Hierarchies of probability logics

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Logic and applications
Dubrovnik, Sept. 16-20, 2013.

Coauthors



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Outline

- Probabilistic logics, overview
- Hierarchies of PLs
- Conclusion

What are PLs?

Logic:

- syntax (language, well formed formulas)
- axiomatic system (axioms, rules)
- proof
- semantics (models, satisfiability)
- consequence relation

How to obtain PLs?

- keep syntax and extend semantics
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- keep syntax and extend semantics
    - $v : For \mapsto [0, 1]$
- extend syntax (new symbols in the language)
    - add/replace quantifiers
    - add new operators

History (1)

- Leibnitz (1646 – 1716)
- Bernoullies, Bayes, Lambert, Bolzano, De Morgan, MacColl, Peirce, Poretskiy, . . .
- Laplace (1749 – 1827)
- George Boole (1815 – 1864), An Investigation into the Laws of Thought, on which are founded the Mathematical Theories of Logic and Probabilities (1854):

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- George Boole (1815 – 1864), An Investigation into the Laws of Thought, on which are founded the Mathematical Theories of Logic and Probabilities (1854):

  logical functions: $f_1(x_1, \ldots, x_m)$, ..., $f_k(x_1, \ldots, x_m)$, $F(x_1, \ldots, x_m)$
  probabilities:
  $p_1 = P(f_1(x_1, \ldots, x_m))$, ..., $p_k = P(f_k(x_1, \ldots, x_m))$,
  solve: $P(F(x_1, \ldots, x_m))$ using p_1, ..., p_k

  corrected by T. Hailperin ('80.)

History (3)

XX century:

- progress of theories concerning derivations of truth in Math. logic
- measure theory, formal calculus of probability, Kolmogorov
- Keynes, Reichenbach, De Finetti, Carnap, Cox, ...

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- '60, '70: Keisler, Geifmann, Scott, Adams
- '80: applications in AI

Degrees of beliefs

- The probability that a particular bird A flies is at least 0.75.

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$$P_{\geq 0.75} Fly(A)$$

Early papers

- N. Nilsson, Probabilistic logic, *Artificial intelligence* 28, 71 – 87, 1986.
- H. Gaifman. A Theory of Higher Order Probabilities. In: *Proceedings of the Theoretical Aspects of Reasoning about Knowledge* (edts. J.Y. Halpern), Morgan-Kaufmann, San Mateo, California, 275–292. 1986.
- M. Fattorosi-Barnaba and G. Amati. Modal operators with probabilistic interpretations I. *Studia Logica* 46(4), 383–393. 1989.
- R. Fagin, J. Halpern and N. Megiddo. A logic for reasoning about probabilities. *Information and Computation* 87(1-2):78 – 128. 1990.
- M. Rašković. Classical logic with some probability operators. *Publications de l'Institut Mathématique*, n.s. 53(67), 1 – 3. 1993.
- R. Fagin and J. Halpern. Reasoning about knowledge and probability. *Journal of the ACM*, 41(2):340–367, 1994.
- A. Frish and P. Haddawy. Anytime deduction for probabilistic logic. *Artificial Intelligence* 69, 93 – 122. 1994.

Motivating example (1)

Example

Knowledge base:

if A_1 then B_1

if A_2 then B_2

if A_3 then B_3

...

Motivating example (1)

Example

Knowledge base:

if A_1 then B_1 (cf c_1)

if A_2 then B_2 (cf c_2)

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...

Uncertain knowledge: from statistics, our experiences and beliefs, etc.

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Uncertain knowledge: from statistics, our experiences and beliefs, etc.

- To check consistency of (finite) sets of sentences.
- To deduce probabilities of conclusions from uncertain premisses.

- The probabilistic logics allow strict reasoning *about* probabilities using well-defined syntax and semantics.
- Formulas in these logics remain either true or false.
- Formulas do not have probabilistic (numerical) truth values.

