Logics for probabilistic spatio-temporal reasoning

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Logic and Applications 2014  
Dubrovnik, September 23th 2014
Outline

1 Probabilistic logics
   • About probabilistic logics
   • Syntax and semantics
   • Non-compactness as an axiomatization issue
   • Variants

2 PST logics
   • The PST framework for probabilistic spatiotemporal databases
   • $L_{ST+PST}^Q$: syntax and semantics
   • $L_{ST+PST}^Q$: a complete axiomatization

3 Future work
   • adding temporal operators
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   - Variants

2 PST logics
   - The PST framework for probabilistic spatiotemporal databases
   - $L^Q_{ST+PST}$: syntax and semantics
   - $L^Q_{ST+PST}$: a complete axiomatization

3 Future work
   - Adding temporal operators
What are PLs?

Logic:
- syntax (language, well formed formulas)
What are PLs?

Logic:
- syntax (language, well formed formulas)
- semantics (models, satisfiability)
- consequence relation
Logic:
- syntax (language, well formed formulas)
- semantics (models, satisfiability)
- consequence relation
- axiomatic system (axioms, rules)
- proof
The probabilistic logics allow strict reasoning about probabilities using well-defined syntax and semantics.

Formulas in these logics remain either true or false.

Formulas do not have probabilistic (numerical) truth values.
Probabilistic quantifiers and operators
About probabilistic logics

Probabilistic quantifiers and operators

Quantifiers – statistical probability:
Probabilistic logics

Quantifiers – statistical probability:
- Model theory (Keisler, mid 70’s)
Probabilistic quantifiers and operators

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- generalization of $\forall$, $\exists$
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- $P_x \geq r \alpha(x)$
Probabilistic logics

Probabilistic quantifiers and operators

Quantifiers – statistical probability:
- Model theory (Keisler, mid 70’s)
- Generalization of $\forall$, $\exists$
- $P_{x \geq r} \alpha(x)$
- Semantics: $\mu(\{a \mid \mathcal{M} \models \alpha(a)\}) \geq r$
Probabilistic quantifiers and operators

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Operators – subjective probability:
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Operators – subjective probability:
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- $P_{\geq r} \alpha \quad (P(\alpha) \geq r)$
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Operators – subjective probability:
- Theoretical computer science (Fagin, Halpern, Megiddo, 1990)
- generalization of $\Box, \Diamond$
- $P_{\geq r} \alpha \ (P(\alpha) \geq r)$
- semantics: modal semantics – measure of all worlds in which $\alpha$ holds is at least $r$
Examples of probability formulas
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- \( P(\alpha) \leq \frac{1}{3} \land P(\beta) = \frac{1}{5} \rightarrow P(\alpha \lor \beta) < \frac{8}{15} \)
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Examples of probability formulas

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- $2P(\alpha) - 3P(\beta) + \frac{1}{2} \geq \frac{1}{3} P(\gamma)$ (LWF)
- $P(\alpha \land \beta) \geq \frac{1}{2} P(\beta)$
Examples of probability formulas

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- \( 2P(\alpha) - 3P(\beta) + \frac{1}{2} \geq \frac{1}{3} P(\gamma) \) (LWF)
- \( P(\alpha \land \beta) \geq \frac{1}{2} P(\beta) \)
- \( P(\alpha) + 2P(\beta) P(\gamma) \geq \frac{2}{3} \) (PWF)
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   - The PST framework for probabilistic spatiotemporal databases
     - $L_{ST+PST}^Q$: syntax and semantics
     - $L_{ST+PST}^Q$: a complete axiomatization

3. Future work
   - adding temporal operators
Syntax

\[ r \in \mathbb{Q} \cup [0, 1]; \text{ probability operator } P_{\geq r} \alpha \quad (P_{\geq 1} \approx \Box) \]
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- Language:
Syntax and semantics

Syntax

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- Language:
  - propositional letters \( \{p, q, r, \ldots\} \)
  - Boolean connectives \( \neg, \wedge \)
  - a list of probability operators \( P_{\geq r} \)
Syntax and semantics

Syntax

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- Language:
  - propositional letters \{p, q, r, \ldots\}
  - Boolean connectives \(\neg\), \(\land\)
  - a list of probability operators \(P_{\geq r}\)

