The unessential in classical logic and computation*

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Abstract

We present a congruence relation on classical proofs, which identifies proofs up to trivial rule permutation. The study is performed in the framework of ${}^*\mathcal{X}$ calculus, designed to provide a Curry-Howard correspondence for classical logic, therefore the terms can be seen as proofs. Each congruence class has a single diagrammatic representation.

1 Introduction

We first present a higher order rewrite system, the ${}^*\mathcal{X}$ calculus, which represents a computational interpretation of standard Gentzen's formulation for classical logic (the sequent system G1 [1]). This system is characterized by the presence of structural rules, namely weakening and contraction. In this calculus, which encompasses all the details of classical computation, we define which syntactically different terms are in essence the same.

The history of computational interpretations of classical logic is recent. The first one relying on sequents was presented by Herbelin [2], while a more direct correspondence with a standard sequent formulation of classical logic was presented in [3]. This research first lead to the $\mathcal{X}$ calculus [4] which served as a base to implement explicit erasure and duplication, yielding the ${}^*\mathcal{X}$ calculus [5].

2 ${}^*\mathcal{X}$ calculus, the syntax

Intuitively when we speak about ${}^*\mathcal{X}$-terms we speak about classical proofs in sequent system with explicit structural rules. Terms are built from *names*.

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This concept differs from that applied in λ-calculus, where *variable* is the basic notion. The difference lies in the fact that a variable can be substituted by an arbitrary term, while a name can be only *renamed* (that is, substituted by another name). The presences of hats over certain names denotes the binding of a name.

Definition 1 ($^*\mathcal{X}$-syntax) *The syntax of $^*\mathcal{X}$-calculus is presented in Figure 1, where $x, y, z...$ range over an infinite set of in-names and $\alpha, \beta, \gamma...$ range over an infinite set of out-names.*

P, Q	$::=$	$\langle x.\alpha \rangle$	*capsule*	(axiom rule)
	$\mid$	$\widehat{x}\, P\, \widehat{\beta} \cdot \alpha$	*exporter*	(right arrow-introduction)
	$\mid$	$P\, \widehat{\alpha}\, [x]\, \widehat{y}\, Q$	*importer*	(left arrow-introduction)
	$\mid$	$P \widehat{\alpha} \dagger \widehat{x} Q$	*cut*	(cut)
	$\mid$	$x \odot P$	*left-eraser*	(left weakening)
	$\mid$	$P \odot \alpha$	*right-eraser*	(right weakening)
	$\mid$	$z{<}^{\widehat{x}}_{\widehat{y}}\langle P]$	*left-duplicator*	(left contraction)
	$\mid$	$[P\rangle^{\widehat{\alpha}}_{\widehat{\beta}}{>}\gamma$	*right-duplicator*	(right contraction)

Figure 1: The syntax of $^*\mathcal{X}$

3 Assigning types to terms

The type system presents the way constructors are linked with logic, i.e., with proofs. Expressions of the form $P \vdots \Gamma \vdash \Delta$ represent the *type assignment* [1].

Definition 2 (Typable terms) *We say that a term P is* typable *if there exist contexts Γ and Δ such that $P \vdots \Gamma \vdash \Delta$ holds in the system of inference rules given by Figure 2.*

Reduction rules are numerous and capture the richness and complexity of classical cut elimination, but we will not be dealing with the dynamic of the system here (see [4, 5] for details).

4 The congruence relation

We introduce the congruence relation on terms, denoted $\equiv$, represented by a list of equations (the list is very partial due to limited space). For the complete

[1] Technically we assign contexts, which are sets of pairs (name, formula), to terms. If we forget about labels and consider only types, we are going back to Gentzen's classical system $G1$ (see Figure 2), where contexts are multisets of formulas.

$$\frac{}{\langle x.\alpha\rangle \vdots\cdot\ x:A \vdash \alpha:A}\ (ax)$$

$$\frac{P \vdots\cdot\ \Gamma \vdash \alpha:A,\Delta \qquad Q \vdots\cdot\ \Gamma', y:B \vdash \Delta'}{P\,\widehat{\alpha}\,[x]\,\widehat{y}\,Q \vdots\cdot\ \Gamma,\Gamma', x:A\to B \vdash \Delta,\Delta'}\ (\to L) \qquad \frac{P \vdots\cdot\ \Gamma, x:A \vdash \alpha:B,\Delta}{\widehat{x}\,P\,\widehat{\alpha}\cdot\beta \vdots\cdot\ \Gamma \vdash \beta:A\to B,\Delta}\ (\to R)$$

$$\frac{P \vdots\cdot\ \Gamma \vdash \alpha:A,\Delta \qquad Q \vdots\cdot\ \Gamma', x:A \vdash \Delta'}{P\widehat{\alpha}\,\dagger\,\widehat{x}Q \vdots\cdot\ \Gamma,\Gamma' \vdash \Delta,\Delta'}\ (cut)$$

