Information Frames

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Aim To present a logic-oriented representation of L-domains.

Advantage Among others this allows talking about things like higher type functionals or the semantics of programming language in proof assistants.

Let

- A be a set of atomic propositions.

Our language will have no logical connectives. Finite conjunctions will be represented by finite sets of atomic propositions. To state when such finite sets will contain consistent information or not, the logic includes

- a consistency predicate Con

satisfying the natural requirement

$$X \in \mathrm{Con} \wedge Y \subseteq X \Longrightarrow Y \in \mathrm{Con}$$

In addition, there is

- an entailment relation $\vdash \subseteq \mathrm{Con} \times A$

saying what atomic propositions are entailed by which finite consistent sets of atomic proposition. Entailment is extended to finite sets of atomic propositions in the obvious way:

$$X \vdash Y \iff (\forall a \in Y) X \vdash a$$

The following conditions have to be satisfied:

$$X \vdash Y \implies Y \in \mathrm{Con} \qquad \text{(Entailment preserves consistency)}$$
$$X, Y \in \mathrm{Con} \wedge Y \supseteq X \wedge X \vdash a \implies Y \vdash a \qquad \text{(Weakening)}$$
$$X \vdash Y \wedge Y \vdash a \implies X \vdash a \qquad \text{(Transitivity, Cut)}$$

Finally, we assume that A contains an element $\texttt{true}$ which is entailed under any assumption

$$\emptyset \vdash \texttt{true}$$

In the next step, let us assume that we are given

- a set I the elements of which are called possible worlds and
- a binary relation R on I, the accessability relation such that
- for every $i \in I$ there is a consistency predicate Con_i and an entailment relation $\vdash_i$ satisfying the requirements introduced so far.

When moving from i to j according to R, we expect to gain more knowledge about the subject our language is related to. So, the following monotonicity conditions appear very naturally:

$$iRj \Longrightarrow \mathrm{Con}_i \subseteq \mathrm{Con}_j$$

$$iRj \wedge X \in \mathrm{Con}_i \wedge X \vdash_i a \Longrightarrow X \vdash_j a$$

$$iRj \wedge X \in Con_i \wedge X \vdash_j a \Longrightarrow X \vdash_i a$$

The last requirement says that world j is conservative over world i.

Finally, let us suppose the

$$I = A,$$

that is, world a is the world in which statement a "holds".

Con_i should then be thought of as the collection of those finite sets of elementary statements that are consistent with statement i.

So, it is natural to require that

$$\{i\} \in \mathrm{Con}_i \,.$$

As a consequence we obtain

$$\frac{\{i\} \in \mathrm{Con}_i \qquad iRj \Rightarrow \mathrm{Con}_i \subseteq \mathrm{Con}_j}{iRj \Rightarrow \{i\} \in \mathrm{Con}_j}$$

We now choose the accessibility relation R to be as weak as possible, i.e., we require that also

$$\{i\} \in \mathrm{Con}_j \Longrightarrow iRj$$

So, we have

$$iRj \Longleftrightarrow \{i\} \in \mathrm{Con}_j$$

which says:

World j is accessible from world i, exactly if statement i is consistent with statement j.

Definition

Let A be a set, R be a binary relation on A, $\texttt{true} \in A$, $(\mathrm{Con}_i)_{i \in A}$ be a family of subsets of $\mathcal{P}_f(A)$, and $(\vdash_i)_{i \in A}$ be a family of relations $\vdash_i \subseteq \mathrm{Con}_i \times A$. Then $(A, R, (\mathrm{Con}_i)_{i \in A}, (\vdash_i)_{i \in A}, \texttt{true})$ is an information frame if the following conditions hold, for all $i, j, a \in A$ and all finite subsets X, Y, F of A:

- $\{i\} \in \mathrm{Con}_i$
- $Y \subseteq X \wedge X \in \mathrm{Con}_i \Rightarrow Y \in \mathrm{Con}_i$
- $\emptyset \vdash_i \texttt{true}$
- $X \vdash_i Y \Rightarrow Y \in \mathrm{Con}_i$
- $X, Y \in \mathrm{Con}_i \wedge Y \supseteq X \wedge X \vdash_i a \Rightarrow Y \vdash_i a$
- $X \vdash_i Y \wedge Y \vdash_i a \Rightarrow X \vdash_i a$

- $iRj \Rightarrow \mathrm{Con}_i \subseteq \mathrm{Con}_j$
- $\{i\} \in \mathrm{Con}_j \Rightarrow iRj$.
- $iRj \wedge X \in \mathrm{Con}_i \wedge X \vdash_i a \Rightarrow X \vdash_j a$
- $iRj \wedge X \in \mathrm{Con}_i \wedge X \vdash_j a \Rightarrow X \vdash_i a$
- $X \vdash_i F \Rightarrow (\exists e \in A) X \vdash_i e \wedge \{e\} \vdash_e F$.

