Cut-Elimination for Modal Fixed Point Logics

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based on joint work with
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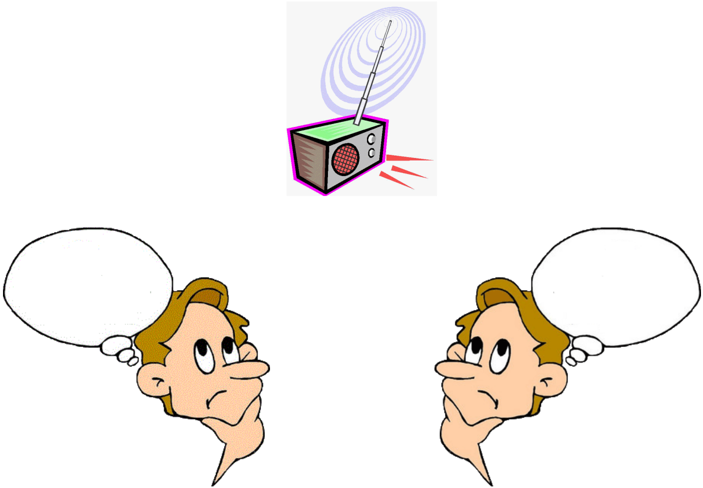
Roadmap

- Hilbert system for common knowledge
- Infinitary system based on an ω-rule
- Syntactic cut-elimination
- Infinite branches
- The situation for the μ-calculus
- Justification logic and common knowledge

Common knowledge



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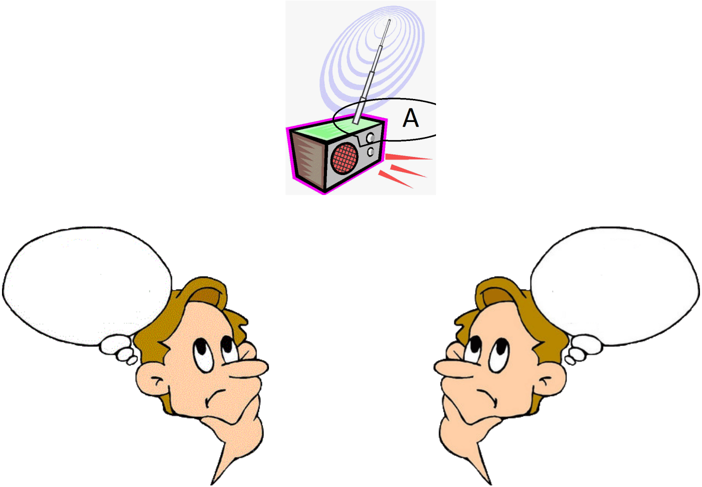

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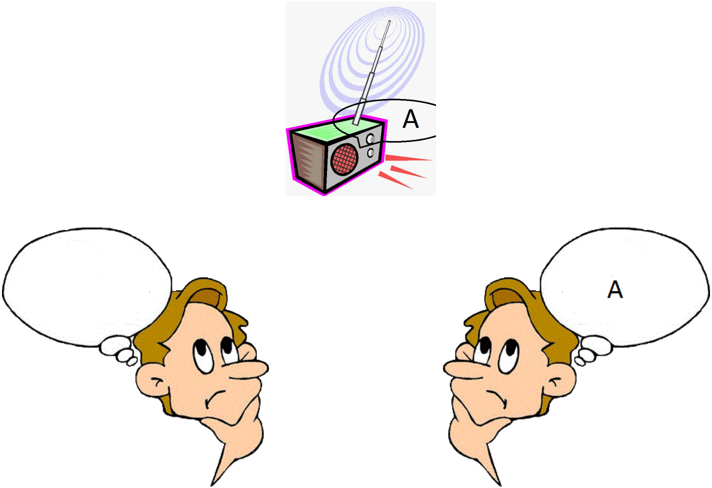