- The set of formulas is the smallest set containing propositional letters and closed under \(\neg\), \(\land\) and \(P_{\geq r}\)
Syntactic and semantics

Semantics

⟨W, Prob, v⟩
W ≠ ∅ – worlds
v: W × P → {⊤, ⊥}
Prob assigns to every w ∈ W a probability space
Prob(w) = ⟨W(w), H(w), µ(w)⟩:
W(w) – a non-empty subset of W,
H(w) – an algebra of subsets of W(w),
µ(w): H(w) → [0, 1] – a finitely additive probability measure on H(w).
Semantics

\[ \langle W, \text{Prob}, v \rangle \]
Semantics

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- \( W \neq \emptyset \) – worlds
- \( v : W \times \mathcal{P} \rightarrow \{\top, \bot\} \) – valuations
Semantics

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- \( \text{Prob} \) assigns to every \( w \in W \) a probability space
  \( \text{Prob}(w) = \langle W(w), H(w), \mu(w) \rangle \):
Semantics

\[ \langle W, \text{Prob}, v \rangle \]

- \( W \neq \emptyset \) – worlds
- \( v : W \times \mathcal{P} \rightarrow \{\top, \bot\} \) – valuations
- \text{Prob} assigns to every \( w \in W \) a probability space \( \text{Prob}(w) = \langle W(w), H(w), \mu(w) \rangle \):
  - \( W(w) \) – a non empty subset of \( W \),
  - \( H(w) \) – an algebra of subsets of \( W(w) \)
  - \( \mu(w) : H(w) \rightarrow [0, 1] \) – a finitely additive probability measure on \( H(w) \).
Satisfiability relation

- $M, w \models \alpha$ iff $\nu(w)(p) = T$, 

- $M, w \models \alpha \land \beta$ iff $M, w \models \alpha$ and $M, w \models \beta$,

- $M, w \models P \geq s \alpha$ iff $\mu(w)([\alpha]_M, w) \geq s [\alpha]_M$, 

- $\{u \in W(w) : M, u \models \alpha\}$
Satisfiability relation

- $M, w \models \alpha$ iff $\nu(w)(p) = \top$,
- $M, w \models \neg \alpha$ iff $M, w \not\models \alpha$,
- $M, w \models \alpha \land \beta$ iff $M, w \models \alpha$ and $M, w \models \beta$, and
Satisfiability relation

- $M, w \models \alpha$ iff $v(w)(p) = \top$,
- $M, w \models \neg \alpha$ iff $M, w \not\models \alpha$,
- $M, w \models \alpha \land \beta$ iff $M, w \models \alpha$ and $M, w \models \beta$, and
- $M, w \models P_{\geq s} \alpha$ iff $\mu(w)([\alpha]_{M, w}) \geq s$

$[\alpha]_{M, w} = \{ u \in W(w) : M, u \models \alpha \}$
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3. **Future work**
   - adding temporal operators
Non-compactness as an axiomatization issue

Example

- Inherent non-compactness:

\[ T = \{ \neg P \leq 0 \} \cup \{ P < \frac{1}{n}p : n \text{ is a positive integer} \} \]
Non-compactness as an axiomatization issue

Example

- Inherent non-compactness:
  \[ T = \{ \neg P = 0 p \} \cup \{ P_{<1/n} p : n \text{ is a positive integer} \} \]

- \[ T_k = \{ \neg P = 0 p, P_{<1/1} p, P_{<1/2} p, \ldots, P_{<1/k} p \} \]

- \( c: 0 < c < \frac{1}{k}, \quad \mu[p] = c \)

- \( M \) satisfies every \( T_k \), but does not satisfy \( T \)
Example

- Inherent non-compactness:
  \[ T = \{ \neg P = 0 \} \cup \{ P < 1/n : n \text{ is a positive integer} \} \]

- \( T_k = \{ \neg P = 0, P < 1/1p, P < 1/2p, \ldots, P < 1/kp \} \)
- \( c : 0 < c < \frac{1}{k}, \quad \mu[p] = c \)
- \( M \) satisfies every \( T_k \), but does not satisfy \( T \)

