Combinatorial aspects of Interpretability Logic

Counting formulas that have GL equivalents

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September 18th, 2013
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Quick introduction to Interpretability logic

Visser introduced in 1990 a generalization of Provability logic (GL) to better capture relative strength of various arithmetical formulas.

We work in closed fragment (no propositional variables) and minimized language (no superfluous connectives) so all formulas are given by grammar rule:

$$F ::= \bot \mid (F \to F) \mid F \rhd F$$

$\top$, $\neg$, $\vee$, $\wedge$, $\leftrightarrow$, $\Box$, $\Diamond$ are expressible from these
e.g. $\Box\varphi :\Longleftrightarrow (\varphi \to \bot) \rhd \bot$
"T proves φ" $=$ "T extended with $\neg\varphi$ interpretes contradiction"

Formulating the question. . .

In GL, closed formulas have normal forms (Artemov, 1987)
In ILF, they have the *same* normal forms (Hájek & Švejdar, 1991)
In IL, many of them (but not all!) also do (Čačić & Vuković, 2011)
How many?

Formulating the question...

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What does that question even mean? (Is $\aleph_0$ a valid answer?☻)

How many pairs of integers are coprime?

$$6\pi^{-2} = \lim_n \frac{\#\{(x,y) \in \{-n..n\}^2 : gcd(x,y) = 1\}}{\#\{-n..n\}^2}$$

Integers are *ordered*. Formulas are ordered, too: by complexity.

Definition ("*share*")

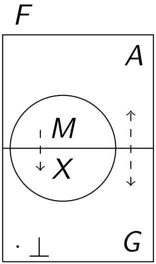
$$\mu_Z := \lim_n \frac{\#\{\varphi \in IL_0^{(n)} : \varphi \text{ is of the form } Z\} \;=: z_n}{\#IL_0^{(n)} \;=: f_n}$$

where $IL_0^{(n)}$ is the set of all closed IL formulas with n connectives.

Some classes of formulas we will consider

(dashed arrows show action of negation▼)

all	$F ::= \bot \mid F \to F \mid F \rhd F$
affirmative	$A ::= F \to A \mid G \to F \mid F \rhd F$
negative	$G ::= \bot \mid A \to G$
direct	$M ::= A \rhd G$
cross	$X ::= M \to \bot$
	$B_1 ::= \bot \rhd F$
	$B_2 ::= F \rhd A$
	$B_3 ::= A \rhd \bot$
	$B_4 ::= G \rhd X$
	$B_5 ::= X \rhd \bot$
basic normal	$B ::= B_1 \mid B_2 \mid B_3 \mid B_4 \mid B_5$
locally normal	$L ::= \bot \mid B \mid L \to L$
normal	$N ::= G \mid M \mid L \mid (N \to \bot) \rhd \bot$



Explanations of classes

Affirmative formulas hold on all terminal worlds in Veltman models.
Negative formulas hold on *no* terminal world.
Direct formulas are the "modalization" of negative formulas.
Cross formulas are negations of direct formulas.
$B_{1..5}$ we've shown to have local GL equivalents. B is their union.
L is the Boolean closure of B.

So, formulas in L also have local GL equivalents (by substitution).
Also, negative and direct formulas have *global* GL equivalents.
And finally, normal formulas are the $\Box$-closure of all that.

So, all normal formulas have the same normal forms as in GL.
We want to know their share, μ_N.

Multiplication lemma: crucial for calculation

Definition (*“spread”*: a very useful auxiliary parameter)

$$\sigma_Z := \sum_{n=0}^{\infty} \frac{z_n}{8^{n+1}} = \lim_{t \uparrow \frac{1}{8}} t\, z(t) \qquad \Big(z \text{ is the ogf for } (z_n)_n\Big)$$

Lemma (“Multiplication”; Mnemonic: $\mu_Z = (\sigma_Z)'$.)

$\sigma_{U \rhd V} = \sigma_U \sigma_V$ *and* $\mu_{U \rhd V} = \mu_U \sigma_V + \sigma_U \mu_V$ *(and same for* $\rightarrow$*)*.

Proof for σ (for μ the proof is much more complicated ↯).

