Approaching substructural term calculi via the resource control calculus

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Outline

- motivation;
- the resource control calculus - $\lambda_{\circledR}$;
- substructural restrictions - $\lambda_{\bar{t}}$ and $\lambda_{\bar{c}}$ calculi.

Computational interpretations of logics

The Curry-Howard correspondence captures the computational content of logic, by observing a multilevel connection between formal systems (logic) and some term calculi (computation).

- the λ-calculus $\leftrightsquigarrow$ intuitionistic natural deduction
- the $\lambda\mu$-calculus (Parigot, 1992) $\leftrightsquigarrow$ classical natural deduction
- the $\overline{\lambda}\mu\tilde{\mu}$-calculus (Herbelin, Curien, 2000) $\leftrightsquigarrow$ classical sequent calculus
- the λ^{Gtz}-calculus (Espírito Santo, 2006) $\leftrightsquigarrow$ intuitionistic sequent calculus

In all these formal systems, structural rules are implicit.

ND with explicit structural rules

$$\frac{}{A \vdash A}\,(Ax)$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\,(\to_{intro}) \qquad \frac{\Gamma \vdash A \quad \Delta \vdash A \to B}{\Gamma, \Delta \vdash B}\,(\to_{elim})$$

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}\,(Thin) \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,(Cont)$$

Implicit SR

- contexts are sets
- Axiom: $\Gamma, A \vdash A$
- additive (context-sharing) rules

Explicit SR

- contexts are multisets
- Axiom: $A \vdash A$
- multiplicative (context-splitting) rules

- goal?
  To propose term calculi (CH) corresponding to the formal systems with exp. str. rules.
- motivation?
  - theoretical - to obtain an insight into a part of the computation process that is usually hidden;
  - practical - controlling enables optimization.
- solution:
  - $\lambda_{®}$ (resource control lambda calculus) $\leftrightsquigarrow$ ND with exp.str.rules
  - $\lambda_{®}^{\mathrm{Gtz}}$ (resource control lambda Gentzen calculus) $\leftrightsquigarrow$ LJ with exp.str.rules (will not be presented here)
- why "resource control"?
  structural rules in logic $\leftrightsquigarrow$ duplication and erasure of variables in terms.

Syntax of the $\lambda_{®}$-calculus

Two ways of defining the syntax:

1. indirect way: extracting the set of terms out from the larger set of pre-terms;
2. direct way: mutual recursive definition of terms and free variables.

- pre-terms of the $\lambda_{®}$-calculus:

$$\text{Pre-terms} \quad f \quad ::= \quad x \mid \lambda x.f \mid ff \mid x \odot f \mid x <^{x_1}_{x_2} f$$

- $\lambda x.f$ is an abstraction, ff is an application, $x \odot f$ is an erasure and $x <^{x_1}_{x_2} f$ is a duplication.
- $\lambda_{®}$-terms, ranged over $M, N, P ...$, are only those pre-terms that satisfy the following two conditions:
  - every free variable occurs exactly once in a term;
  - every binder binds exactly one occurrence of a free variable.

$\lambda_{\circledR}$-terms - formally

The set $\Lambda_{\circledR}$ is defined by the following inference rules:

$$\frac{}{x \in \Lambda_{\circledR}} \qquad \frac{M \in \Lambda_{\circledR} \quad x \in Fv(M)}{\lambda x.M \in \Lambda_{\circledR}}$$

$$\frac{M \in \Lambda_{\circledR} \quad N \in \Lambda_{\circledR} \quad Fv(M) \cap Fv(N) = \emptyset}{MN \in \Lambda_{\circledR}}$$

$$\frac{M \in \Lambda_{\circledR} \quad x \notin Fv(M)}{x \odot M \in \Lambda_{\circledR}} \qquad \frac{M \in \Lambda_{\circledR} \quad x_1 \neq x_2, \quad x_1, x_2 \in Fv(M) \quad x \notin Fv(M) \setminus \{x_1, x_2\}}{x <^{x_1}_{x_2} M \in \Lambda_{\circledR}}$$

Example

pre-terms

$$\lambda x.y \quad , \quad \lambda x.xx \quad , \quad x <^{y}_{z} (xy)$$

are not $\lambda_{\circledR}$-terms.