- finitary (recursive) axiomatization + strong completeness \( \Rightarrow \) compactness
- finitary axiomatization for real valued probabilistic logics: there are consistent sets that are not satisfiable
Non-compactness as an axiomatization issue

Approaches

- weak completeness
Non-compactness as an axiomatization issue

Approaches

- weak completeness

- Restrictions on ranges of probabilities: \( \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\} \)
Non-compactness as an axiomatization issue

Approaches

- weak completeness

- Restrictions on ranges of probabilities: \( \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\} \)

- infinitary axiomatization
Infinitary inference rule

Infinitary formula:

“if a-probability of $\alpha$ is infinitely close to the rational number $r \in [0, 1]$, then it must be equal to $r$”
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Intuitive form of the rule:
Infinitary inference rule

Infinitary formula:

“if $a$-probability of $\alpha$ is infinitely close to the rational number $r \in [0, 1]$, then it must be equal to $r$”

Intuitive form of the rule:

\[
\frac{\{P_{>r-\frac{1}{n}}\alpha \mid n \in \omega\}}{P_{\geq r}\alpha}
\]
Infinitary inference rule

Infinitary formula:

“if a-probability of $\alpha$ is infinitely close to the rational number $r \in [0, 1]$, then it must be equal to $r$”

Intuitive form of the rule:

\[
\frac{\{P_{> r - \frac{1}{n}} \alpha \mid n \in \omega\}}{P_{\geq r} \alpha}
\]

+ implicative form of the rule (for proving Deduction theorem)
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3 Future work
   - adding temporal operators
Basic variants

- without iteration of probabilities
  \[ P_{\geq s} P_{\geq t} \alpha, \quad \beta \lor P_{\geq s} \alpha \notin For \]
Basic variants

- without iteration of probabilities
  \[ P_{\geq s} P_{\geq t} \alpha, \quad \beta \lor P_{\geq s} \alpha \not\in For \]
- first order logic
Variants

Basic variants

- without iteration of probabilities
  \[ P_{\geq s} P_{\geq t} \alpha, \quad \beta \lor P_{\geq s} \alpha \not\in \text{For} \]
- first order logic
- values of probability functions in non-Archimedean structures
Basic variants

- without iteration of probabilities
  \[ P_{\geq s} P_{\geq t} \alpha, \quad \beta \lor P_{\geq s} \alpha \notin \text{For} \]
- first order logic
- values of probability functions in non-Archimedean structures
- change underlying logic
(Probabilistic) extension of syntax

- conditional probabilities
Variants

(Probabilistic) extension of syntax

- conditional probabilities
- $\preceq$ – qualitative probability operator

If $\alpha, \beta \in \text{For}_C$, $M \models \alpha \preceq \beta$ iff $\mu([\alpha]) \leq \mu([\beta])$. 

Variants

(Probabilistic) extension of syntax

- conditional probabilities
- \( \preceq \) – qualitative probability operator
  
  If \( \alpha, \beta \in For_C \), \( M \models \alpha \preceq \beta \) iff \( \mu([\alpha]) \leq \mu([\beta]) \),

- independency operator (todo)

- probability of a formula belongs to a countable set
The PST framework for probabilistic spatiotemporal databases

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PST atom

- GPS systems – possibility of tracking moving objects (vehicles, cell phones...)
- AI – representing such information
  - involve space and time
  - probability (uncertainty about the identity of an object, its exact location or time value)
GPS systems – possibility of tracking moving objects (vehicles, cell phones...)

AI – representing such information

- involve space and time
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ST (SpatioTemporal) atom: $\text{loc}(id, r, t)$

a particular object $id$ is in a particular region $r$ at a particular time $t$
GPS systems – possibility of tracking moving objects (vehicles, cell phones...)