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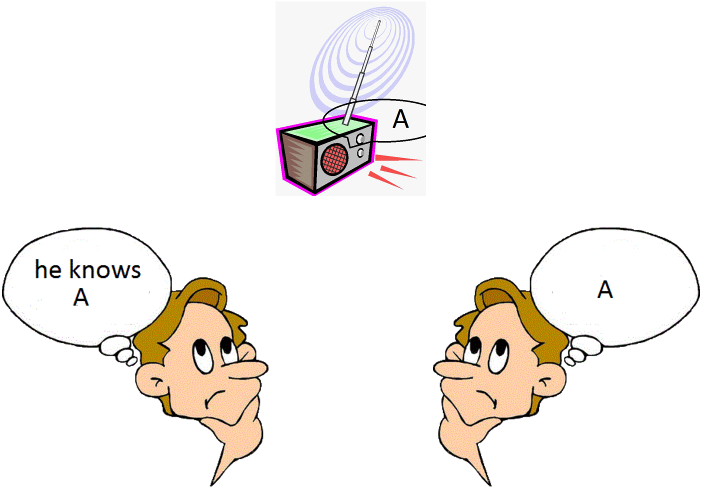

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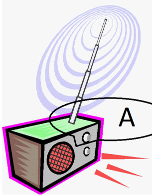

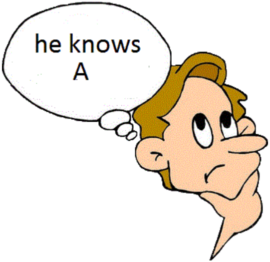

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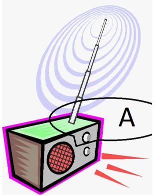
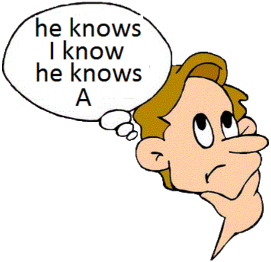




Common knowledge

Informally, common knowledge of a proposition A is defined as the infinitary conjunction

everybody knows A and

everybody knows that everybody knows A and

everybody knows that everybody knows that everybody knows A and

...

This is equivalent to:
Common knowledge of A is the greatest fixed point of

$$\lambda X.\text{everybody knows } A \text{ and everybody knows } X.$$

The Language

$$A ::= p \mid \bar{p} \mid (A \vee A) \mid (A \wedge A) \mid \Diamond_i A \mid \Box_i A \mid \circledast A \mid \boxast A$$

Abbreviations:

$$\begin{aligned} \Box A &= \Box_1 A \wedge \ldots \wedge \Box_h A \\ \Diamond A &= \Diamond_1 A \vee \ldots \vee \Diamond_h A \\ \Box^n A &= \underbrace{\Box \ldots \Box}_{n-\text{times}} A \end{aligned}$$

Negation and implication are defined as usual

The Hilbert System H_R

(TAUT) all instances of propositional tautologies

$$(\mathsf{MP})\ \frac{A \quad A \to B}{B}$$

$$(\mathsf{K})\quad \Box_i A \wedge \Box_i (A \to B) \to \Box_i B \qquad (\mathsf{NEC})\ \frac{A}{\Box_i A}$$

$$(\mathsf{CCL})\quad \circledast A \to (\Box A \wedge \Box \circledast A)$$

$$(\mathsf{I\text{-}R})\ \frac{B \to (\Box A \wedge \Box B)}{B \to \circledast A}$$

Theorem

H_R *is a sound and complete deductive system for common knowledge.*

The ω-rule: System $\mathsf{G_C}$

$$\Gamma, p, \bar{p} \qquad \wedge\ \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \vee\ \frac{\Gamma, A, B}{\Gamma, A \vee B}$$

$$\Box_i\ \frac{\Gamma, \circledast\Delta, A}{\Diamond_i\Gamma, \circledast\Delta, \Box_i A, \Sigma}$$

$$\boxast\ \frac{\Gamma, \Box^k A \quad \text{for all } k \geq 1}{\Gamma, \boxast A} \qquad \circledast\ \frac{\Gamma, \circledast A, \Diamond A}{\Gamma, \circledast A}$$

Theorem

$\mathsf{G_C}$ is a sound and complete deductive system for common knowledge.

