Logics for probabilistic spatio-temporal reasoning

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Outline

1. Probabilistic logics
   - About probabilistic logics
   - Syntax and semantics
   - Non-compactness as an axiomatization issue
   - Variants
2. PST logics
   - The PST framework for probabilistic spatiotemporal databases
   - $L^{\mathbb{Q}}_{ST+PST}$: syntax and semantics
   - $L^{\mathbb{Q}}_{ST+PST}$: a complete axiomatization
3. Future work
   - adding temporal operators

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- syntax (language, well formed formulas)

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- syntax (language, well formed formulas)
- semantics (models, satisfiability)
- consequence relation
- axiomatic system (axioms, rules)
- proof

- The probabilistic logics allow strict reasoning *about* probabilities using well-defined syntax and semantics.
- Formulas in these logics remain either true or false.
- Formulas do not have probabilistic (numerical) truth values.

Probabilistic quantifiers and operators

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- $P_{\geqslant r} \alpha \quad (P(\alpha) \geqslant r)$
- semantics: modal semantics – measure of all worlds in which α holds is at least r

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- $P(\alpha) + 2P(\beta)P(\gamma) \geqslant \frac{2}{3}$ (PWF)

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- Language:
  - propositional letters $\{p, q, r, \ldots\}$
  - Boolean connectives $\neg$, $\wedge$
  - a list of probability operators $P_{\geq r}$
- The set of formulas is the smallest set containing propositional letters and closed under $\neg$, $\wedge$ and $P_{\geq r}$

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  - $Prob$ assigns to every $w \in W$ a probability space $Prob(w) = \langle W(w), H(w), \mu(w)\rangle$:
    - $W(w)$ – a non empty subset of W,
    - $H(w)$ – an algebra of subsets of $W(w)$
    - $\mu(w) : H(w) \rightarrow [0, 1]$ – a finitely additive probability measure on $H(w)$.

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- $\mathbf{M}, w \models \alpha \wedge \beta$ iff $\mathbf{M}, w \models \alpha$ and $\mathbf{M}, w \models \beta$, and
- $\mathbf{M}, w \models P_{\geq s}\alpha$ iff $\mu(w)([\alpha]_{\mathbf{M},w}) \geq s$

$[\alpha]_{\mathbf{M},w} = \{u \in W(w) : \mathbf{M}, u \models \alpha\}$

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- c: $0 < c < \frac{1}{k}$, $\quad \mu[p] = c$
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- finitary (recursive) axiomatization + strong completeness $\Rightarrow$ compactness
- finitary axiomatization for real valued probabilistic logics: there are consistent sets that are not satisfiable

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$+$ implicative form of the rule (for proving Deduction theorem)

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Basic variants

- without iteration of probabilities
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- values of probability functions in non-Archimedean structures
- change underlying logic

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- $\preceq$ – qualitative probability operator

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(Probabilistic) extension of syntax

- conditional probabilities
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  If $\alpha, \beta \in For_C$, $\mathbf{M} \models \alpha \preceq \beta$ iff $\mu([\alpha]) \leqslant \mu([\beta])$,
- independency operator (todo)
- probability of a formula belongs to a countable set

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PST atom

- GPS systems – possibility of tracking moving objects (vehicles, cell phones...)
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    - involve space and time
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- PST (Probabilistic SpatioTemporal) atom: $loc(id, r, t)[\ell, u]$

PST database

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- Semantics:
  - (possible) world – mapping of objects (for every time instance) in space (+ reachability constraints)
  - interpretation – probability distribution over worlds

Limitations of PST formalism

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- $loc(Bus1, Q, 5)$ and $loc(Bus2, R, 6)[.4, 1]$
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ST formula

Definition (ID,S,T)

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Definition (ST formula)

- An *ST atom*: a formula of the form $loc(id, r, t)$, where $id \in ID$, $t \in T$, and $r \subseteq S$
- *ST formula*: a Boolean combination of ST atoms; connectives: $\sim$ (negation), & (conjunction), $|$ (disjunction), $\supset$ (implication), and $\equiv$ (equivalence).
- Notation: $\mathcal{A}$ – the set of ST formulas; α, β – ST formulas.