If $W := U \underset{(\rightarrow)}{\rhd} V$, then $w_0 = 0$ and $w_{n+1} = \sum_{k=0}^{n} u_k v_{n-k}$, so

$$\sigma_W = \sum_{n=0}^{\infty} \frac{w_n}{8^{n+1}} = \sum_{n=1}^{\infty} \frac{w_n}{8^{n+1}} = \sum_{n=0}^{\infty} \frac{w_{n+1}}{8^{n+2}} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} u_k v_{n-k} 8^{-n-2}$$

$$= \sum\sum_{0 \leqslant \underline{k} \leqslant \underline{n}} \frac{u_k}{8^{k+1}} \frac{v_{n-k}}{8^{n-k+1}} = \sum_{k=0}^{\infty} \frac{u_k}{8^{k+1}} \sum_{n=k}^{\infty} \frac{v_{n-k}}{8^{n-k+1}} = \sigma_U \sigma_V. \qquad \square$$

Here be dragons. . .

For the class $\{\bot\}$, which we write without braces, we can calculate $\sigma_\bot = \frac{1}{8} + \frac{0}{64} + \frac{0}{512} + \cdots = \frac{1}{8}$ and $\mu_\bot = \lim\,(1, 0, 0, \ldots) = 0$. Also, for whole F, $\mu_F = \lim_n \frac{f_n}{f_n} = 1$. But even σ_F is a big problem.

ϟ Warning: don't try this in peer-reviewed texts!

Rigorous treatment of the following calculations is very hard. I'm greatly thankful to my colleague V. Kovač for writing that part of the article correctly. Here we'll proceed carelessly♫.

Now let's calculate σ_F. From $F = \bot + (F \to F) + (F \rhd F)$ (writing $+$ for disjoint union, then μ and σ are "additive") we get $\sigma_F = \frac{1}{8} + 2\sigma_F^2$: a quadratic equation with only solution $\sigma_F = \frac{1}{4}$.

This *doesn't* prove σ_F exists, but if we know it, then we *do* know all other spreads exist (majorized series with nonnegative terms), and $0 \leq \sigma_Z \leq \sigma_F = \frac{1}{4}$. For shares it's more complicated: we have to prove each exists "separately". Here we'll just assume they do.

Corollaries of multiplication: Boolean closure

Now let's consider the Boolean closure of some class V, whose all formulas have $\rhd$ as their main connective.

Corollary ("Boolean closure")

If $W ::= \bot \mid V \mid W \to W$ (disjoint union), then

$$\sigma_W = \frac{1}{2}\left(1 - \sqrt{\frac{1}{2} - 4\,\sigma_V}\right) \quad \textit{and} \quad \mu_W = \frac{\mu_V}{1 - 2\sigma_W}\ .$$

Proof.

For σ_W we get quadratic equation $\sigma_W = \frac{1}{8} + \sigma_V + \sigma_W^2$, and one solution (with $+$ in place of blue $-$) is $\geq \frac{1}{2} > \frac{1}{4}$, so it cannot be the "true" σ_W. The other solution is as given above.
Also, multiplication lemma for μ gives us linear equation $\mu_W = 0 + \mu_V + 2\mu_W\sigma_W$, with solution as above. □

Corollaries of multiplication: negativity

Negative formulas can be defined in any Boolean closed class U.

Corollary ("Negativity")

If $W ::= \bot \mid (U \setminus W) \to W$ *(where* $W \subseteq U$*), then*

$$\sigma_W = \frac{1}{2}\left(\sigma_U - 1 + \sqrt{(\sigma_U - 1)^2 + \frac{1}{2}}\right) \text{ and } \mu_W = \frac{\sigma_W \mu_U}{1 - \sigma_U + 2\sigma_W}.$$

Proof.

Since $W \subseteq U$, we have $W + (U \setminus W) = U$, so $\mu_{U \setminus W} = \mu_U - \mu_W$ and same for σ (they are subtractive, not only additive). Now multiplication lemma gives quadratic $\sigma_W = \frac{1}{8} + (\sigma_U - \sigma_W)\,\sigma_W$ with one solution negative (since $\sigma_U \leq \frac{1}{4}$) and the other as above, and linear equation $\mu_W = 0 + (\sigma_U - \sigma_W)\,\mu_W + (\mu_U - \mu_W)\,\sigma_W$, with solution as above. □

Starting steps

So far we have handled (found spread and share for) $\perp$ and F.
Negativity handles G, and then subtractivity handles $A = F - G$.

