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Problems in formulating the consecution calculus of contraction–less relevant logics

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RW

- *RW* is the relevant logic *R* [A. ANDERSON, N. BELNAP JR., Entailment: the logic of relevance and necessity, vol. 1, Princeton University Press, Princeton, New Jersey, 1975. p. 341] without the contraction axiom:

$$(W) \quad (\alpha \to .\alpha \to \beta) \to .\alpha \to \beta$$

- Brady: *RW* is decidable logic [R. T. BRADY, *The Gentzenization and decidability of RW*, Journal of Philosophical Logic, 19, 35-73, 1990.R. T. BRADY, *The Gentzenization and decidability of RW*, Journal of Philosophical Logic, 19, 35-73, 1990.]

The Hilbert–type formulation of *RW*

*Ax*1. $\alpha \to \alpha$

*Ax*2. $\alpha \to \beta \to .\beta \to \gamma \to .\alpha \to \gamma$

*Ax*3. $\alpha \to .(\alpha \to \beta) \to \beta$

*Ax*4. $\alpha \wedge \beta \to \alpha$

*Ax*5. $\alpha \wedge \beta \to \beta$

*Ax*6. $\alpha \to \beta \wedge \alpha \to \gamma. \to .\alpha \to (\beta \wedge \gamma)$

*Ax*7. $\alpha \to \alpha \vee \beta$

*Ax*8. $\beta \to \alpha \vee \beta$

*Ax*9. $(\alpha \to \gamma) \wedge (\beta \to \gamma). \to .(\alpha \vee \beta) \to \gamma$

*Ax*10. $\alpha \wedge (\beta \vee \gamma) \to (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$

*Ax*11. $\alpha \to \sim \beta. \to .\beta \to \sim \alpha$

*Ax*12. $\sim\sim \alpha \to \alpha$

R′. $\dfrac{\vdash \alpha \qquad \vdash \alpha \to \beta}{\vdash \beta}$

R″. $\dfrac{\vdash \alpha \qquad \vdash \beta}{\vdash \alpha \wedge \beta}$

The first problem

How to enable the inference of the distribution law:

$$\alpha \wedge (\beta \vee \gamma). \rightarrow .(\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$$

in the absence of the thinning rule?

The proof of the distribution law in Gentzen's *LK* and *LJ*

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\alpha, \beta \vdash \alpha}\,(\text{Thinning}) \qquad \dfrac{\beta \vdash \beta}{\alpha, \beta \vdash \beta}\,(\text{Thinning})}{\alpha, \beta \vdash \alpha \wedge \beta}\,(\vdash \wedge)}{\alpha, \beta \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}\,(\vdash \vee) \qquad \begin{matrix}\vdots \\ \alpha, \gamma \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)\end{matrix}}{\dfrac{\alpha, \beta \vee \gamma \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}{\dfrac{\alpha \wedge (\beta \vee \gamma), \alpha \wedge (\beta \vee \gamma) \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}{\dfrac{\alpha \wedge (\beta \vee \gamma) \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}{\vdash \alpha \wedge (\beta \vee \gamma). \rightarrow .(\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}\,(\vdash \rightarrow)}\,(\text{Contraction})}\,(\wedge \vdash)}\,(\vee \vdash)$$

A puzzle

$$\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\alpha, \beta \vdash \alpha}\ (\text{Thinning})}{\alpha \vdash \beta \to \alpha}\ (\vdash\to)}{\vdash \alpha \to (\beta \to \alpha)}\ (\vdash\to)$$

- The solution: we need two types of sequences of formulae [J. M. Dunn, *A 'Gentzen system' for positive relevant implication*, The Journal of Symbolic Logic 38, pp. 356-357, 1973.] [G. Minc, *Cut elimination theorem for relevant logics*, Journal of Soviet Mathematics 6, 422-428, 1976.]

The informal meaning of a Gentzen sequent

$$\alpha_1, \ldots, \alpha_n \vdash \beta_1, \ldots, \beta_n$$

$$(\alpha_1 \wedge \ldots \wedge \alpha_n) \rightarrow (\beta_1 \vee \ldots \vee \beta_n)$$

- A sequence of formulae appearing in the antecedent of a sequent in *LJ* and *LK*, represents the conjunction of those formulae.