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ST (SpatioTemporal) atom: \( \text{loc}(id, r, t) \)

a particular object \( id \) is in a particular region \( r \) at a particular time \( t \)

PST (Probabilistic SpatioTemporal) atom: \( \text{loc}(id, r, t)[\ell, u] \)
PST database

- PST database is any set of PST atoms
PST database

- PST database is any set of PST atoms
- Semantics:
  - (possible) world – mapping of objects (for every time instance) in space (+ reachability constraints)
  - interpretation – probability distribution over worlds
Limitations of PST formalism

- "Dragan is in Luxembourg", "Dragan is in Dubrovnik"
- but not "Dragan is in Luxembourg or Dubrovnik"
Limitations of PST formalism

- "Dragan is in Luxembourg", "Dragan is in Dubrovnik"
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- \( \text{loc}(Bus1, Q, 5) \) and \( \text{loc}(Bus2, R, 6)[.4, 1] \)
- but not \( \text{loc}(Bus1, Q, 5) \) or \( \text{loc}(Bus2, R, 6)[.4, 1] \)
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Definition \((ID, S, T)\)

- \(ID\) is a finite set of objects.
- \(S\) is a finite set of points in space.
- \(T = \{1, \ldots, N\}\) is a finite set of time instances.
**ST formula**

**Definition (\(ID, S, T\))**

- \(ID\) is a finite set of objects.
- \(S\) is a finite set of points in space.
- \(T = \{1, \ldots, N\}\) is a finite set of time instances.

**Definition (ST formula)**

- An **ST atom**: a formula of the form \(\text{loc}(id, r, t)\), where \(id \in ID\), \(t \in T\), and \(r \subseteq S\).
- **ST formula**: a Boolean combination of ST atoms; connectives: \(\sim\) (negation), \& (conjunction), \(\mid\) (disjunction), \(\supset\) (implication), and \(\equiv\) (equivalence).
- Notation: \(\mathcal{A}\) – the set of ST formulas; \(\alpha, \beta\) – ST formulas.
PST formula

**Definition (PST formula)**

- **Basic PST atom**: a formula of the form $\alpha[0, u]$, where $\alpha \in \mathcal{A}$.
- **PST formula**: a Boolean combination of basic PST atoms; connectives: $\neg$, $\land$, $\lor$, $\rightarrow$, and $\leftrightarrow$.
- **Notation**: $\mathcal{P}$ – the set of all PST formulas; $\rho$ and $\sigma$ – PST formulas.
PST formula

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**Abreviations:**

- $\alpha(\ell, 1]$ is $\neg \alpha[0, \ell]$.
- $\alpha[\ell, 1]$ is $\sim \alpha[0, 1 - \ell]$.
- $\alpha[0, u)$ is $\neg \alpha[u, 1]$.
- if $0 \leq \ell \leq u \leq 1$, then $\alpha[\ell, u]$ is $\alpha[0, u] \land \alpha[\ell, 1]$.
- if $0 \leq \ell < u \leq 1$, we define $\alpha[\ell, u)$, $\alpha(\ell, u]$, and $\alpha(\ell, u)$ similarly as above.
Example of ST formula:

\( loc(id_2, \{p_2, p_4\}, 2) \& loc(id_2, \{p_2, p_4\}, 3) \)
Example of ST formula:
\( \text{loc}(id_2, \{p_2, p_4\}, 2) \& \text{loc}(id_2, \{p_2, p_4\}, 3) \)

Example of PST formula:
\( \text{loc}(id_1, \{p_2, p_3\}, 1)[0, .5] \lor \\
(\text{loc}(id_2, \{p_2, p_4\}, 2) \& \text{loc}(id_2, \{p_2, p_4\}, 3))[.5, 1] \)
Formula

Example of ST formula:
\[ \text{loc}(id_2, \{p_2, p_4\}, 2) \& \text{loc}(id_2, \{p_2, p_4\}, 3) \]

Example of PST formula:
\[ \text{loc}(id_1, \{p_2, p_3\}, 1)[0, .5] \lor \text{loc}(id_2, \{p_2, p_4\}, 1)[.5, 1] \]

**Definition (Formula)**

- \( \mathcal{F} = \mathcal{A} \cup \mathcal{P} \)
- Notation: \( \phi \) and \( \psi \) -formulas; set of arbitrary formulas is indicated by \( \Phi \)
Proabilistic logics

PST logics

Future work

$L_{ST\perp PST}$: syntax and semantics

World

- Reachability definition $RD \subseteq S \times S$
- $(p, p') \in RD$ – an $id$ can reach $p'$ from $p$ in one unit of time.
World

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- $(p, p') \in RD$ – an id can reach $p'$ from $p$ in one unit of time.