The problem of cut-elimination

$$
\dfrac{
  \square_i \; \dfrac{\dfrac{\pi_1}{A, \Gamma, \circledast \bar{B}}}{\square_i A, \Diamond_i \Gamma, \Sigma, \circledast \bar{B}}
  \qquad
  \boxed{*} \; \dfrac{\cdots \quad \dfrac{\pi_{2k}}{\square^k B, \Delta} \quad \cdots}{\boxed{*} B, \Delta} \; 1 \leq k < \omega
}{\square_i A, \Diamond_i \Gamma, \Sigma, \Delta} \; \text{cut}
$$

Typical cut-elimination procedure yields:

$$
\square_i \; \dfrac{
  \dfrac{
    \dfrac{\pi_1}{A, \Gamma, \circledast \bar{B}}
    \qquad
    \boxed{*} \; \dfrac{\cdots \quad \dfrac{\pi_{2k}}{\square^k B, \Delta} \quad \cdots}{\boxed{*} B, \Delta} \; 1 \leq k < \omega
  }{A, \Gamma, \Delta} \; \text{cut}
}{\square_i A, \Diamond_i \Gamma, \Sigma, \Diamond_i \Delta}
$$

System $\mathsf{D_C}$

Nested sequents:

- make $\Box_i$ a structural rule
- allow deep application of rules

Ex: $A, B, [C, [D]_i]_j$ corresponds to $A \vee B \vee \Box_j(C \vee \Box_i D)$

$$\Gamma\{p, \bar{p}\} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \qquad \vee\ \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}$$

$$\Box_i\ \frac{\Gamma\{[A]_i\}}{\Gamma\{\Box_i A\}} \qquad \Diamond_i\ \frac{\Gamma\{\Diamond_i A, [\Delta, A]_i\}}{\Gamma\{\Diamond_i A, [\Delta]_i\}}$$

$$\boxast\ \frac{\Gamma\{\Box^k A\} \quad \text{for all } k \geq 1}{\Gamma\{\boxast A\}} \qquad \circledast\ \frac{\Gamma\{\circledast A, \Diamond^k A\}}{\Gamma\{\circledast A\}}$$

Properties of $\mathsf{D_C}$

Lemma (Structural rules and invertibility)

(i) The rules necessitation, weakening and contraction are admissible for system $\mathsf{D_C}$*.*
(ii) All rules in $\mathsf{D_C}$ *are invertible for* $\mathsf{D_C}$*.*

Theorem (Cut-elimination for the deep system)

If $\mathsf{D_C} \vdash^{\alpha}_{\omega \cdot n} \Gamma$*, then* $\mathsf{D_C} \vdash^{\varphi_1^n(\alpha)}_{0} \Gamma$*.*

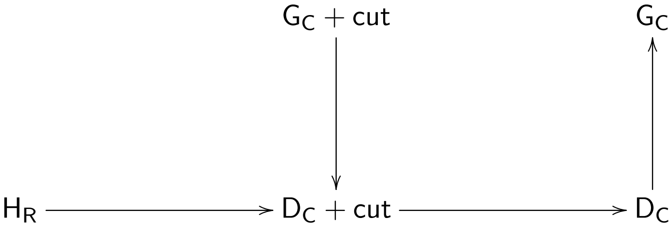
Theorem (Cut-elimination for the shallow system)

If $\mathsf{G_C} \vdash^{\alpha}_{\omega \cdot n} \Gamma$*, then* $\mathsf{G_C} \vdash^{\omega \cdot (\varphi_1^n(\omega \cdot \alpha)+1)}_{0} \Gamma$

Theorem (Upper bounds)

If A *is a valid formula, then* $\mathsf{D_C} \vdash^{<\varphi_2 0}_{0} A$ *and* $\mathsf{G_C} \vdash^{<\varphi_2 0}_{0} A$*.*