Two different conjunctions in R (and in R_+)

An effect of the absence of thinning:

classical connectives split into dual pairs.

- extensional (truth–functional) conjunction $\wedge$
- intensional conjunction (fusion) $\circ$

$$\frac{\Gamma[\alpha] \vdash \Delta}{\Gamma[\alpha \wedge \beta] \vdash \Delta} \qquad \frac{\Gamma[\beta] \vdash \Delta}{\Gamma[\alpha \wedge \beta] \vdash \Delta} \qquad \frac{\Gamma[\alpha, \beta] \vdash \Delta}{\Gamma[\alpha \circ \beta] \vdash \Delta}$$

Consequently, two types of sequences would be required to gentzenise R_+:

- *extensional* sequences to stand in for ordinary 'extensional' conjunction $\wedge$
- *intensional* sequences to stand in for fusion $\circ$.

Intensional and *extensional* sequnces of formulae in LR_+

$$\frac{\alpha \vdash \alpha}{\alpha, \beta \vdash \alpha} \text{ (Thinning)}$$

$$\frac{\alpha \vdash \alpha}{\alpha, \beta \vdash \alpha} \text{ (extensional thinning)} \quad \text{and} \quad \frac{\alpha \vdash \alpha}{\alpha; \beta \vdash \alpha} \text{ (intensional thinning)}$$

$\alpha, \beta \vdash \alpha$ is interpreted as $\alpha \wedge \beta \rightarrow \alpha$

$\alpha; \beta \vdash \alpha$ is interpreted as $\alpha \circ \beta \rightarrow \alpha$

$(\alpha \circ \beta) \rightarrow \gamma. \rightarrow .\alpha \rightarrow (\beta \rightarrow \gamma)$ is valid, but

$(\alpha \wedge \beta) \rightarrow \gamma. \rightarrow .\alpha \rightarrow (\beta \rightarrow \gamma)$ is not

$$\frac{\alpha; \beta \vdash \alpha}{\alpha \vdash \beta \rightarrow \alpha} \text{ is valid, but } \frac{\alpha, \beta \vdash \alpha}{\alpha \vdash \beta \rightarrow \alpha} \text{ is not.}$$

The proof of the distribution law, in a two–sided sequent system with both intensional and extensional sequences

$$
\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\alpha, \beta \vdash \alpha}\ (\mathrm{KE}\vdash) \qquad \dfrac{\beta \vdash \beta}{\alpha, \beta \vdash \beta}\ (\mathrm{KE}\vdash)}{\alpha, \beta \vdash \alpha \wedge \beta}\ (\vdash \wedge)}{\alpha, \beta \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}\ (\vdash \vee) \qquad \begin{matrix} \vdots \\ \alpha, \gamma \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \end{matrix}}{\dfrac{\dfrac{\alpha, \beta \vee \gamma \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}{\alpha \wedge (\beta \vee \gamma), \alpha \wedge (\beta \vee \gamma) \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}\ (\wedge \vdash)}{\alpha \wedge (\beta \vee \gamma) \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}\ (\mathrm{WE}\vdash)}\ (\vee \vdash)
$$

The second problem

How to disable the inference of the modal fallacy:

$$\alpha \to (\beta \to \beta)$$

in the presence of the cut rule?

- The empty left–hand side in the left premise of the cut rule:

$$\frac{\Gamma \vdash \alpha \qquad \Sigma[\alpha] \vdash \beta}{\Sigma[\Gamma] \vdash \beta}$$

can lead to irrelevance:

$$\dfrac{\dfrac{\vdash \beta \to \beta \qquad \dfrac{\beta \to \beta \vdash \beta \to \beta}{\alpha, \beta \to \beta \vdash \beta \to \beta}\ (\text{KE})}{\alpha \vdash \beta \to \beta}\ (\text{cut})}{\vdash \alpha \to .(\beta \to \beta)}\ (\to \text{r})$$