Formal language

- $\mathrm{Var} = \{p, q, r, \ldots\}$, connectives $\neg$ and $\wedge$ and

$$P_{\geq s}, \quad s \in Q \cap [0, 1]$$

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- $P_{\geq s}P_{\geq t}\alpha$, $\quad \beta \vee P_{\geq s}\alpha \notin For$

Semantics (1)

- A probabilistic model $M = \langle W, H, \mu, v \rangle$:
  - W is a nonempty set of elements called worlds,
  - H is an algebra of subsets of W,
  - $\mu : H \to [0, 1]$ is a finitely additive probability measure, and
  - $v : W \times \mathrm{Var} \to \{\top, \bot\}$ is a valuation

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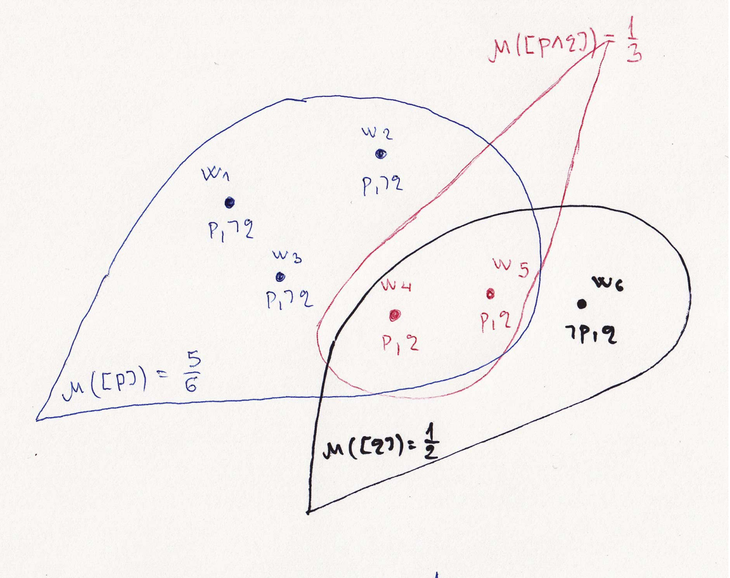
- Measurable models
  - $\alpha \in For_C$
  - $[\alpha] = \{w \in W : w \models \alpha\}$
  - $[\alpha] \in H$

Semantics (2)

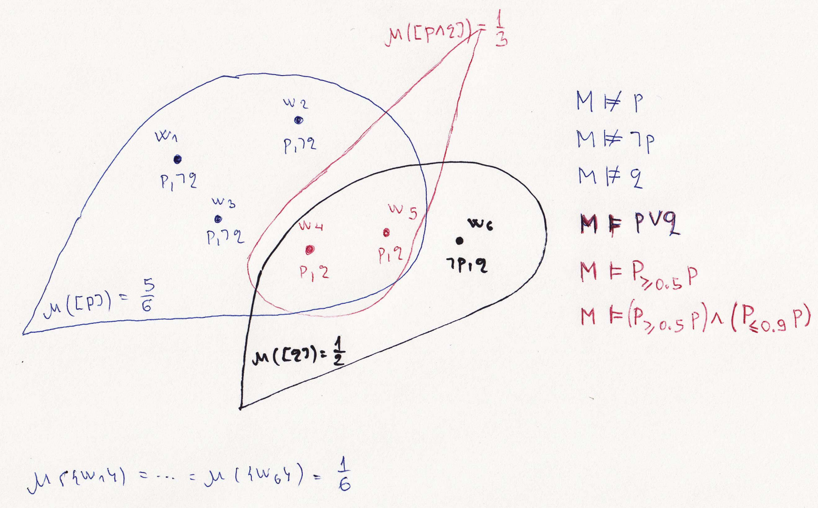


Satisfiability (1)

- if $\alpha \in For_C$, $M \models \alpha$ if $(\forall w \in W) v(w)(\alpha) = \top$
- $M \models P_{\geq s}\alpha$ if $\mu([\alpha]_M) \geq s$,
- if $A \in For_P$, $M \models \neg A$ if $M \not\models A$,
- if $A, B \in For_P$, $M \models A \wedge B$ if $M \models A$ and $M \models B$.

A set of formulas $F = \{A_1, A_2, \ldots\}$ is satisfiable if there is a model M, $M \models A_i$, $i = 1, 2, \ldots$.

Satisfiability (2)



Logical issues (1)

- Providing a sound and complete axiomatic system
  - simple completeness (every consistent formula is satisfiable, $\models A$ iff $\vdash A$)
  - extended completeness (every consistent set of formulas is satisfiable)
- Decidability (there is a procedure which decides if an arbitrary formula formula is valid)

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- Compactness (a set of formulas is satisfiable iff every finite subset is satisfiable).