Definition (World)

An RD-compliant world is a function $w : ID \times T \rightarrow S$, satisfying the condition:

if $w(id, t) = p_1$ and $w(id, t + 1) = p_2$ then $(p_1, p_2) \in RD$. 
World

- Reachability definition $RD \subseteq S \times S$
- $(p, p') \in RD$ – an $id$ can reach $p'$ from $p$ in one unit of time.

**Definition (World)**

An RD-compliant world is a function $w : ID \times T \rightarrow S$, satisfying the condition:

$$\text{if } w(id, t) = p_1 \text{ and } w(id, t + 1) = p_2 \text{ then } (p_1, p_2) \in RD.$$ 

**Definition (Valuation)**

Given $W$, the valuation $v_W : A \times W \rightarrow \{0, 1\}$ is defined as follows:

- $v_W(loc(id, r, t), w) = 1$ iff $w(id, t) \in r$,
- $v_W(\alpha \& \beta, w) = 1$ iff $v_W(\alpha, w) = 1$ and $v_W(\beta, w) = 1$,
- $v_W(\sim \alpha, w) = 1$ iff $v_W(\alpha, w) = 0$. 
Definition (Interpretation)

An interpretation $I : W \rightarrow [0, 1] \cap \mathbb{Q}$ is a probability distribution over $W$, i.e., a nonnegative function such that

$$\sum_{w \in W} I(w) = 1.$$
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$$\sum_{w \in W} I(w) = 1.$$ 

Definition (PST Structure)

A PST structure is a pair $\langle W, I \rangle$ where $W$ is a (nonempty) set of worlds and $I$ is an interpretation.
Satisfiability

Definition (Satisfiability)

Let \( M = \langle W, I \rangle \) be a PST structure. We define the satisfiability relation \( \models \) recursively as follows:

- \( M \models \alpha \) iff \( \nu_W(\alpha, w) = 1 \) for all \( w \in W \).
- \( M \models \alpha[0, u] \) iff \( \sum_{\nu_W(\alpha, w) = 1} I(w) \leq u \).
- \( M \models \neg \rho \) iff \( M \not\models \rho \).
- \( M \models \rho \land \sigma \) iff \( M \models \rho \) and \( M \models \sigma \).
### Satisfiability

**Definition (Satisfiability)**

Let $M = \langle W, I \rangle$ be a PST structure. We define the satisfiability relation $\models$ recursively as follows:

- $M \models \alpha$ iff $\nu_W(\alpha, w) = 1$ for all $w \in W$.
- $M \models \alpha[0, u]$ iff $\sum_{\nu_W(\alpha, w)=1} I(w) \leq u$.
- $M \models \neg \rho$ iff $M \not\models \rho$.
- $M \models \rho \land \sigma$ iff $M \models \rho$ and $M \models \sigma$.

**Definition (Entailment)**

- $M$ is a *model* of $\Phi$, $M \models \Phi$, iff $M \models \phi$ for every $\phi \in \Phi$.
- $\Phi \models \phi$, $\Phi$ *entails* $\phi$, iff all models of $\Phi$ are models of $\phi$.
- $\phi$ is *valid*, that is, it is satisfied in every PST structure.
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Axioms

Propositional reasoning

- All instances of classical propositional tautologies for both ST and PST formulas.
Axioms

Propositional reasoning

- All instances of classical propositional tautologies for both ST and PST formulas.

Spatio-temporal axioms

- \( \text{loc}(id, S, t) \).
- \( \text{loc}(id, r, t) \equiv \|_{p \in r} \text{loc}(id, \{p\}, t) \).
- \( \text{loc}(id, \{p\}, t) \supset \sim \text{loc}(id, \{p'\}, t), p \neq p' \).
- \( \sim (\text{loc}(id, \{p\}, t) \& \text{loc}(id, \{p'\}, t + 1)), (p, p') \notin RD, t < N. \)
Axioms

Probabilistic axioms

- $\alpha[0, 1]$.
- $\alpha[\ell, u] \rightarrow \alpha[\ell, u]$.
- $\alpha[\ell, u] \rightarrow \alpha[\ell, u'], u < u'$.
- $(\alpha[\ell, u] \land \beta[\ell', u'] \land \neg (\alpha \& \beta)[1, 1]) \rightarrow \alpha|\beta[\ell'', u'']$,  
  \[
  \ell'' = \min\{\ell + \ell', 1\}, \quad u'' = \min\{u + u', 1\}.
  \]
- $\alpha[0, u] \land \beta[0, u') \rightarrow \alpha|\beta[0, u''], u'' = u + u', u + u' \leq 1$. 

