Certain applications of ultraproducts

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- Filters;

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3.$A \in \mathcal{F} \wedge A \subseteq B \Rightarrow B \in \mathcal{F}$ (bigger set then a big set is big).

Example

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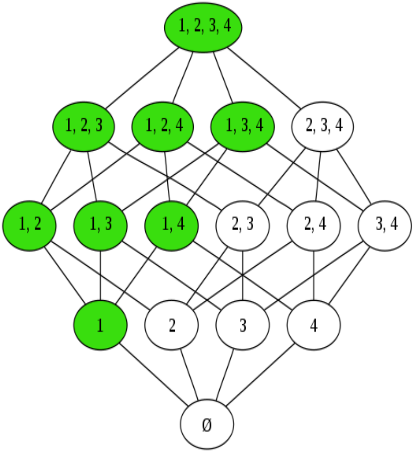

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Lemma

Let U be a filter on I. Then $\sim_U$ is equivalence relation on $\prod A_i$. Conversely, if $\sim_U$ is equivalence relation on $\prod A_i$ then U filter, if $card(A_i) \geqslant 3$, $\forall i \in I$.

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- If R is relation symbol, $ar(R) = n$, then

$$\langle f_{1_U}, \ldots, f_{n_U}\rangle \in R^{\mathcal{A}} \quad \textit{iff} \quad \{i \in I \mid \langle f_1(i), \ldots, f_n(i)\rangle \in R^{\mathcal{A}_i}\} \in U;$$

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$$F^{\mathcal{A}}(f_{1_U}, \ldots, f_{n_U}) = g_U \quad iff \quad \{i \in I \mid F^{\mathcal{A}_i}(f_1(i), \ldots, f_n(i)) = g(i)\} \in U;$$

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- If c is constant, then

$$c^{\mathcal{A}} = \{c^{\mathcal{A}_i} \mid i \in I\}_U.$$

Theorem

(Łoś's theorem)
For every formula $\varphi(x_1, \ldots, x_n)$ *of language* $\mathcal{L}$ *we have:*

$$\mathcal{A} \models_{\nu} \varphi(x_1, \ldots, x_n) \quad \textit{iff} \quad \{i \in I \mid \mathcal{A}_i \models_{\nu_i} \varphi(x_1, \ldots, x_n)\} \in U.$$

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Every colletion of subsets of X *with f.i.p. can be extended to ultrafilter on* X.

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Let U be an ultrafilter containing $\mathcal{J}$.Then, $\prod_{P \in S_\omega(T)} M_P/U \models T$,because for all $\varphi \in T$

$$\{P \in S_\omega(T) \mid M_P \models \varphi\} = J_{\{\varphi\}} \in U.$$

□

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Thank you for your attention!