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- c: $0 < c < \frac{1}{k}$, $\quad \mu[p] = c$
- M satisfies every F_k, but does not satisfy F

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- finitary (recursive) axiomatization + extended completeness $\Rightarrow$ compactness
- finitary axiomatization for real valued probabilistic logics: there are consistent sets that are not satisfiable

Logical issues (3)

- Restrictions on ranges of probabilities: $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$
- infinitary axiomatization

$LPP_2^{\mathrm{Fr}(n)}$

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$$\models_{LPP_2^{\mathrm{Fr}(n)}} P_{>\frac{1}{2}} p \to P_{\geq \frac{1+1}{2}} p$$

- $n = 3$, $LPP_2^{\mathrm{Fr}(3)}$, $\mu : H \to \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, $\mu(p) = \frac{2}{3}$

$$\not\models_{LPP_3^{\mathrm{Fr}(n)}} P_{>\frac{1}{2}} p \to P_{\geq \frac{1+1}{2}} p$$

LPP_2 (1)

Axioms

- all instances of classical propositional tautologies
- axioms for probabilistic reasoning
  - $P_{\geq 0}\alpha$
  - $P_{\leq r}\alpha \rightarrow P_{<s}\alpha$, $s > r$
  - $P_{<s}\alpha \rightarrow P_{\leq s}\alpha$
  - $(P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge P_{\geq 1}(\neg(\alpha \wedge \beta))) \rightarrow P_{\geq \min(1,r+s)}(\alpha \vee \beta)$
  - $(P_{\leq r}\alpha \wedge P_{<s}\beta) \rightarrow P_{<r+s}(\alpha \vee \beta)$, $r + s \leq 1$

LPP_2 (2)

Rules

- From Φ and $\Phi \rightarrow \Psi$ infer Ψ.
- From α infer $P_{\geq 1}\alpha$.
- From
$$\{A \rightarrow P_{\geq s-\frac{1}{k}}\alpha, \text{ for } k \geq \frac{1}{s}\}$$
infer
$$A \rightarrow P_{\geq s}\alpha.$$

LPP_2 (3)

- Proof from the set of formulas ($F \vdash \varphi$):
  - at most denumerable sequence of formulas $\varphi_0, \varphi_1, \ldots, \varphi$,
  - φ_i is an axiom or a formula from the set F,
  - or φ_i is derived from the preceding formulas by an inference rule
- A formula φ is a *theorem* ($\vdash \varphi$) if it is deducible from the empty set.

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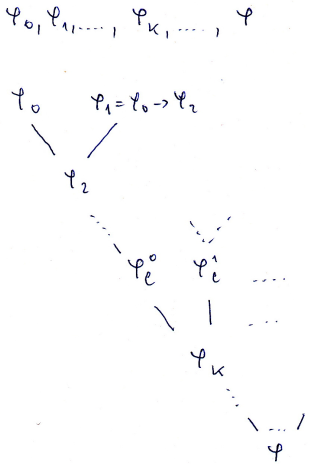
  - or φ_i is derived from the preceding formulas by an inference rule
- A formula φ is a *theorem* ($\vdash \varphi$) if it is deducible from the empty set.
- A set F of formulas is *consistent* if there are at least a classical formula and at least a probabilistic formula that are not deducible from F.

LPP_2 (4)



$LPP_2^{\mathrm{Fr}(n)}$ vs LPP_2

- $LPP_2^{\mathrm{Fr}(n)}$-Axiom

$$\bigvee_{k=0}^{n} P_{=\frac{k}{n}}\alpha$$

  i.e., $\mu([\alpha]) \in \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$

- instead of LPP_2-Rule:
  From
  $\{A \rightarrow P_{\geq s-\frac{1}{k}}\alpha$, for $k \geq \frac{1}{s}\}$
  infer
  $A \rightarrow P_{\geq s}\alpha$

New probabilistic operators

$$P_{\geq s}\alpha \wedge P_{\leq r}\alpha \qquad \ldots \qquad \mu([\alpha]) \in [s, r]$$

$$\bigvee_{k=0}^{n} P_{=\frac{k}{n}}\alpha \qquad \ldots \qquad \mu([\alpha]) \in \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$$