Then multiplication lemma handles $B_{1..5}$, M and X.
The expressions do become more and more complicated (we won't even bother writing them here), but *Mathematica* helps.☼

What about B? We can't just sum the shares of $B_{1..5}$, since they are not disjoint. But barely: of $\binom{5}{2} = 10$ pairs, 8 *are* disjoint. Remaining two intersections are easily analyzed using

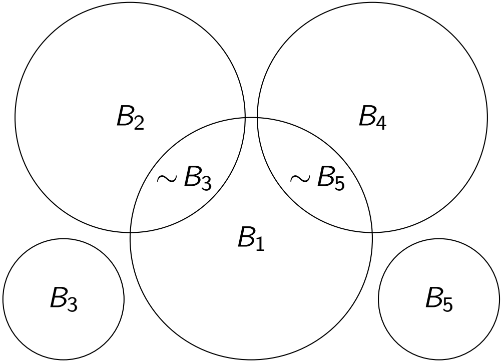
Definition ("*equal distribution*")

$Z \sim Y$ means $z_n = y_n$ for all n.

Of course, $Z \sim Y$ implies $\mu_Z = \mu_Y$ and $\sigma_Z = \sigma_Y$.

$$B_1 \cap B_2 = (\bot \rhd F) \cap (F \rhd A) = \bot \rhd A \sim A \rhd \bot = B_3$$
$$B_1 \cap B_4 = (\bot \rhd F) \cap (G \rhd X) = \bot \rhd X \sim X \rhd \bot = B_5$$

diagram of B :

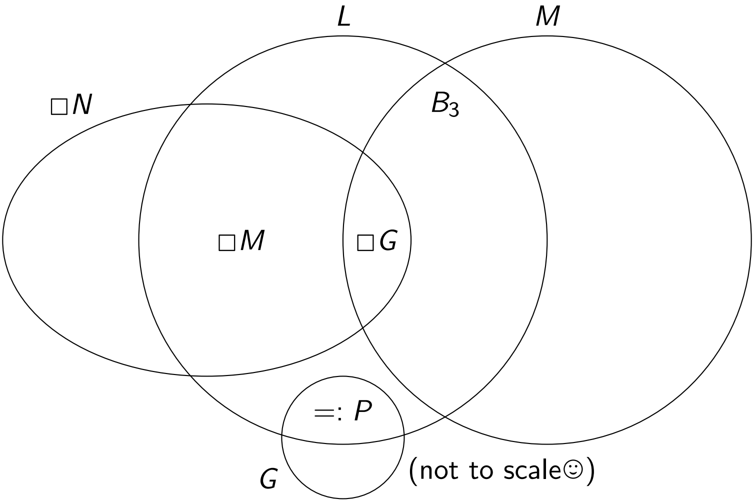


So, $\mu_B = \mu_{B_1} + \mu_{B_2} + \cancel{\mu_{B_3}} + \mu_{B_4} + \cancel{\mu_{B_5}} - \cancel{\mu_{B_1 \cap B_2}} - \cancel{\mu_{B_1 \cap B_4}}$

$= \mu_{B_1} + \mu_{B_2} + \mu_{B_4}$, and same for σ.

One more complication

Now we can use the Boolean closure to handle L.
But N presents the same problem as B, only much harder.
[. . . calculation. . .]



(not to scale☺)

Example of a [. . . calculation. . .] for the previous slide

$$
\begin{aligned}
L \cap \Box N &= (\cancel{\bot} \mid B \mid \cancel{L \to L}) \cap ((N \to \bot) \rhd \bot) \\
&= (\cancel{\bot \rhd F} \mid \cancel{F \rhd A} \mid A \rhd \bot \mid \cancel{G \rhd X} \mid X \rhd \bot) \cap ((N \to \bot) \rhd \bot) \\
&= (A \rhd \bot \mid X \rhd \bot) \cap ((N \to \bot) \rhd \bot) \\
&= ((A \mid X) \cap (N \to \bot)) \rhd \bot \\
&= ((\cancel{F \to A} \mid G \to F \mid \cancel{F \rhd F} \mid M \to \bot) \cap (N \to \bot)) \rhd \bot \\
&= (((G \mid M) \cap N) \to (F \cap \bot)) \rhd \bot \\
&= (((G \mid M) \cap N) \to ((\bot \mid F \to F \mid F \rhd F) \cap \bot)) \rhd \bot \\
&= (((G \mid M) \cap (L \mid G \mid M \mid \Box N)) \to \bot) \rhd \bot \\
&= ((G \mid M) \to \bot) \rhd \bot \\
&= \Box (G \mid M) \\
&= \Box G \mid \Box M
\end{aligned}
$$

Automatizing this reasoning is more complicated than it seems. ☹

Stitching it all together

Applying multiplication lemma twice with $V := \bot$, we have

Corollary ($\Box$)

$$\mu_{\Box Z} = \frac{\mu_Z}{64} \qquad \textit{and} \qquad \sigma_{\Box Z} = \frac{\sigma_Z}{64} .$$