Although some λ-terms are not $\lambda_{®}$-terms (like $\lambda x.y$, xx,...), every λ-term has a corresponding $\lambda_{®}$-term, obtained using a mapping $[\;]_{rc} : \Lambda \rightarrow \Lambda_{®}$ in the following way:

$$
\begin{array}{rcl}
[x]_{rc} & = & x \\
[\lambda x.M]_{rc} & = & \left\{ \begin{array}{ll} \lambda x.[M]_{rc}, & x \in Fv(M) \\ \lambda x.x \odot [M]_{rc}, & x \notin Fv(M) \end{array} \right. \\
[MN]_{rc} & = & \left\{ \begin{array}{ll} [M]_{rc}[N]_{rc}, & Fv(M) \cap Fv(N) = \emptyset \\ x <^{x_1}_{x_2} [M[x_1/x]N[x_2/x]]_{rc}, & x \in Fv(M) \cap Fv(N) \end{array} \right.
\end{array}
$$

Example

$$
\begin{array}{rcl}
\lambda - term & \rightsquigarrow & \lambda_{®} - term \\
\lambda x.y & \rightsquigarrow & \lambda x.x \odot y \\
\lambda x.xx & \rightsquigarrow & \lambda x.x <^{x_1}_{x_2} (x_1 x_2)
\end{array}
$$

Computing in the $\lambda_{\circledR}$-calculus

Operational semantics:

reduction rules + substitution + equivalencies

- β-reduction - the key step: $(\beta) \quad (\lambda x.M)N \quad \rightarrow \quad M[N/x]$
- substitution - implicit i.e. meta-operator:

$$
\begin{array}{rcl}
x[N/x] & \triangleq & N \\
(\lambda y.M)[N/x] & \triangleq & \lambda y.M[N/x], \;\; x \neq y \\
(MP)[N/x] & \triangleq & M[N/x]P, \;\; x \notin Fv(P) \\
(MP)[N/x] & \triangleq & MP[N/x], \;\; x \notin Fv(M) \\
(y \odot M)[N/x] & \triangleq & y \odot M[N/x], \;\; x \neq y \\
(x \odot M)[N/x] & \triangleq & Fv(N) \odot M \\
(y <^{y_1}_{y_2} M)[N/x] & \triangleq & y <^{y_1}_{y_2} M[N/x], \;\; x \neq y \\
(x <^{x_1}_{x_2} M)[N/x] & \triangleq & Fv(N) <^{Fv(N_1)}_{Fv(N_2)} M[N_1/x_1, N_2/x_2]
\end{array}
$$

- Substitution is linear and satisfies interface preservation.

Computing in the $\lambda_{®}$-calculus

- γ-reductions - contraction propagation:

$$\begin{array}{lrcl}
(\gamma_1) & x <^{x_1}_{x_2} (\lambda y.M) & \rightarrow & \lambda y.x <^{x_1}_{x_2} M \\
(\gamma_2) & x <^{x_1}_{x_2} (MN) & \rightarrow & (x <^{x_1}_{x_2} M)N, \text{ if } x_1, x_2 \notin Fv(N) \\
(\gamma_3) & x <^{x_1}_{x_2} (MN) & \rightarrow & M(x <^{x_1}_{x_2} N), \text{ if } x_1, x_2 \notin Fv(M)
\end{array}$$

- ω-reductions - thinning extraction:

$$\begin{array}{lrcl}
(\omega_1) & \lambda x.(y \odot M) & \rightarrow & y \odot (\lambda x.M), \; x \neq y \\
(\omega_2) & (x \odot M)N & \rightarrow & x \odot (MN) \\
(\omega_3) & M(x \odot N) & \rightarrow & x \odot (MN)
\end{array}$$

- $\gamma\omega$-reductions - interaction of the structural operators:

$$\begin{array}{lrcl}
(\gamma\omega_1) & x <^{x_1}_{x_2} (y \odot M) & \rightarrow & y \odot (x <^{x_1}_{x_2} M), \; y \neq x_1, x_2 \\
(\gamma\omega_2) & x <^{x_1}_{x_2} (x_1 \odot M) & \rightarrow & M[x/x_2]
\end{array}$$