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New probabilistic operators

$P_{\geq s}\alpha \wedge P_{\leq r}\alpha$ ... $\mu([\alpha]) \in [s, r]$

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$\bigvee_{k=0}^{\infty} P_{=s_k}\alpha$

New probabilistic operators

$P_{\geq s}\alpha \wedge P_{\leq r}\alpha$ ... $\mu([\alpha]) \in [s, r]$

$\vee_{k=0}^{n} P_{=\frac{k}{n}}\alpha$ ... $\mu([\alpha]) \in \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$

$\mathbf{Q}_{\{\mathbf{s_0}, \mathbf{s_1}, \ldots, \mathbf{s_n}, \ldots\}}\alpha$ ... $\mu([\alpha]) \in \{s_0, s_1, \ldots, s_n, \ldots\}$

$\Leftrightarrow$

$\vee_{k=0}^{\infty} P_{=s_k}\alpha$

$LPP_{2,P,Q,O}$

Extension of LPP_2:

- O - recursive family of recursive subsets of $[0,1]_{\mathbb{Q}}$
- Q_F, $F \in O$
- $M \models Q_F p$ iff $\mu([p]) \in F$

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- Q_F, $F \in O$
- $M \models Q_F p$ iff $\mu([p]) \in F$

- Q_F's and $P_{\geq s}$'s are mutually undefinable

$$LPP_2^{\mathrm{Fr}(n)},\ \mu : H \to \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\},\ P_{\geq s}$$

$$LPP_2,\ \mu : H \to [0,1],\ P_{\geq s}$$

$$LPP_{2,P,Q,O},\ \mu : H \to [0,1],\ P_{\geq s},\ Q_F$$

$$LPP_2^{\mathrm{Fr}(n)},\ \mu : H \to \{0, \tfrac{1}{n}, \dots, \tfrac{n-1}{n}, 1\},\ P_{\geq s}$$

same language

diff. models

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same models
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$$LPP_{2,P,Q,O},\ \mu : H \to [0,1],\ P_{\geq s},\ Q_F$$

- (soundness) If $T \vdash \phi$, then $T \models \phi$;
- (deduction theorem) $T \vdash \phi \to \psi$ iff $T, \phi \vdash \psi$;
- (strong completeness) Every consistent theory is satisfiable;
- decidability: $LPP_2^{\mathrm{Fr}(n)}$, LPP_2
- (un)decidability: $LPP_{2,P,Q,O}$-logic is (un)decidable.

Hierarchies: LPP_2 and $LPP_{2,P,Q,O}$ (1)

- Measurable models: every $[\alpha] = \{w \in W : w \models \alpha\} \in H$
- $\mathcal{M}(\phi)$ is the set of all $M \in \mathcal{M}$ such that $M \models \phi$.

Hierarchies: LPP_2 and $LPP_{2,P,Q,O}$ (2)

- $F_1 = \{\frac{1}{2^i} : i = k, k+1, \ldots\},\ k > 0$
- $F_2 = \{\frac{1}{2^i} : i = 1, 2, \ldots\}$

Hierarchies: LPP_2 and $LPP_{2,P,Q,O}$ (2)

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- $F_2 = \{\frac{1}{2^i} : i = 1, 2, \ldots\}$

- $F_1 = F_2 \cap [0, \frac{1}{2^k}]$

- $\mathcal{M}(Q_{F_1}\alpha) = \mathcal{M}(Q_{F_2}\alpha \wedge P_{\leq \frac{1}{2^k}}\alpha)$

Hierarchies: LPP_2 and $LPP_{2,P,Q,O}$ (3)

$F \subseteq [0,1]_{\mathbb{Q}}$

quasi complement: $1 - F = \{1 - s : s \in F\}$

Definition

O_1 is **representable** in O_2 if every $F_1 \in O_1$ can be expressed as:

a finite union of

finite intersections of sets, differences between sets and quasi complements of

sets from O_2 and $[r, s]$, $[r, s)$, $(r, s]$ and (r, s), $r, s \in [0,1]_{\mathbb{Q}}$

Hierarchies: LPP_2 and $LPP_{2,P,Q,O}$ (4)