$L^Q_{ST \perp PST}$: a complete axiomatization
Inference rules

- (a) From $\alpha$ and $\alpha \supset \beta$ infer $\beta$
- (b) From $\rho$ and $\rho \rightarrow \sigma$ infer $\sigma$.
- From $\alpha$ infer $\alpha[1, 1]$. 
Inference rules

(a) From $\alpha$ and $\alpha \supset \beta$ infer $\beta$
(b) From $\rho$ and $\rho \rightarrow \sigma$ infer $\sigma$.
From $\alpha$ infer $\alpha[1, 1]$.
From the set of premises

$$\{\rho \rightarrow \neg \alpha[q, q] \mid q \in \mathbb{Q} \cap [0, 1]\}$$

infer $\neg \rho$. 
• $\phi$ is deducible from $\Phi$ ($\Phi \vdash \phi$) if there is an at most countable sequence of formulas $\phi_0, \phi_1, \ldots, \phi$, such that every $\phi_i$ is an axiom or a formula from the set $\Phi$, or is derived from the preceding formulas by an inference rule.

• $\phi$ is a theorem if $\emptyset \vdash \phi$
\( \phi \) is deducible from \( \Phi \) (\( \Phi \vdash \phi \)) if there is an at most countable sequence of formulas \( \phi_0, \phi_1, \ldots, \phi \), such that every \( \phi_i \) is an axiom or a formula from the set \( \Phi \), or is derived from the preceding formulas by an inference rule.

\( \phi \) is a theorem if \( \emptyset \vdash \phi \)

\( \Phi \) is consistent if there is no PST formula \( \rho \) such that
\[ \Phi \vdash \rho \land \neg \rho \]

\( \Phi \) is maximal consistent if it is consistent and for all \( \psi \in \mathcal{F} \setminus \Phi \), \( \Phi \cup \{\psi\} \) is inconsistent
Some theorems of $L_{ST+PST}^Q$

**Theorem (Deduction theorem)**

Let $\Phi$ be a set of formulas.

(a) $\Phi \cup \{\alpha\} \vdash \beta$ iff $\Phi \vdash \alpha \supset \beta$.

(b) $\Phi \cup \{\rho\} \vdash \sigma$ iff $\Phi \vdash \rho \rightarrow \sigma$. 
Some theorems of $L_{ST+PST}^Q$

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**Lemma**

(a) $\{\alpha \equiv \beta\} \vdash \alpha[\ell, u] \leftrightarrow \beta[\ell, u]$

(b) $\vdash \parallel_{w \in W} \&_{id \in ID} \&_{t=1}^{N} loc(id, \{w(id, t)\}, t)$
Some theorems of $L_{ST+PST}$

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**Lemma**

For a maximal consistent set $\Phi$ and ST formula $\alpha$ there is a unique $\mu(\alpha)$ such that $\Phi \vdash \alpha[\mu(\alpha), \mu(\alpha)]$. 
Completion

\(\Phi\) – a theory
\(\{\alpha_i \mid i \in \omega\}\) – an enumeration of all ST-formulas
\(\{\rho_i \mid i \in \omega\}\) – an enumeration of all ST-formulas
Completion

\( \Phi \) – a theory
\( \{ \alpha_i \mid i \in \omega \} \) – an enumeration of all ST-formulas
\( \{ \rho_i \mid i \in \omega \} \) – an enumeration of all ST-formulas