A solution: The addition of t as primitive $\begin{cases} t \\ t \to (\alpha \to \alpha) \end{cases}$

Two approaches:

- Cut:

$$\frac{\Gamma \vdash \alpha \qquad \Sigma[\alpha] \vdash \beta}{\Sigma[\Gamma] \vdash \beta} \ (\Gamma \text{ non–empty}) \qquad \frac{\vdash \alpha \qquad \Sigma[\alpha] \vdash \beta}{\Sigma[t] \vdash \beta}$$

$$\dfrac{\vdash \beta \to \beta \qquad \dfrac{\beta \to \beta \vdash \beta \to \beta}{\alpha, \beta \to \beta \vdash \beta \to \beta}\ (\mathrm{KE})}{\alpha, t \vdash \beta \to \beta}\ (\mathrm{cut})$$

- Sequents must have non–empty antecedents (they have t there instead).

$$\dfrac{t \vdash \beta \to \beta \qquad \dfrac{\beta \to \beta \vdash \beta \to \beta}{\alpha, \beta \to \beta \vdash \beta \to \beta}\ (\mathrm{KE})}{\alpha, t \vdash \beta \to \beta}\ (\mathrm{cut})$$

A note: With the cut rule restricted to instances where Γ is non–empty, only, the Cut–Elimination would not be provable

The proper form of the cut rule:

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \Gamma; \alpha \vdash \beta\end{array}}{\Gamma \vdash \alpha \to \beta}\,(\vdash\to) \qquad \dfrac{\begin{array}{cc}\pi_2 & \pi_3\\ \vdash \alpha & \beta \vdash\end{array}}{\alpha \to \beta \vdash}\,(\to\vdash)}{\Gamma \vdash}\,(\text{cut})$$

Improper forms:

$$\dfrac{\vdash \alpha \qquad \Gamma; \alpha \vdash \beta}{\Gamma \vdash \beta} \quad \text{and} \quad \dfrac{\vdash \alpha \qquad \Gamma; \alpha \vdash}{\Gamma \vdash}$$

Giambrone's Gentzen system for RW_+

- The cut–free Gentzen system $LRW^{\circ t}$ of $RW^{\circ t}$
  - t is added as a place filter to ensure the non–emptiness of antecedents
  - $\circ$ is added to provide the interpretation for intensional sequences
- The expansion of $LRW_+^{\circ t}$ to include empty antecedents.
- The Gentzen system of RW_+.

The third problem

How to add negation to Giambrone's LRW_+?

- Brady's [R. T. BRADY, *The Gentzenization and decidability of RW*, Journal of Philosophical Logic, 19, 35-73, 1990.] solution: *LRW* is the single–conclusion sequent system based on signed formulae.

Originally, to enable the inference of $\sim\sim \alpha \to \alpha$, Gentzen allowed multiple–conclusion sequents:

$$\cfrac{\cfrac{\cfrac{\alpha \vdash \alpha}{\vdash \sim \alpha; \alpha}\ (\vdash\sim)}{\sim\sim \alpha \vdash \alpha}\ (\sim\vdash)}{\vdash \sim\sim \alpha \to \alpha}\ (\vdash\to)$$

Problems with multiple–conclusion sequents

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma_1 \vdash \alpha; \Delta_1} \qquad \overset{\pi_2}{\Gamma_1 \vdash \beta \lor \gamma; \Delta_1}}{\Gamma_1 \vdash \alpha \land (\beta \lor \gamma); \Delta_1}\,(\vdash \land) \qquad \dfrac{\dfrac{\overset{\pi_3}{\Gamma_2; (\alpha, \beta \lor \gamma) \vdash \Delta_2}}{(\Gamma_2; \alpha \land (\beta \lor \gamma)), (\Gamma_2; \alpha \land (\beta \lor \gamma)) \vdash \Delta_2}\,(\land \vdash)}{\Gamma_2; \alpha \land (\beta \lor \gamma) \vdash \Delta_2}\,(\mathrm{WE} \vdash)}{\Gamma_1; \Gamma_2 \vdash \Delta_1; \Delta_2}\,(\mathrm{cut})$$

