Cut-elimination for modal fixed point logics

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1 Introduction

Modal fixed point logics with additional constructors for fixed points occur in many different places in computer science. For instance, there are temporal logics with an always operator, epistemic logics with a common knowledge operator, program logics with an iteration operator, and the propositional modal μ-calculus with fixed points for arbitrary positive formulas.

While the model-theoretic side of modal fixed point logics is very well investigated, we do not know much about the proof theory of these logics. In this talk we will survey syntactic cut-elimination results for modal logics with fixed points.

Most of these results make use of deep inference where rules may not only be applied to outermost connectives but also deeply inside formulas. The first result of this kind has been obtained by Pliuskevicius [10] who presents a syntactic cut-elimination procedure for linear time temporal logic. Brünnler and Studer [1] employ nested sequents to develop a cut-elimination procedure for the logic of common knowledge. Hill and Poggiolesi [6] use a similar approach to establish effective cut-elimination for propositional dynamic logic. A generalization of this method is studied in [2] where, however, it is also shown that it cannot be extended to fixed points that have a $\Box$-operator in the scope of a μ-operator. Fixed points of this kind occur, for instance, in CTL in the form of universal path quantifiers.

Thus we need a more general approach to obtain syntactic cut-elimination for the modal μ-calculus. A standard proof-theoretic technique to deal with inductive definitions and fixed points is Buchholz' Ω-rule [3, 5]. Jäger and Studer [7] present a formulation of the Ω-rule for non-iterated modal fixed point logic and they obtain cut-elimination for positive formulas of this logic. In order to overcome this restriction to positive formulas, Mints [8] introduces an Ω-rule that has a wider set of premises, which enables him to obtain full cut-elimination for non-iterated modal fixed point logic.

Mints' cut-elimination algorithm makes use of, in addition to ideas from [4], a new tool presented in [8]. It is based on the distinction, see [11], between implicit and explicit occurrences of formulas in a derivation with cut. If an occurrence of a formula is traceable to the endsequent of the derivation, then it is called explicit. If it is traceable to a cut-formula, then it is an implicit occurrence.

Implicit and explicit occurrences of greatest fixed points are treated differently in the translation of the induction rule to the infinitary system. An instance of the induction rule that derives a sequent $\nu X.A, B$ goes to an instance of the ω-rule if $\nu X.A$ is explicit. Otherwise, if $\nu X.A$ is traceable to a cut-formula, the induction rule is translated to an instance of the Ω-rule that is preserved until the last stage of cut-elimination. At that stage, called collapsing, the Ω-rule is eliminated completely. Recently, Mints and Studer [9] showed that this method can be extended to a μ-calculus with iterated fixed points. Hence they obtain complete syntactic cut-elimination for the one-variable fragment of the modal μ-calculus.

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