Definition

L_2 is **more expressive than** L_1
if for every formula $\phi \in For(P, Q, O_1)$ there is a formula $\psi \in For(P, Q, O_2)$ such that

$$\mathcal{M}(\phi) = \mathcal{M}(\psi)$$

Theorem

O_1 is representable in O_2 iff L_2 is more expressive than L_1

Hierarchies: LPP_2 and $LPP_{2,P,Q,O}$ (5)

Definition

Let O be a recursive family of recursive subsets of $[0,1]_{\mathbb{Q}}$. The family of all recursive subsets of $[0,1]_{\mathbb{Q}}$ that are representable in O is denoted by $\overline{O}$.

$$\mathcal{O}^* = \{\overline{O}_o : o \in \mathcal{O}_{/\sim}\}$$

Theorem

The structure $(\mathcal{O}^*, \subseteq)$ is a non-modular non-atomic lattice, with the smallest element which is σ-incomplete and without any maximal element.

Hierarchies: $LPP_2^{\mathrm{Fr}(n)}$ (1)

- $LPP_2^{\mathrm{Fr}(2)}$, $\mu : H \to \{0, \frac{1}{2}, 1\}$
- $LPP_2^{\mathrm{Fr}(4)}$, $\mu : H \to \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$

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- $\models_{LPP_2^{\mathrm{Fr}(2)}} P_{>\frac{1}{2}} p \rightarrow P_{\geq 1} p$
- $\not\models_{LPP_4^{\mathrm{Fr}(2)}} P_{>\frac{1}{2}} p \rightarrow P_{\geq 1} p$

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- $\models_{LPP_2^{\mathrm{Fr}(2)}} (\mathbf{P_{=0}p} \vee \mathbf{P_{=\frac{1}{2}}p} \vee \mathbf{P_{=1}p}) \to \quad (P_{>\frac{1}{2}} p \to P_{\geq 1} p)$
- $\models_{LPP_4^{\mathrm{Fr}(2)}} (\mathbf{P_{=0}p} \vee \mathbf{P_{=\frac{1}{2}}p} \vee \mathbf{P_{=1}p}) \to \quad (P_{>\frac{1}{2}} p \to P_{\geq 1} p)$

Hierarchies: $LPP_2^{\mathrm{Fr}(n)}$ (2)

Theorem

L_2 is more expressive than L_1 iff $\mathrm{Fr}(n_1) \subseteq \mathrm{Fr}(n_2)$

Theorem

The hierarchy is atomic and non-modular lattice with minimum and without a maximal element.

Two hierarchies

$$\mathbf{Q}_{\{\mathbf{0},\frac{\mathbf{1}}{\mathbf{2}},\mathbf{1}\}}\mathbf{p} \rightarrow (P_{>\frac{1}{2}}p \rightarrow P_{\geq 1}p)$$

Two hierarchies

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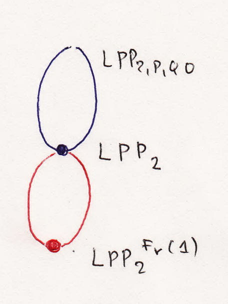

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2. *What kind of "weighted" logic are developed?*
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- values of probability functions in non-Archimedean structures
- intuitionistic logic, temporal logic, . . .
- conditional probabilities, first order logic

3. *For what purpose?*
    - checking consistency of finite sets of rules in expert systems
    - deducing probabilities of conclusions from uncertain premisses
    - modelling non-monotonic reasoning, spatial-temporal-uncertain reasoning
    - modelling situations when classical reasoning is not adequate (intuitionistic logic)

List of related publications

http://www.mi.sanu.ac.rs/~zorano/papers.html

An example

- reasoning about discrete sample spaces
- experiment: tossing a fair coin an arbitrary, but finite number of times
- α: only heads (i.e. no tails) are observed in the experiment
- $Q_{\{\frac{1}{2},\frac{1}{2^2},\frac{1}{2^3},\ldots\}}\alpha$

Lattice

- Lattice: a partially ordered set with unique least upper bounds and greatest lower bounds
- σ-Complete lattice: every set has a supremum and infimum
- Atomic lattice: for each non-zero element x, there exists an atom $a \leq x$
- Modularity law: $a \leq c$ implies $a \vee (b \wedge c) = (a \vee b) \wedge c$