Computing in the $\lambda_{®}$-calculus

- equivalencies:

$$
\begin{array}{lrcl}
(\epsilon_1) & x \odot (y \odot M) & \equiv & y \odot (x \odot M) \\
(\epsilon_2) & x <^{x_1}_{x_2} M & \equiv & x <^{x_2}_{x_1} M \\
(\epsilon_3) & x <^{y}_{z} (y <^{u}_{v} M) & \equiv & x <^{y}_{u} (y <^{z}_{v} M) \\
(\epsilon_4) & x <^{x_1}_{x_2} (y <^{y_1}_{y_2} M) & \equiv & y <^{y_1}_{y_2} (x <^{x_1}_{x_2} M), \quad x \neq y_1, y_2, \; y \neq x_1, x_2
\end{array}
$$

- α-equivalence - for both binders:

$$
\begin{array}{rcl}
\lambda x.M & \equiv_\alpha & \lambda y.M[y/x] \\
x <^{y}_{z} M & \equiv_\alpha & x <^{y_1}_{z_1} M[y_1/y, z_1/z]
\end{array}
$$

The type system $\lambda_{®} \rightarrow$

$$\frac{}{x : \alpha \vdash x : \alpha}\ (Ax)$$

$$\frac{\Gamma, x : \alpha \vdash M : \beta}{\Gamma \vdash \lambda x.M : \alpha \to \beta}\ (\to_I) \qquad \frac{\Gamma \vdash M : \alpha \to \beta \quad \Delta \vdash N : \alpha}{\Gamma, \Delta \vdash MN : \beta}\ (\to_E)$$

$$\frac{\Gamma, x : \alpha, y : \alpha \vdash M : \beta}{\Gamma, z : \alpha \vdash z <^{x}_{y} M : \beta}\ (Cont) \qquad \frac{\Gamma \vdash M : \alpha}{\Gamma, x : \beta \vdash x \odot M : \alpha}\ (Thin)$$

Theorem (Ghilezan et al. (2009))

If a $\lambda_{®}$-term is typeable in the system $\lambda_{®} \rightarrow$, then it is strongly normalizing (terminating).

The type system $\lambda_{®} \rightarrow$

$$\frac{}{x : \alpha \vdash x : \alpha}\ (Ax)$$

$$\frac{\Gamma, x : \alpha \vdash M : \beta}{\Gamma \vdash \lambda x.M : \alpha \rightarrow \beta}\ (\rightarrow_I) \qquad \frac{\Gamma \vdash M : \alpha \rightarrow \beta \quad \Delta \vdash N : \alpha}{\Gamma, \Delta \vdash MN : \beta}\ (\rightarrow_E)$$

$$\frac{\Gamma, x : \alpha, y : \alpha \vdash M : \beta}{\Gamma, z : \alpha \vdash z <^x_y M : \beta}\ (Cont) \qquad \frac{\Gamma \vdash M : \alpha}{\Gamma, x : \beta \vdash x \odot M : \alpha}\ (Thin)$$

Theorem (Ghilezan et al. (2009))

If a $\lambda_{®}$-term is typeable in the system $\lambda_{®} \rightarrow$, then it is strongly normalizing (terminating).

The type system $\lambda_{®}\cap$

$$\frac{}{x : \sigma \vdash x : \sigma}\ (Ax) \qquad \frac{\Gamma, x : \alpha \vdash M : \sigma}{\Gamma \vdash \lambda x.M : \alpha \to \sigma}\ (\to_I)$$

$$\frac{\Gamma \vdash M : \cap_{i=1}^{n} \tau_i \to \sigma \quad \Delta_0 \vdash N : \tau_0 \quad \ldots \quad \Delta_n \vdash N : \tau_n}{\Gamma, \Delta_0^{\top} \sqcap \Delta_1 \sqcap \ldots \sqcap \Delta_n \vdash MN : \sigma}\ (\to_E)$$

$$\frac{\Gamma, x : \alpha, y : \beta \vdash M : \sigma}{\Gamma, z : \alpha \cap \beta \vdash z <_{y}^{x} M : \sigma}\ (Cont) \qquad \frac{\Gamma \vdash M : \sigma}{\Gamma, x : \top \vdash x \odot M : \sigma}\ (Thin)$$

Theorem (Ghilezan et al. (2011))

A $\lambda_{®}$-term is strongly normalizing (terminating) if and only if it is typeable in the system $\lambda_{®}\cap$.

Towards substructural term calculi

In order to obtain term calculi corresponding in the CH way to the intuitionistic implicative logic without (either explicit or implicit) weakening / contraction, we:

- start from the $\lambda_{®}$-calculus;
- remove the thinning / contraction operator;
- remove all corresponding reduction, substitution and equivalence rules;
- but keep the related constraints in the definition of terms and in the type assignment rules.