1. \( \Phi_0 = \Phi \).
2. If \( \rho_i \) is consistent with \( \Phi_{3i} \), then \( \Phi_{3i+1} = \Phi_{3i} \cup \{ \rho_i \} \), otherwise \( \Phi_{3i+1} = \Phi_{3i} \).
3. \( \Phi_{3i+2} = \Phi_{3i+1} \cup \{ \alpha_i[q, q] \} \), where \( q \in \mathbb{Q} \cap [0, 1] \) is a number such that \( \Phi_{3i+2} \) is consistent.
4. If \( \alpha_i \) is consistent with \( \Phi_{3i+2} \), then \( \Phi_{3i+3} = \Phi_{3i+2} \cup \{ \alpha_i \} \), otherwise \( \Phi_{3i+3} = \Phi_{3i+2} \)
5. \( \Phi^* = \bigcup_{n \in \omega} \Phi_n \).
Completion

Φ – a theory
\{α_i \mid i ∈ ω\} – an enumeration of all ST-formulas
\{ρ_i \mid i ∈ ω\} – an enumeration of all ST-formulas

1. \(Φ_0 = Φ\).

2. If \(ρ_i\) is consistent with \(Φ_{3i}\), then \(Φ_{3i+1} = Φ_{3i} \cup \{ρ_i\}\), otherwise \(Φ_{3i+1} = Φ_{3i}\).

3. \(Φ_{3i+2} = Φ_{3i+1} \cup \{α_i[q, q]\}\), where \(q ∈ Q \cap [0, 1]\) is a number such that \(Φ_{3i+2}\) is consistent.

4. If \(α_i\) is consistent with \(Φ_{3i+2}\), then \(Φ_{3i+3} = Φ_{3i+2} \cup \{α_i\}\), otherwise \(Φ_{3i+3} = Φ_{3i+2}\)

5. \(Φ^* = \bigcup_{n∈ω} Φ_n\).

Finite consistency ≠ consistency!
Canonical model

\[ \Phi \subseteq \Phi^* \]
\[ M^* = \langle W, I \rangle: \]
Canonical model

\[ \Phi \subseteq \Phi^* \]

\[ M^* = \langle W, I \rangle: \]

- \[ W = \{ w \in W \mid \forall \alpha \in \Phi^*, \Phi^* \vdash \alpha_w \supset \alpha \} \]
- \[ I(w) = \mu(\alpha_w), \text{ where } \Phi^* \vdash \alpha[\mu(\alpha), \mu(\alpha)] \]
Probabilistic logics

PST logics

Future work

$\mathcal{L}_{\mathcal{PST}}^\mathcal{Q}$: a complete axiomatization

Canonical model

$\Phi \subseteq \Phi^*$

$M^* = \langle W, I \rangle$:

- $W = \{ w \in \overline{W} \mid \forall \alpha \in \Phi^*, \Phi^* \vdash \alpha_w \supset \alpha \}$.
- $I(w) = \mu(\alpha_w)$, where $\Phi^* \vdash \alpha[\mu(\alpha), \mu(\alpha)]$

Theorem

$M^*$ is a PST structure.
Strong completeness theorem

\[ M^* \models \rho \text{ iff } \Phi^* \vdash \rho \]

It follows that:
Strong completeness theorem

\[ M^* \models \rho \text{ iff } \Phi^* \vdash \rho \]

It follows that:

**Theorem**

Every consistent set \( \Phi \) of formulas has a model.

**Corollary**

If \( \Phi \models \phi \) then \( \Phi \vdash \phi \).
Real-valued logic $L_{ST+PST}^R$

R3a. From the set of premises

$$\{\rho \rightarrow \beta[\ell - \frac{1}{n}, 1] \mid n \in \omega \setminus \{0\}, \ell - \frac{1}{n} \geq 0\}$$

infer $\rho \rightarrow \beta[\ell, 1]$. 

I($w$) = sup $\{\ell \in [0, 1] \cap \mathbb{Q} \mid \Phi^* \vdash \alpha[w][\ell, 1]\}$
Real-valued logic $L_{ST+PST}^R$

R3a. From the set of premises

$$\{\rho \rightarrow \beta[\ell - \frac{1}{n}, 1] \mid n \in \omega \setminus \{0\}, \ell - \frac{1}{n} \geq 0\}$$

infer $\rho \rightarrow \beta[\ell, 1]$.