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- *Basic PST atom*: a formula of the form $\alpha[0, u]$, where $\alpha \in \mathcal{A}$.
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Abreviations:

- $\alpha(\ell, 1]$ is $\neg\, \alpha[0, \ell]$.
- $\alpha[\ell, 1]$ is $\sim\alpha[0, 1 - \ell]$.
- $\alpha[0, u)$ is $\neg\alpha[u, 1]$.
- if $0 \leq \ell \leq u \leq 1$, then $\alpha[\ell, u]$ is $\alpha[0, u] \wedge \alpha[\ell, 1]$.
- if $0 \leq \ell < u \leq 1$, we define $\alpha[\ell, u)$, $\alpha(\ell, u]$ and $\alpha(\ell, u)$ similarly as above.

Formula

Example of ST formula:
$loc(id_2, \{p_2, p_4\}, 2) \& loc(id_2, \{p_2, p_4\}, 3)$

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Example of PST formula:
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Definition (Formula)

- $\mathcal{F} = \mathcal{A} \cup \mathcal{P}$
- Notation: ϕ and ψ -formulas; set of arbitrary formulas is indicated by Φ

World

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Definition (World)

An RD-compliant *world* is a function $w : ID \times T \longrightarrow S$, satisfying the condition:

$$\text{if } w(id, t) = p_1 \text{ and } w(id, t+1) = p_2 \text{ then } (p_1, p_2) \in RD.$$

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Definition (Valuation)

Given W, the valuation $v_W : \mathcal{A} \times W \longrightarrow \{0, 1\}$ is defined as follows:

- $v_W(loc(id, r, t), w) = 1$ iff $w(id, t) \in r$,
- $v_W(\alpha \& \beta, w) = 1$ iff $v_W(\alpha, w) = 1$ and $v_W(\beta, w) = 1$,
- $v_W(\sim\alpha, w) = 1$ iff $v_W(\alpha, w) = 0$.

Model

Definition (Interpretation)

An *interpretation* $I : W \longrightarrow [0,1] \cap \mathbb{Q}$ is a probability distribution over W, i.e., a nonnegative function such that

$$\sum_{w \in W} I(w) = 1.$$

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Definition (PST Structure)

A *PST structure* is a pair $\langle W, I \rangle$ where W is a (nonempty) set of worlds and I is an interpretation.

Satisfiability

Definition (Satisfiability)

Let $M = \langle W, I \rangle$ be a PST structure. We define the satisfiability relation $\models$ recursively as follows:

- $M \models \alpha$ iff $v_W(\alpha, w) = 1$ for all $w \in W$.
- $M \models \alpha[0, u]$ iff $\sum_{v_W(\alpha,w)=1} I(w) \leq u$.
- $M \models \neg\rho$ iff $M \not\models \rho$.
- $M \models \rho \wedge \sigma$ iff $M \models \rho$ and $M \models \sigma$.

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- $M \models \neg\rho$ iff $M \not\models \rho$.
- $M \models \rho \wedge \sigma$ iff $M \models \rho$ and $M \models \sigma$.

Definition (Entailment)

- M is a *model* of Φ, $M \models \Phi$, iff $M \models \phi$ for every $\phi \in \Phi$
- $\Phi \models \phi$, Φ *entails* ϕ, iff all models of Φ are models of ϕ.
- ϕ is *valid*, that is, it is satisfied in every PST structure.

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Axioms

Propositional reasoning

- All instances of classical propositional tautologies for both ST and PST formulas.

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Spatio-temporal axioms

- $loc(id, S, t)$.
- $loc(id, r, t) \equiv \; \|_{p \in r} loc(id, \{p\}, t)$.
- $loc(id, \{p\}, t) \supset \sim loc(id, \{p'\}, t)$, $p \neq p'$.
- $\sim (loc(id, \{p\}, t) \& loc(id, \{p'\}, t+1))$, $(p, p') \notin RD$, $t < N$.

Axioms

Probabilistic axioms

- $\alpha[0,1]$.
- $\alpha[\ell,u) \to \alpha[\ell,u]$.
- $\alpha[\ell,u] \to \alpha[\ell,u'),\ u<u'$.
- $(\alpha[\ell,u] \wedge \beta[\ell',u'] \wedge \sim(\alpha\&\beta)[1,1]) \to \alpha|\beta[\ell'',u'']$,
  $\ell''=\min\{\ell+\ell',1\},\ u''=\min\{u+u',1\}$.
- $\alpha[0,u] \wedge \beta[0,u') \to \alpha|\beta[0,u''),\ u''=u+u',\ u+u'\leq 1$.

Inference rules

- (a) From α and $\alpha \supset \beta$ infer β
  (b) From ρ and $\rho \rightarrow \sigma$ infer σ.
- From α infer $\alpha[1,1]$.