The obtained calculi are:

the $\lambda_{\bar{t}}$-calculus, corresponding to a variant of the relevant logic,
the $\lambda_{\bar{c}}$-calculus, corresponding to a variant of the affine logic.

The calculus without thinning - $\lambda_{\bar{t}}$

- Pre-terms: $f \quad ::= \quad x \mid \lambda x.f \mid ff \mid x <^{x_1}_{x_2} f$
- Terms:

$$\frac{}{x \in \Lambda_{\bar{t}}} \qquad \frac{M \in \Lambda_{\bar{t}} \quad x \in Fv(M)}{\lambda x.M \in \Lambda_{\bar{t}}}$$

$$\frac{M \in \Lambda_{\bar{t}} \quad N \in \Lambda_{\bar{t}} \quad Fv(M) \cap Fv(N) = \emptyset}{MN \in \Lambda_{\bar{t}}}$$

$$\frac{M \in \Lambda_{\bar{t}} \quad x_1 \neq x_2, \quad x_1, x_2 \in Fv(M) \quad x \notin Fv(M) \setminus \{x_1, x_2\}}{x <^{x_1}_{x_2} M \in \Lambda_{\bar{t}}}$$

- Operational semantics:
  - reductions: β, γ_1, γ_2, γ_3;
  - equivalencies: α, ϵ_2, ϵ_3, ϵ_4;
  - reduced substitution definition.
- This is a strict sub-calculus of $\lambda_{\circledR}$, hence there are λ-terms and $\lambda_{\circledR}$-terms that cannot be represented in the $\lambda_{\bar{t}}$ calculus, i.e. $\lambda x.y$ and $z <_y^x x$.

The type system $\lambda_{\bar{t}} \rightarrow$

$$\frac{}{x : \alpha \vdash x : \alpha}\ (Ax)$$

$$\frac{\Gamma, x : \alpha \vdash M : \beta}{\Gamma \vdash \lambda x.M : \alpha \rightarrow \beta}\ (\rightarrow_I) \qquad \frac{\Gamma \vdash M : \alpha \rightarrow \beta \quad \Delta \vdash N : \alpha}{\Gamma, \Delta \vdash MN : \beta}\ (\rightarrow_E)$$

$$\frac{\Gamma, x : \alpha, y : \alpha \vdash M : \beta}{\Gamma, z : \alpha \vdash z <^{x}_{y} M : \beta}\ (Cont)$$

Simply typed $\lambda_{\bar{t}}$-calculus corresponds in the Curry-Howard way to the intuitionistic natural deduction with explicit contraction and without thinning.

The type system $\lambda_{\bar{t}} \rightarrow$

$$\frac{}{x : \alpha \vdash x : \alpha}\ (Ax)$$

$$\frac{\Gamma, x : \alpha \vdash M : \beta}{\Gamma \vdash \lambda x.M : \alpha \rightarrow \beta}\ (\rightarrow_I) \qquad \frac{\Gamma \vdash M : \alpha \rightarrow \beta \quad \Delta \vdash N : \alpha}{\Gamma, \Delta \vdash MN : \beta}\ (\rightarrow_E)$$

$$\frac{\Gamma, x : \alpha, y : \alpha \vdash M : \beta}{\Gamma, z : \alpha \vdash z <^{x}_{y} M : \beta}\ (Cont)$$

Simply typed $\lambda_{\bar{t}}$-calculus corresponds in the Curry-Howard way to the intuitionistic natural deduction with explicit contraction and without thinning.

The type system $\lambda_{\bar{\mathrm{t}}}\cap$

$$\frac{}{x : \sigma \vdash x : \sigma} \; (Ax)$$

$$\frac{\Gamma, x : \alpha \vdash M : \sigma}{\Gamma \vdash \lambda x.M : \alpha \to \sigma} \; (\to_I)$$

$$\frac{\Gamma \vdash M : \cap_{i=1}^{n} \tau_i \to \sigma \quad \Delta_1 \vdash N : \tau_1 \quad \ldots \quad \Delta_n \vdash N : \tau_n}{\Gamma, \Delta_1 \sqcap \ldots \sqcap \Delta_n \vdash MN : \sigma} \; (\to_E)$$

$$\frac{\Gamma, x : \alpha, y : \beta \vdash M : \sigma}{\Gamma, z : \alpha \cap \beta \vdash z <^{x}_{y} M : \sigma} \; (Cont)$$

To be proved: A $\lambda_{\bar{\mathrm{t}}}$-term is strongly normalizing if and only if it is typeable in the system $\lambda_{\bar{\mathrm{t}}}\cap$.