The construction of $\Phi^*$: $\mathcal{F} = \{ \psi_i \mid i = 0, 1, 2, \ldots \}$

1. $\Phi_0 = \Phi$.
2. If $\psi_i$ is consistent with $\Phi_i$, then $\Phi_{i+1} = \Phi_i \cup \{\psi_i\}$.
3. If $\psi_i$ is not consistent with $\Phi_i$, then there are two cases:
   1. If $\psi_i = \rho \rightarrow \beta[\ell, 1]$, then
      $$\Phi_{i+1} = \Phi_i \cup \{\rho \rightarrow \beta[0, \ell - \frac{1}{n}]\},$$
      where $n$ is a positive integer such that $\Phi_{i+1}$ is consistent.
   2. Otherwise, $\Phi_{i+1} = \Phi_i$. 
R3a. From the set of premises

\[ \{ \rho \rightarrow \beta[\ell - \frac{1}{n}, 1] \mid n \in \omega \setminus \{0\}, \ell - \frac{1}{n} \geq 0 \} \]

infer \( \rho \rightarrow \beta[\ell, 1] \).

The construction of \( \Phi^* \):

1. \( \Phi_0 = \Phi \).
2. If \( \psi_i \) is consistent with \( \Phi_i \), then \( \Phi_{i+1} = \Phi_i \cup \{\psi_i\} \).
3. If \( \psi_i \) is not consistent with \( \Phi_i \), then there are two cases:
   1. If \( \psi_i = \rho \rightarrow \beta[\ell, 1] \), then
      \[
      \Phi_{i+1} = \Phi_i \cup \{\rho \rightarrow \beta[0, \ell - \frac{1}{n}]\},
      \]
      where \( n \) is a positive integer such that \( \Phi_{i+1} \) is consistent.
   2. Otherwise, \( \Phi_{i+1} = \Phi_i \).

\[
I(w) = \sup \{ \ell \in [0, 1] \cap \mathbb{Q} \mid \Phi^* \vdash \alpha_w[\ell, 1] \}
\]
Outline

1. Probabilistic logics
   - About probabilistic logics
   - Syntax and semantics
   - Non-compactness as an axiomatization issue
   - Variants

2. PST logics
   - The PST framework for probabilistic spatiotemporal databases
   - $L_{ST+PST}^Q$: syntax and semantics
   - $L_{ST+PST}^Q$: a complete axiomatization

3. Future work
   - adding temporal operators
Temporal operators

Temporal operators (CTL*)

Basic:

- □ – next, $U$ – until
  - □$\alpha$: $\alpha$ has to hold at the next state
  - $\alpha U \beta$: $\alpha$ has to hold at least until $\beta$, which holds at the current or a future moment

- $A$ – universal path operator (branching time)
Temporal operators (CTL*)

Basic:

- $\bigcirc$ – next, $U$ – until
  - $\bigcirc \alpha$: $\alpha$ has to hold at the next state
  - $\alpha U \beta$: $\alpha$ has to hold at least until $\beta$, which holds at the current or a future moment
- $A$ – universal path operator (branching time)

Other:

- $F \alpha$ is $\top U \alpha$ – sometime
- $G \alpha$ is $\neg F \neg \alpha$ – always
- $E \alpha$ is $\neg A \neg \alpha$ – existential path operator
Temporal base

Axiomatization issues

- non-compactness
  
  \[ T = \{ \neg G\alpha \} \cup \{ \Box^n\alpha \mid n \in \omega \} \]
Temporal base

Axiomatization issues

- non-compactness
  \[ T = \{ \neg G \alpha \} \cup \{ \bigcirc^n \alpha \mid n \in \omega \} \]
- probability of \( \alpha U \beta \) – beyond finite additivity (todo)
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- Semantical property: if \( T \models \alpha \), then \( AT \models A\alpha \).
Axiomatization issues

- non-compactness
  \[ T = \{ \neg G \alpha \} \cup \{ \square^n \alpha \mid n \in \omega \} \]
- probability of \( \alpha U \beta \) – beyond finite additivity (todo)
- Semantical property: if \( T \models \alpha \), then \( AT \models A\alpha \).
- \( T \vdash \alpha \Rightarrow AT \vdash A\alpha \) must be theorem