The mix rule:

$$\dfrac{\Gamma_1 \vdash \alpha; \Delta_1 \qquad \Gamma_2; (\alpha, \Gamma_3) \vdash \Delta_2}{\Gamma_2; (\Gamma_1, \Gamma_3) \vdash \Delta_1; \Delta_2}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\pi_2}{\Gamma_1 \vdash \beta \lor \gamma; \Delta_1} \qquad \dfrac{\overset{\pi_1}{\Gamma_1 \vdash \alpha; \Delta_1} \qquad \overset{\pi_3}{\Gamma_2; (\alpha, \beta \lor \gamma) \vdash \Delta_2}}{\Gamma_2; (\Gamma_1, \beta \lor \gamma) \vdash \Delta_1; \Delta_2}\,(\mathrm{mix})}{\Gamma_2; (\Gamma_1, \Gamma_1) \vdash \Delta_1; \Delta_1; \Delta_2}\,(\mathrm{mix})}{\Gamma_2; \Gamma_1 \vdash \Delta_1; \Delta_1; \Delta_2}\,(\mathrm{WE} \vdash)}{\dfrac{\ldots}{\Gamma_1; \Gamma_2 \vdash \Delta_1; \Delta_1; \Delta_2}}\,\text{permutations}}{\dfrac{\ldots}{\Gamma_1; \Gamma_2 \vdash \Delta_1; \Delta_2}}\,\text{??? for non–empty } \Delta_1$$

Brady's *LRW*

$$\dfrac{\dfrac{\dfrac{T\alpha \vdash T\alpha}{F\sim\alpha \vdash T\alpha}\,(F\sim\vdash)}{T\sim\sim\alpha \vdash T\alpha}\,(T\sim\vdash) \qquad \dfrac{\dfrac{F\alpha \vdash F\alpha}{T\sim\alpha \vdash F\alpha}\,(T\sim\vdash)}{F\sim\sim\alpha \vdash F\alpha}\,(F\sim\vdash)}{\vdash T\sim\sim\alpha \to \alpha}\,(\vdash T\to)$$

The unusual second premise of the rule $(\vdash T \to)$ is needed to pair with the rule $(T \to\vdash)$ in the mix–elimination argument.

$$
\dfrac{\begin{array}{c}\pi_2\\ \Gamma' \vdash T\beta\vee\gamma\end{array} \qquad \dfrac{\begin{array}{c}\pi_1\\ \Gamma' \vdash T\alpha\end{array} \qquad \begin{array}{c}\pi_3\\ \Gamma''; (T\alpha, T\beta\vee\gamma) \vdash S\gamma\end{array}}{\Gamma''; (\Gamma', T\beta\vee\gamma) \vdash S\gamma}\ (\mathrm{cut})}{\dfrac{\dfrac{\dfrac{\Gamma''; (\Gamma', \Gamma') \vdash S\gamma}{\Gamma''; \Gamma' \vdash S\gamma}\ (\mathrm{WE}\vdash)}{\dots}\ \text{permutations}}{\Gamma'; \Gamma'' \vdash S\gamma}}\ (\mathrm{cut})
$$

Brady's system contains over 50 inference rules. E. g. there are 8 different rules for $\rightarrow$:

- four of them for the rule $(T \rightarrow\vdash)$,
- two of them for $(F \rightarrow\vdash)$,
- one for $(\vdash F \rightarrow)$ and
- one rule for $(\vdash T \rightarrow)$, which is, unusually, two–premise rule.

We ask

How to obtain a Gentzen system of *RW* directly?

- We formulate the right–handed sequent system *GRW*, based on *multisets* instead of sequences.
  - A two–sided sequent system is not suitable for *RW*.
  - *RW* is commutative logic, therefore the use of multisets as data structures appears as most natural in this context.

Our system

1. We obtain the cut–free Gentzen system *GRW* for *RW* directly (we do not proceed via extended system $RW^{\circ t}$).
2. The inference rules of *GRW* are simple and natural.
3. The proof of the Cut–Elimination Theorem in *GRW* is clear and simple.

Thank you!