The calculus without contraction - $\lambda_{\overline{c}}$

- Pre-terms: $f \quad ::= \quad x \mid \lambda x.f \mid ff \mid x \odot f$
- Terms:

$$\frac{}{x \in \Lambda_{\overline{c}}} \qquad \frac{M \in \Lambda_{\overline{c}} \quad x \in Fv(M)}{\lambda x.M \in \Lambda_{\overline{c}}}$$

$$\frac{M \in \Lambda_{\overline{c}} \quad N \in \Lambda_{\overline{c}} \quad Fv(M) \cap Fv(N) = \emptyset}{MN \in \Lambda_{\overline{c}}}$$

$$\frac{M \in \Lambda_{\overline{c}} \quad x \notin Fv(M)}{x \odot M \in \Lambda_{\overline{c}}}$$

- Operational semantics:
  - reductions: β, ω_1, ω_2, ω_3;
  - equivalencies: α (just for lambda abstraction), ϵ_1;
  - reduced substitution definition.
- This is a strict sub-calculus of $\lambda_{®}$, hence there are λ-terms and $\lambda_{®}$-terms that cannot be represented in the $\lambda_{\overline{c}}$ calculus, i.e. $\lambda x.xx$ and $x <^{x_1}_{x_2} (x_1 \odot y)x_2$.

The type system $\lambda_{\overline{c}} \rightarrow$

$$\frac{}{x : \alpha \vdash x : \alpha}\ (Ax)$$

$$\frac{\Gamma, x : \alpha \vdash M : \beta}{\Gamma \vdash \lambda x.M : \alpha \rightarrow \beta}\ (\rightarrow_I) \qquad \frac{\Gamma \vdash M : \alpha \rightarrow \beta \quad \Delta \vdash N : \alpha}{\Gamma, \Delta \vdash MN : \beta}\ (\rightarrow_E)$$

$$\frac{\Gamma \vdash M : \beta}{\Gamma, x : \alpha \vdash x \odot M : \beta}\ (Thin)$$

Simply typed $\lambda_{\overline{c}}$-calculus corresponds in the Curry-Howard way to the intuitionistic natural deduction with explicit weakening and without contraction.

The type system $\lambda_{\bar{c}} \rightarrow$

$$\frac{}{x : \alpha \vdash x : \alpha}\ (Ax)$$

$$\frac{\Gamma, x : \alpha \vdash M : \beta}{\Gamma \vdash \lambda x.M : \alpha \rightarrow \beta}\ (\rightarrow_I) \qquad \frac{\Gamma \vdash M : \alpha \rightarrow \beta \quad \Delta \vdash N : \alpha}{\Gamma, \Delta \vdash MN : \beta}\ (\rightarrow_E)$$

$$\frac{\Gamma \vdash M : \beta}{\Gamma, x : \alpha \vdash x \odot M : \beta}\ (Thin)$$

Simply typed $\lambda_{\bar{c}}$-calculus corresponds in the Curry-Howard way to the intuitionistic natural deduction with explicit weakening and without contraction.

The type system $\lambda_{\overline{c}}\cap$

$$\frac{}{x : \sigma \vdash x : \sigma}\ (Ax)$$

$$\frac{\Gamma, x : \alpha \vdash M : \sigma}{\Gamma \vdash \lambda x.M : \alpha \to \sigma}\ (\to_I)$$

$$\frac{\Gamma \vdash M : \cap_{i=1}^{n} \tau_i \to \sigma \quad \Delta_0 \vdash N : \tau_0 \quad \ldots \quad \Delta_n \vdash N : \tau_n}{\Gamma, \Delta_0^{\top} \sqcap \Delta_1 \sqcap \ldots \sqcap \Delta_n \vdash MN : \sigma}\ (\to_E)$$

$$\frac{\Gamma \vdash M : \sigma}{\Gamma, x : \top \vdash x \odot M : \sigma}\ (Thin)$$

To be proved: A $\lambda_{\overline{c}}$-term is strongly normalizing if and only if it is typeable in the system $\lambda_{\overline{c}}\cap$.

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