Cut-elimination on one slide



Infinite branches

The infinitary system S:

$$\Gamma, p, \bar{p} \qquad \wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B}$$

$$\Box_i\,\frac{\Gamma, A}{\Diamond_i \Gamma, \Box_i A, \Sigma}$$

$$\boxast\,\frac{\Gamma, \Box A \wedge \Box \boxast A}{\Gamma, \boxast A} \qquad \circledast\,\frac{\Gamma, \Diamond A \vee \Diamond \circledast A}{\Gamma, \circledast A}$$

Global condition: every infinite branch contains a $\boxast$-thread, i.e. there is a $\boxast A$ unfolded infinitely often.

An S-proof for the induction axiom

$$
\dfrac{\vdots}{\neg A, \Diamond\neg A, \diamondast(A \wedge \Diamond\neg A), \underline{\boxast A}}\ (\boxast)
$$

$$
\dfrac{\dfrac{(\text{ax'})}{\neg A, A, \diamondast(A \wedge \Diamond\neg A), \boxast A} \qquad \neg A, \Diamond\neg A, \diamondast(A \wedge \Diamond\neg A), \underline{\boxast A}}{\neg A, A \wedge \Diamond\neg A, \diamondast(A \wedge \Diamond\neg A), \underline{\boxast A}}\ (\wedge)
$$

$$
\dfrac{\neg A, A \wedge \Diamond\neg A, \diamondast(A \wedge \Diamond\neg A), \underline{\boxast A}}{\Diamond\neg A, \Diamond(A \wedge \Diamond\neg A), \Diamond\diamondast(A \wedge \Diamond\neg A), \underline{\Box\boxast A}}\ (\Box)
$$

$$
\dfrac{\Diamond\neg A, \Diamond(A \wedge \Diamond\neg A), \Diamond\diamondast(A \wedge \Diamond\neg A), \underline{\Box\boxast A}}{\Diamond\neg A, \Diamond(A \wedge \Diamond\neg A) \vee \Diamond\diamondast(A \wedge \Diamond\neg A), \underline{\Box\boxast A}}\ (\vee)
$$

$$
\dfrac{\Diamond\neg A, \Diamond(A \wedge \Diamond\neg A) \vee \Diamond\diamondast(A \wedge \Diamond\neg A), \underline{\Box\boxast A}}{\Diamond\neg A, \diamondast(A \wedge \Diamond\neg A), \underline{\Box\boxast A}}\ (\diamondast)
$$

$$
\dfrac{\dfrac{\dfrac{(\text{ax'})}{\neg A, A}}{\Diamond\neg A, \diamondast(A \wedge \Diamond\neg A), \Box A}\ (\Box) \qquad \Diamond\neg A, \diamondast(A \wedge \Diamond\neg A), \underline{\Box\boxast A}}{\Diamond\neg A, \diamondast(A \wedge \Diamond\neg A), \underline{\Box A \wedge \Box\boxast A}}\ (\wedge)
$$

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$$

Completeness for S

Let $\mathcal{T}$ be a proof search tree for Γ. Define an infinite game on it where player I tries to show that Γ is provable.

1. at any $(\Box')$ node, player I chooses one of the children,
2. at any $(\wedge)$ node, player II chooses one of the children,

Such a game results in a path in $\mathcal{T}$. Finite path: player I wins if the path ends in an axiom. Infinite path: player I wins if the path contains a $\circledast$-thread.

Theorem

1. *There is a winning strategy for player I if and only if there is an S-proof for Γ contained in $\mathcal{T}$.*
2. *There is a winning strategy for player II if and only if there is an S_{Dis}-disproof for Γ contained in $\mathcal{T}$.*
3. *The game is determined, i.e. one of the players has a winning strategy.*

Completeness for S

Theorem

S *is a complete deductive system for common knowledge.*

Proof. Let A be a formula that is not provable in S.
The proof search tree for A does not contain a proof for A.
There is no winning strategy for player I.
There must be a winning strategy for player II.
The proof search tree for A contains a $\mathsf{S}_{\mathsf{Dis}}$-disproof for A.
That disproof induces a counter model for A.

