \quad (\alpha \to .\alpha \to \beta) \to .\alpha \to \beta$$

The first problem in formulating a consecution calculus of $RW^{\circ t}$ is common to all relevant systems: how to enable the inference of $\alpha \wedge (\beta \vee \gamma). \to .(\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$, in the absence of thinning. This problem is solved by Dunn [4] and Minc [5], by allowing two types of sequences of formula: intensional (usually denoted by $(\Gamma_1; \ldots; \Gamma_n)$) and extensional ones (usually denoted by $(\Gamma_1, \ldots, \Gamma_n)$), which must be allowed to be nested within another. Due to the presence of these two types of sequences, every Gentzen structural rule can be formulated either as the intensional or as the extensional one. The contraction axiom (W) corresponds to Intensional Contraction. The missing thinning rule corresponds to Intensional Thinning structural rule, therefore in a sequent system of $RW^{\circ t}$, in addition to the structural rule Intensional Thinning, we also lack Intensional Contraction. The extensional variants of those rules should be present. The proof of the above distribution rule, in the single conclusion sequent system, is then:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\alpha, \beta \vdash \alpha}\,(\mathrm{KE}\vdash) \quad \dfrac{\beta \vdash \beta}{\alpha, \beta \vdash \beta}\,(\mathrm{KE}\vdash)}{\alpha, \beta \vdash \alpha \wedge \beta}\,(\vdash \wedge)}{\alpha, \beta \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}\,(\vdash \vee) \qquad \begin{matrix}\vdots \\ \alpha, \gamma \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)\end{matrix}}{\alpha, \beta \vee \gamma \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}\,(\vee \vdash)}{\alpha \wedge (\beta \vee \gamma), \alpha \wedge (\beta \vee \gamma) \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}\,(\wedge \vdash)}{\alpha \wedge (\beta \vee \gamma) \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}\,(\mathrm{WE}\vdash)}{\vdash \sim\sim \alpha \wedge (\beta \vee \gamma). \to .(\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}\,(\vdash\to)$$

Following that idea, Giambrone [3] established the (single–conclusion) cut–free Gentzenisation of $RW_+^{\circ t}$ (positive $RW^{\circ t}$).

The second problem is how to enable the inference of $\sim\sim \alpha \to \alpha$. Originally, Gentzen allowed multiple-conclusion sequents:

$$\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\vdash \sim \alpha; \alpha}\,(\vdash\sim)}{\sim\sim \alpha \vdash \alpha}\,(\sim\vdash)}{\vdash \sim\sim \alpha \to \alpha}\,(\vdash\to)$$

But, for RW it won't work. Brady [2] found that the Gentzenisation of RW requires careful addition of negation to Gentzenisation of $RW_+^{\circ t}$. Really, in the multiple–conclusion sequent system, in the style of Gentzen, but with intensional and extensional sequences of formula, we would have:

$$\dfrac{\dfrac{\begin{matrix}\pi_1 \\ \Gamma_1 \vdash \alpha; \Delta_1\end{matrix} \qquad \begin{matrix}\pi_2 \\ \Gamma_1 \vdash \beta \vee \gamma; \Delta_1\end{matrix}}{\Gamma_1 \vdash \alpha \wedge (\beta \vee \gamma); \Delta_1}\,(\vdash \wedge) \qquad \dfrac{\dfrac{\begin{matrix}\pi_3 \\ \Gamma_2; (\alpha, \beta \vee \gamma) \vdash \Delta_2\end{matrix}}{(\Gamma_2; \alpha \wedge (\beta \vee \gamma)), (\Gamma_2; \alpha \wedge (\beta \vee \gamma)) \vdash \Delta_2}\,(\wedge \vdash)}{\Gamma_2; \alpha \wedge (\beta \vee \gamma) \vdash \Delta_2}\,(\mathrm{WE}\vdash)}{\Gamma_1; \Gamma_2 \vdash \Delta_1; \Delta_2}\,(\mathrm{cut})$$

Our attempt to transform this proof to a cut–free proof, would lead to:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\pi_2}{\Gamma_1 \vdash \beta \vee \gamma; \Delta_1} \qquad \dfrac{\overset{\pi_1}{\Gamma_1 \vdash \alpha; \Delta_1} \qquad \overset{\pi_3}{\Gamma_2; (\alpha, \beta \vee \gamma) \vdash \Delta_2}}{\Gamma_2; (\Gamma_1, \beta \vee \gamma) \vdash \Delta_1; \Delta_2}\ (\mathrm{cut})}{\Gamma_2; (\Gamma_1, \Gamma_1) \vdash \Delta_1; \Delta_1; \Delta_2}\ (\mathrm{cut})}{\Gamma_2; \Gamma_1 \vdash \Delta_1; \Delta_1; \Delta_2}\ (\mathrm{WE} \vdash)}{\dfrac{\dots}{\Gamma_1; \Gamma_2 \vdash \Delta_1; \Delta_1; \Delta_2}}\ \text{permutations}$$

However, in the absence of extensional thinning, from here it is not possible to derive a sequent $\Gamma_1; \Gamma_2 \vdash \Delta_1; \Delta_2$, for non–empty Δ_1.

Brady solved this problem by formulating the single–conclusion sequent system of $RW^{\circ t}$ based on *signed* formulae $T\alpha$ and $F\alpha$, instead of just formulae α, with logical rules for both types of signed formulae. In his system, instead of the above derivation, we would have the following ($S\gamma$ stands in either for $T\gamma$ or $F\gamma$, or is empty):

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\pi_2}{\Gamma' \vdash T\beta \vee \gamma} \qquad \dfrac{\overset{\pi_1}{\Gamma' \vdash T\alpha} \qquad \overset{\pi_3}{\Gamma''; (T\alpha, T\beta \vee \gamma) \vdash S\gamma}}{\Gamma''; (\Gamma', T\beta \vee \gamma) \vdash S\gamma}\ (\mathrm{cut})}{\Gamma''; (\Gamma', \Gamma') \vdash S\gamma}\ (\mathrm{cut})}{\Gamma''; \Gamma' \vdash S\gamma}\ (\mathrm{WE} \vdash)}{\dfrac{\dots}{\Gamma'; \Gamma'' \vdash S\gamma}}\ \text{permutations}$$

However, Brady's system is very complicated (e. g., there are 8 different forms of the rule for $\rightarrow$, i. e., it has four shemata for the rule ($T \rightarrow\vdash$), two for ($F \rightarrow\vdash$), one for ($\vdash F \rightarrow$) and one rule for ($\vdash T \rightarrow$), which is, unusually, two–premise rule).

We propose another solution. Namely, we change the vocabulary. Instead of $\sim$, we take a propositional constant f as primitive (and we define negation, as usual, via $\sim \alpha =_{\mathrm{df}} \alpha \rightarrow f$). We define a cut–free right–handed sequent system of $RW^{\circ tf}$, based on (just) formulae. $RW^{\circ tf}$ is obtained by adding co–tenability $\circ$ and propositional constants t and f to positive RW via the axioms given in [1] pp. 343.-344.

References

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[2] R. T. Brady, *The Gentzenization and decidability of RW,* Journal of Philosophical Logic, 19, 35-73, 1990.

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[4] J. M. Dunn, *A 'Gentzen system' for positive relevant implication,* The Journal of Symbolic Logic 38, pp. 356-357, 1973.

[5] G. Minc, *Cut elimination theorem for relevant logics,* Journal of Soviet Mathematics 6, 422-428, 1976.