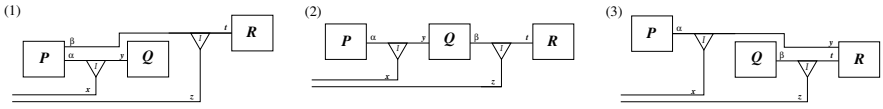
$$\frac{P \vdots\cdot\ \Gamma \vdash \Delta}{x \odot P \vdots\cdot\ \Gamma, x:A \vdash \Delta}\ (weak\text{-}L) \qquad \frac{P \vdots\cdot\ \Gamma \vdash \Delta}{P \odot \alpha \vdots\cdot\ \Gamma \vdash \alpha:A,\Delta}\ (weak\text{-}R)$$

$$\frac{P \vdots\cdot\ \Gamma, x:A, y:A \vdash \Delta}{z{<}^{\widehat{x}}_{\widehat{y}}\langle P\rangle \vdots\cdot\ \Gamma, z:A \vdash \Delta}\ (cont\text{-}L) \qquad \frac{P \vdots\cdot\ \Gamma \vdash \alpha:A,\beta:A,\Delta}{[P)^{\widehat{\alpha}}_{\widehat{\beta}}{>}\gamma \vdots\cdot\ \Gamma \vdash \gamma:A,\Delta}\ (cont\text{-}R)$$

Figure 2: The type assignent for implicative fragment

list see [5]. Congruence relation induced is reflexive, symmetric and transitive relation closed under any context. The motivation for introducing it into the system is to come closer to the essence of classical proofs, and abstract away from unessential in classical proofs. Every rule is associated with one corresponding diagram. A name is assigned to every congruence rule, and thus they are presented in the form: $name : P \equiv Q$.

importer-importer



$ii1$: $(P\,\widehat{\alpha}\,[x]\,\widehat{y}\,Q)\,\widehat{\beta}\,[z]\,\widehat{t}\,R \equiv (P\,\widehat{\beta}\,[z]\,\widehat{t}\,R)\,\widehat{\alpha}\,[x]\,\widehat{y}\,Q$ with $\alpha,\beta \in N(P)$

$ii2$: $(P\,\widehat{\alpha}\,[x]\,\widehat{y}\,Q)\,\widehat{\beta}\,[z]\,\widehat{t}\,R \equiv P\,\widehat{\alpha}\,[x]\,\widehat{y}\,(Q\,\widehat{\beta}\,[z]\,\widehat{t}\,R)$ with $y,\beta \in N(Q)$

$ii3$: $(Q\,\widehat{\beta}\,[z]\,\widehat{t}\,R) \equiv Q\,\widehat{\beta}\,[z]\,\widehat{t}\,(P\,\widehat{\alpha}\,[x]\,\widehat{y}\,R)$ with $y,t \in N(R)$

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cut-importer

$ci1$: $(P\,\widehat{\alpha}\,[x]\,\widehat{y}\,Q)\widehat{\beta}\,\dagger\,\widehat{z}R \;\equiv\; (P\widehat{\beta}\,\dagger\,\widehat{z}R)\,\widehat{\alpha}\,[x]\,\widehat{y}\,Q$ with $\alpha, \beta \in N(P)$

$ci2$: $(P\,\widehat{\alpha}\,[x]\,\widehat{y}\,Q)\widehat{\beta}\,\dagger\,\widehat{z}R \;\equiv\; P\,\widehat{\alpha}\,[x]\,\widehat{y}\,(Q\widehat{\beta}\,\dagger\,\widehat{z}R)$ with $y, \beta \in N(Q)$

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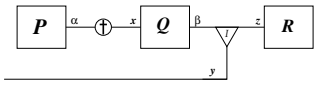
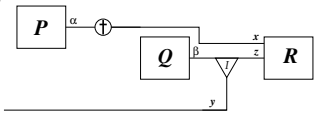
(3)



(4)



$ci3$: $P\widehat{\alpha}\,\dagger\,\widehat{x}(Q\,\widehat{\beta}\,[y]\,\widehat{z}\,R) \;\equiv\; (P\widehat{\alpha}\,\dagger\,\widehat{x}Q)\,\widehat{\beta}\,[y]\,\widehat{z}\,R$ with $x, \beta \in N(Q)$

$ci4$: $P\widehat{\alpha}\,\dagger\,\widehat{x}(Q\,\widehat{\beta}\,[y]\,\widehat{z}\,R) \;\equiv\; Q\,\widehat{\beta}\,[y]\,\widehat{z}\,(P\widehat{\alpha}\,\dagger\,\widehat{x}R)$ with $x, z \in N(R)$

The relation $\equiv$ induces congruence classes on terms. It has been argued in [6] that two sequent proofs induce the same proof net if and only if one can be obtained from the other by a sequence of transpositions of independent rules. At this static level we have proceeded further as we have *explicitly* shown by congruence rules, how exactly this transposition is done.

Basic properties of $\equiv$, and terms as diagrams The congruence relation enjoys important properties. Since it describes the way to perform restructuring of terms, it is important to have preservation of the set of free names. Then, the property of type preservation ensures that term-restructuring defined by $\equiv$ can be seen as proof-transformation.

The reader could already see the diagrams as intuitive illustration of congruence rules. It is possible to define a translation (call it $\mathcal{D}$) from terms to diagrams, inductively on the structure of terms, but due to a lack of space we don't present it here. Based on that definition, ideally we hope to prove that each congruence class (with many terms) has a single diagrammatic representation. Moreover, in the framework of future work and the dynamics of the system, to show that a single reduction step corresponds to a diagrammatic reduction step.

5 Conclusion

By explicitly stating which syntactically different terms should be considered the same, we unveil the unessential part of sequent classical proofs, sometimes referred to as the *syntactic bureaucracy*. This is done in the framework of ${}^*\mathcal{X}$. It is illustrated that such computational model comes close to diagrammatic computation, as the concept of reducing modulo corresponds to diagrammatic reductions which focuses on the essence of classical proofs and drastically reduces the number of reduction steps.

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