The situation for μ

H_μ is a Hilbert system for the modal μ-calculus

Theorem

H_μ *is a sound and complete deductive system for the* μ*-calculus.*

Proof: very involved

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G_μ is a Gentzen system (with an ω-rule) for the modal μ-calculus

Theorem

G_μ *is a sound and complete deductive system for the* μ*-calculus.*

Proof of soundness: uses finite model property
Proof of completeness: canonical model construction

The situation for μ (2)

D_μ is a nested sequent system (with an ω-rule) for the modal μ-calculus

Theorem

1. D_μ *is a sound and complete deductive system for the* $\nu\Box$*-fragment (aka continuous fragement).*
2. D_μ *enjoys syntactic cut-elimination.*
3. D_μ *is* not *complete for the modal* μ*-calculus.*

Proofs:

1. Syntactic embedding of the $\nu\Box$-fragment of G_μ
2. Standard
3. Counter example: accessible part may be larger than ω, i.e. the valid formula $\Box(\mu X.\Box X) \to \mu X.\Box X$ is not derivable.

The situation for μ (3)

S_μ is a system with infinite proof branches for the modal μ-calculus

Theorem

S_μ is a sound and complete deductive system for the μ-calculus.

Proof: using determinacy

Finitary Systems

Lemma (Small model property)

There is a function f such that if a formula A is satisfiable, then there exists a model of size at most $f(A)$.

Definition (The system $\mathsf{G_C}^{<\omega}$)

The system $\mathsf{G_C}^{<\omega}$ is defined by replacing the ω-rule in the system $\mathsf{G_C}$ by the rule

$$\frac{\Gamma, \Box^k A \quad \text{for all } 1 \leq k \leq f(\bigvee \Gamma \vee \boxast A)}{\Gamma, \boxast A, \Sigma}$$

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Other possibilities

- Use induction rule instead of ω-rule (AlberucciJäger05)
- Reformulate focus games as sequent calculi (BrünnlerLange08)
- Tableau systems (AbateGoréWidman07,GorankoShkatov08)

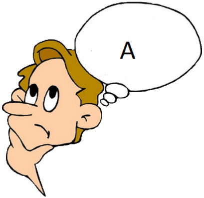
Why is it so difficult?

Theorem

The logic of common knowledge lacks Craig interpolation.

New ideas are needed to design a nice finitary cut-free system.

Modal Logic (without justifications)



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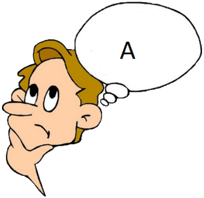

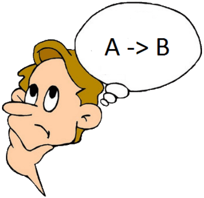

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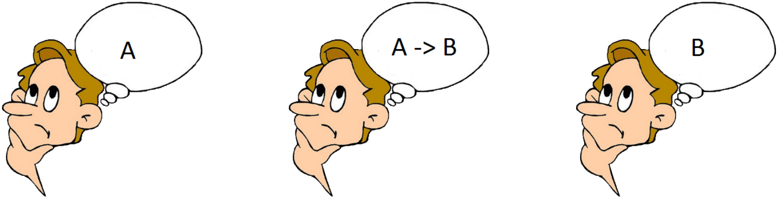

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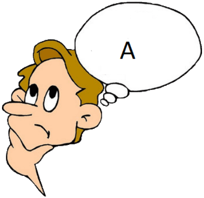
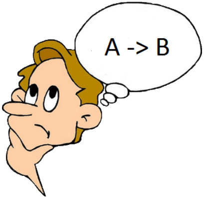
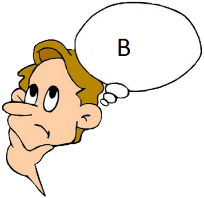
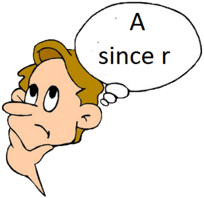






$$\Box A \quad \wedge \quad \Box(A \rightarrow B) \quad \rightarrow \quad \Box B$$

