Non-monotonic extensions of the weak Kleene clone with constants

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- Notation: Let $X \subseteq \mathcal{O}_E$, then $\langle X \rangle$ represents the clone with constants generated by X. If F is a clone, $F^{(n)}$ represents the set of functions of F with n variables.

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- Liar sentence: ‘this sentence is false’ $x = \neg x$
- (1) If this sentence is true, then the following sentence is not true.
  (2) Either the previous sentence is not true or snow is white
- $x_1 = x_1 \to \neg x_2$
  $x_2 = \neg x_1 \vee x_3$
  $x_3 = 1$

The Gupta-Belnap fixed-point property

- A clone $F \subseteq \mathcal{O}_E$ has the *Gupta-Belnap fixed-point property* (f.p.p.) iff every system of equations of the form

$$\begin{aligned} x_1 &= f_1(x_{11}, x_{12}, \ldots, x_{1i_1}) \\ x_2 &= f_2(x_{21}, x_{22}, \ldots, x_{2i_2}) \\ &\vdots \\ x_n &= f_n(x_{n1}, x_{n2}, \ldots, x_{ni_n}) \\ &\vdots \end{aligned}$$

with $f_n \in F$ and $x_{ij} \in \{x_1, x_2, \ldots\}$ for all $i, j, n \in \omega$, has a solution in E.

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  with $f_n \in F$ and $x_{ij} \in \{x_1, x_2, \ldots\}$ for all $i, j, n \in \omega$, has a solution in E.
- Fixed-Point Problem: characterize the clones with constants in $\mathcal{O}_E$ that have the fixed-point property.

Some well known results

- Theorem (Visser): If $(E, \leq)$ is a ccpo and the logical operators of an interpreted language are monotone functions on that order, then the scheme has the f.p.p.

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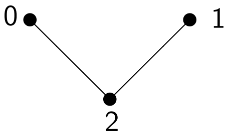

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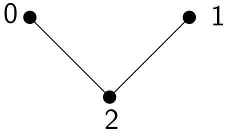

- Corollary (Kripke, Martin, Woodruff): The clones generated by the Kleene strong and weak operators have the f.p.p.

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- The operator of pathologicality:

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- Proof: $x = \neg_k \downarrow (x \wedge_s 2)$

A general characterization for the three-valued case

- Theorem: Let $F \subseteq \mathcal{O}_3$ be a clone with constants. Then F has the f.p.p. iff every function in $F^{(1)}$ has a fixed point. The same characterization is valid for two-valued clones with constants.

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$\odot$	0	1	2
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  - $H_2 = \langle \wedge_w, \vee_w, \wedge_o, \vee_o \rangle$
  - $G_2 = \langle \neg_k, \wedge_w, \odot \rangle$

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- Problem: determine all the clones that can be obtained when we add to the weak Kleene clone a set of functions that include some function which is non-monotonic on the order of information.

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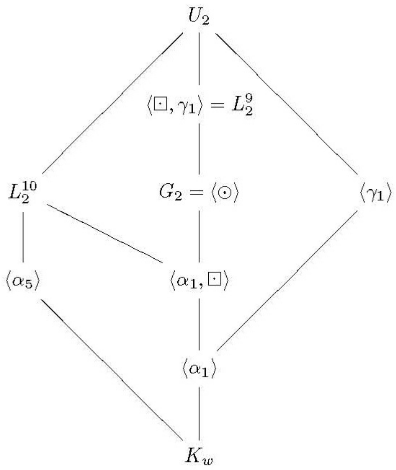
- Facts (Jablonskij): The only maximal three-valued clones that contain the weak Kleene clone are U_2 and C_2.

Graph of non-monotonic expansions of K_w included in U_2:



Characterization of the clones in the graph

-

	α_5	γ_1
0	2	1
1	2	0
2	0	0

$\boxdot$	0	1	2
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- U_2 is the clone of the functions that preserve the relation
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- U_2 is the clone of the functions that preserve the relation $\begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \end{pmatrix}$
- L_2^{10} is the clone of the functions that preserve the relation $\begin{pmatrix} 0 & 1 & 2 & 0 & 0 & 1 & 1 & 2 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & 1 & 1 & 2 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 & 2 & 2 \end{pmatrix}$

- Notation: Let $f \in \mathcal{O}_3$. The derived set of f, denoted $\operatorname{der} f$, is the set of all functions which can be obtained from f with some (all, none) of its variables replaced by constants.

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- We say that the variable x_i is a *contaminant variable* (in f) if, for every $a_1, \ldots, a_n \in E_3$, $f(a_1, \ldots, a_n) = 2$ whenever $a_i = 2$.

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- I_{01} is the the set of all functions that preserve the set $\{0, 1\}$.
- If $f \in I_{01}$, then the restriction of f, denoted $\operatorname{re} f$, is the function $\operatorname{re} f : E_2 \to E_2$ defined as $\operatorname{re} f(a_1, \ldots, a_n) = f(a_1, \ldots, a_n)$, for all $a_1, \ldots, a_n \in E_2$.

Characterization of the clones in the graph

- Let $f \in \mathcal{O}_3$. Then $f \in K_w$ if, and only if, f satisfies the conditions:
  (1) for all $g \in \operatorname{der} f$, if $g \neq c_2$, then $g \in I_{01}$,
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- Let $f \in \mathcal{O}_3$. Then $f \in \langle \neg_2, \wedge_w, \downarrow \rangle$ if, and only if, f satisfies the conditions:
  (1) For every $g \in \operatorname{der} f$, if $g \neq c_2$, then $g \in I_{01}$.
  (2) If $f(a_1, \ldots, a_n) \neq 2$, for some $a_i \in E_3$ and $a_{i_1} = \ldots = a_{i_j} = 2$, for $1 \leq j \leq n$ and $1 \leq i_1 \leq \ldots \leq i_j \leq n$, then the function

$$\operatorname{re} f(a_1, \ldots, a_{i_1-1}, x_1, a_{i_1+1}, \ldots, a_{i_j-1}, x_j, a_{i_j+1}, \ldots, a_n)$$

  is constant.
  (3) If $f \neq c_2$ and there are $a_1, \ldots, a_n \in E_3$ such that $f(a_1, \ldots, a_n) = 2$, then there is a_i, $1 \leq i \leq n$, such that $a_i = 2$ and $f(x_1, \ldots, x_{i-1}, 2, x_{i+1}, \ldots, x_n) = c_2$.

- Let $f \in \mathcal{O}_3$. Then $f \in \langle \neg_2, \wedge_w, \gamma_1 \rangle$ if, and only if, f satisfies the conditions:
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- L_2^9 is the clone of all functions $f \in \mathcal{O}_3$ that satisfy the condition: for all $g \in \operatorname{der} f$, if $g \neq c_2$, then $g \in I_{01}$.