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- (a) From α and $\alpha \supset \beta$ infer β
  (b) From ρ and $\rho \to \sigma$ infer σ.
- From α infer $\alpha[1, 1]$.
- From the set of premises

$$\{\rho \to \neg\alpha[q, q] \mid q \in \mathbb{Q} \cap [0, 1]\}$$

  infer $\neg\rho$.

- ϕ is deducible from Φ ($\Phi \vdash \phi$) if there is an at most countable sequence of formulas $\phi_0, \phi_1, \ldots, \phi$, such that every ϕ_i is an axiom or a formula from the set Φ, or is derived from the preceding formulas by an inference rule.
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- Φ is consistent if there is no PST formula ρ such that $\Phi \vdash \rho \wedge \neg\rho$
- Φ is maximal consistent if it is consistent and for all $\psi \in \mathcal{F} \setminus \Phi$, $\Phi \cup \{\psi\}$ is inconsistent

Some theorems of $L^{\mathbb{Q}}_{ST+PST}$

Theorem (Deduction theorem)

Let Φ be a set of formulas.
(a) $\Phi \cup \{\alpha\} \vdash \beta$ iff $\Phi \vdash \alpha \supset \beta$.
(b) $\Phi \cup \{\rho\} \vdash \sigma$ iff $\Phi \vdash \rho \rightarrow \sigma$.

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(b) $\Phi \cup \{\rho\} \vdash \sigma$ iff $\Phi \vdash \rho \to \sigma$.

Lemma

(a) $\{\alpha \equiv \beta\} \vdash \alpha[\ell, u] \leftrightarrow \beta[\ell, u]$

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Some theorems of $L^{\mathbb{Q}}_{ST+PST}$

Theorem (Deduction theorem)

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Lemma

For a maximal consistent set Φ and ST formula α there is a unique $\mu(\alpha)$ such that $\Phi \vdash \alpha[\mu(\alpha), \mu(\alpha)]$.

Completion

Φ – a theory
$\{\alpha_i \mid i \in \omega\}$ – an enumeration of all ST-formulas
$\{\rho_i \mid i \in \omega\}$ – an enumeration of all ST-formulas

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2. If ρ_i is consistent with Φ_{3i}, then $\Phi_{3i+1} = \Phi_{3i} \cup \{\rho_i\}$, otherwise $\Phi_{3i+1} = \Phi_{3i}$.
3. $\Phi_{3i+2} = \Phi_{3i+1} \cup \{\alpha_i[q, q]\}$, where $q \in \mathbb{Q} \cap [0, 1]$ is a number such that Φ_{3i+2} is consistent.
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Finite consistency $\neq$ consistency!

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M^* is a PST structure.

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Every consistent set Φ of formulas has a model.

Corollary

If $\Phi \models \phi$ then $\Phi \vdash \phi$.

Real-valued logic $L^{\mathbb{R}}_{ST+PST}$

R3a. From the set of premises

$$\{\rho \rightarrow \beta[\ell - \frac{1}{n}, 1] \mid n \in \omega \setminus \{0\}, \ell - \frac{1}{n} \geq 0\}$$

infer $\rho \rightarrow \beta[\ell, 1]$.

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The construction of Φ^*: $\mathcal{F} = \{\psi_i | i = 0, 1, 2, \dots\}$

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$$I(w) = sup\{\ell \in [0, 1] \cap \mathbb{Q} \mid \Phi^* \vdash \alpha_w[\ell, 1]\}$$

Outline

1. Probabilistic logics
   - About probabilistic logics
   - Syntax and semantics
   - Non-compactness as an axiomatization issue
   - Variants
2. PST logics
   - The PST framework for probabilistic spatiotemporal databases
   - $L^{\mathbb{Q}}_{ST+PST}$: syntax and semantics
   - $L^{\mathbb{Q}}_{ST+PST}$: a complete axiomatization
3. Future work
   - adding temporal operators

Temporal operators

Temporal operators (CTL*)

Basic:

- $\bigcirc$ – next, U – until
  - $\bigcirc\alpha$: α has to hold at the next state
  - $\alpha U \beta$: α has to hold at least until β, which holds at the current or a future moment
- A – universal path operator (branching time)

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Other:

- $F\alpha$ is $\top U \alpha$ – sometime
- $G\alpha$ is $\neg F \neg \alpha$ – always
- $E\alpha$ is $\neg A \neg \alpha$ – existential path operator

Temporal base

Axiomatization issues

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