Justification Logic



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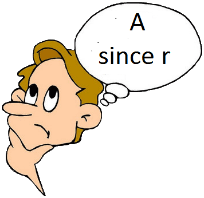

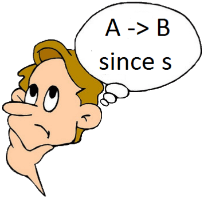

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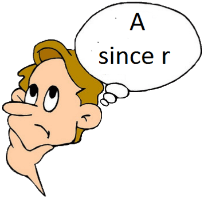

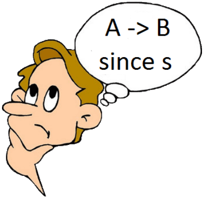

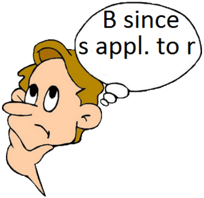

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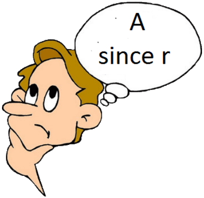

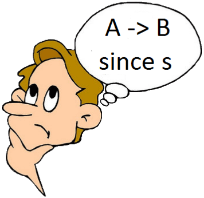

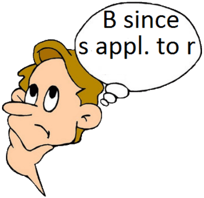

$$r : A \quad \wedge \quad s : (A \to B) \quad \to \quad s \cdot r : B$$

Internalization

Lemma

If

$$F_1, \ldots, F_m \vdash A,$$

then there exists a justification term $t(x_1, \ldots, x_m)$ *for fresh variables* $x_1, \ldots, x_m$ *such that*

$$x_1 : F_1, \ldots, x_m : F_m \vdash t(x_1, \ldots, x_m) : A \quad .$$

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Proof idea: for every rule there is a corresponding operation on terms that reflects that rule, i.e. to internalize

$$\text{(MP)}\ \frac{A \quad A \to B}{B}$$

we have

$$r : A \wedge s : (A \to B) \to s \cdot r : B \quad .$$

Internalizing common knowledge

How can we internalize the induction rule rule

$$\text{(I-R)}\ \frac{B \to (\Box A \land \Box B)}{B \to \boxast A} \quad ?$$

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$$\text{(I-R)}\ \frac{B \to (\Box A \wedge \Box B)}{B \to \boxast A} \quad ?$$

We don't know.

Internalizing common knowledge

How can we internalize the induction rule rule

$$\text{(I-R)}\ \frac{B \to (\Box A \land \Box B)}{B \to \boxast A} \quad ?$$

We don't know. Better use the induction axiom

$$\Box A \land \boxast(A \to \Box A) \to \boxast A \quad .$$

This gives

$$r^E : A \land s^C : (A \to t^E : A) \to \mathsf{ind}(r, s)^C : A \quad .$$

Forgetful projection

Definition (Forgetful projection)

If A is a formula of justification logic, then the modal formula A° is the result of replacing every term in A with the corresponding modal operator.

Theorem

If A is a theorem of justified common knowledge, then A° is a theorem of modal common knowledge.

The problem of realization

A realization is a mapping from modal formulae to justified formulae that replaces modal operators with justification terms.

Is there a realiation r such that A^r is a theorem of justified common knowledge for any theorem A of modal common knowledge?

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A realization is a mapping from modal formulae to justified formulae that replaces modal operators with justification terms.

Is there a realiation r such that A^r is a theorem of justified common knowledge for any theorem A of modal common knowledge?

Usually, realization is proved using a nice cut-free sequent calculus for modal logic. However, G_C does not work since we cannot merge infintely many premises.

Thus, we need a nice finitary cut-free system.

Thank you!