In fact, the previous image is a fractal, since the oval labeled $\Box N$ has the structure of the whole image (N), 64 times smaller.

$$\mu_N = \mu_M + \mu_G + \frac{\mu_N}{64} + \mu_L - \mu_{B_3} - \mu_P - \frac{\mu_M + \mu_G}{64} - \cancel{\frac{\mu_G}{64}} + \cancel{\frac{\mu_G}{64}}$$

$$\mu_N = \mu_M + \mu_G + \frac{64}{63}(\mu_L - \mu_{B_3} - \mu_P)$$

What's P? $P = G \cap L$ is the "relativization" of G to L.
$P ::= \bot \mid (L \setminus P) \to P$, so negativity handles it.
Plugging it all in, and `Simplify`ing, *and* simplifying☺, we get $\leadsto$

The Answer to Life, Universe and share of normal formulas

$$\tfrac{59}{63} - \tfrac{67}{21\sqrt{17}} + \tfrac{2}{7}\left(11 - \tfrac{169}{9\sqrt{17}}\right)\sqrt{\tfrac{2}{23\sqrt{17}-61}}\left(\left(\left(\tfrac{\sqrt{23\sqrt{17}-61}}{16} + \tfrac{1}{\sqrt{2}}\right)^{-2} + 1\right)^{-\frac{1}{2}} + 1\right)$$
$$= \mu_N > 93.77\% > \frac{15}{16}.$$

So, *less than one sixteenth* of closed IL formulas of high enough complexity don't have normal form from GL.

If we wrote one of them on each of these slides,
likely all but one would have GL normal form.☺

For *low enough* complexities it's even better,
for the simplest closed IL formula without GL equivalent is
$\chi := (\bot \to \bot) \rhd ((\bot \to \bot) \rhd \bot \to \bot) \to (\bot \to \bot) \rhd \bot.$

Of course, the real share of formulas having global GL equivalents is probably higher than μ_N: since our grammars are context-free, we don't even recognize e.g. $\chi \to \chi$ as equivalent to $\top$.

But it's not all. . .

In fact, there is a result in my dissertation which can be used here to enlarge μ_N—at the cost of no longer having a radical expression (it is exact, but it is an isolated root of an 8^{th} degree polynomial)

Theorem ("Generalization theorem")

Two closed IL formulas are globally equivalent if and only if their generalizations ("boxes") are locally equivalent.

Since formulas in N have global equivalents, that means $\Box N$ is actually in L (more specifically, in B, since the main connective is $\rhd$), and that enlarges our classes slightly.

It's easily seen that old B_5 is

$$B_5 = X \rhd \bot = (M \to \bot) \rhd \bot = \Box M \subseteq \Box N \ ,$$

so we can "insert" $\Box N$ in place of B_5 in our schema.

New definitions

$$B_5 := \Box N$$
$$N := L \mid M \mid G$$

Other classes have same definitions, but B, L and P still change because their constituents change.

Nontrivial intersections are, as before

$$B_1 \cap B_2 = \bot \rhd A \sim B_3$$
$$B_1 \cap B_4 = \bot \rhd X \sim \Box M \quad (= \text{old } B_5)$$
$$G \cap L =: P$$
$$L \cap M = B_3$$

and a new one: $B_3 \cap B_5 = \Box G$

Complications. . .

Unfortunately, now B, L and N are so interrelated that it's impossible to just solve quadratic equations and get radical expressions. We get a polynomial

$$\begin{aligned}
&66173295868194116705409927972947\\
&+139215430008457910190632065130179t\\
&-173687061284287035102066646514151 2t^2\\
&+1092068190952835376217877321747492t^3\\
&+5149696516234972045870328786482800t^4\\
&-9478785402089921550540672467936576t^5\\
&+6726424050519374538084831392052480t^6\\
&-2250118036025586603533984812523520t^7\\
&+300015738136744880471197975003136t^8
\end{aligned}$$

with 8 distinct roots, four of which are real and only two are between 0 and 1.

Did I justify having 20 slides?

I could even have one more!

$$t_3 = 0.3121\ldots$$

$$t_4 = 0.9533\ldots = \mu_N > 1 - \frac{1}{21} \checkmark$$

How do we know new μ_N is actually t_4 and not t_3? Well, it has to be greater than $\mu_G + \mu_M$, but so is t_3. It really has to be larger than the *old* μ_N, but can we avoid first computing the old one?

Actually, this is a moot discussion. We would have to show that share of new N actually *exists*, and in process we would probably get some quality bounds.

How do do it? I don't know, ask V. Kovač. He's the wizard here.☻