The last requirement is an interpolation property. Among others it says that finite derivable information can be coded into an elementary statement.

Definition

Let $\mathcal{A} = (A, R, (\mathrm{Con}_i)_{i \in A}, (\vdash_i)_{i \in A}, \mathtt{true})$ be an information frame. A subset x of A is a theory of $\mathcal{A}$ if the following three conditions hold:

1. $(\forall F \subseteq_f x)(\exists i \in x) F \in \mathrm{Con}_i$
2. $(\forall i \in x)(\forall X \subseteq_f x)(\forall a \in A)[X \in \mathrm{Con}_i \wedge X \vdash_i a \Rightarrow a \in x]$
3. $(\forall a \in x)(\exists i \in x)(\exists X \subseteq_f x) X \in \mathrm{Con}_i \wedge X \vdash_i a$.

Thus, a theory of $\mathcal{A}$ is

- finitely consistent,
- closed under entailment, and
- every statement is obtained in this way, i.e., it is entailed in some world by finitely many statements that itself have been obtained in this world or worlds from which the present world is accessible.

Let $|\mathcal{A}|$ denote the set of theories of $\mathcal{A}$.

Lemma

$|\mathcal{A}|$ *is directed-complete.*

This means that if T is a collection of theories such any two theories x and y have a common extension in T, i.e., there is a larger theory $z \supseteq x \cup y$, then also $\bigcup T$ is a theory of $\mathcal{A}$.

The consistent sets in $\mathcal{A}$ generate a canonical basis of $|\mathcal{A}|$. For $i \in A$ and $X \in \mathrm{Con}_i$ let

$$[X]_i = \{\, a \in A \mid X \vdash_i a \,\}.$$

Lemma

1. *$[X]_i$ is a theory of $\mathcal{A}$, for each i, X with $X \in \mathrm{Con}_i$.*
2. *For every $z \in |\mathcal{A}|$, the set of all $[X]_i$ with $\{i\} \cup X \subseteq z$ is directed and z is its union.*

Lemma

1. $[\emptyset]_i = [\texttt{true}]_j$, *for all* $i, j \in A$.
2. $[\emptyset]_{\texttt{true}} \subseteq x$, *for all* $x \in |\mathcal{A}|$.

Lemma

Let $x, y, z \in |\mathcal{A}|$ *so that* $x, y \subseteq z$. *Then*

$$\bigcup \{ [Z]_k \mid Z \in \mathrm{Con}_k \wedge k \in z \wedge Z \subseteq_f x \cup y \}$$

is the least upper bound of x *and* y *in* $\downarrow z = \{ u \in |\mathcal{A}| \mid u \subseteq z \}$.

To sum up:

Theorem

Let $(A, R, (\mathrm{Con}_i)_{i \in A}, (\vdash_i)_{i \in A}, \texttt{true})$ *be an information frame. Then* $(|\mathcal{A}|, \subseteq, [\emptyset]_{\texttt{true}})$ *is an L-domain with basis*

$$\widehat{\mathrm{Con}} = \{ [X]_i \mid i \in A \wedge X \in \mathrm{Con}_i \}.$$

Let us now see more generally what we called L-domain.

We had a partially ordered set $\mathcal{D} = (D, \sqsubseteq)$ with least element $\bot$.

Definition

$S \subseteq D$ is directed, if

- S is not empty and
- for all $x, y \in S$ there is some $z \in S$ with $x, y \sqsubseteq z$.

Definition

$\mathcal{D}$ is directed-complete if every directed subset has a least upper bound in D.

Definition

For $x, y \in D$, x approximates y, written $x \ll y$, if for every directed subset S of D,

$$y \sqsubseteq \bigsqcup S \implies (\exists s \in S) x \sqsubseteq s.$$

Lemma

Let $(A, R, (\mathrm{Con}_i)_{i \in A}, (\vdash_i)_{i \in A}, \mathtt{true})$ *be an information frame. For* $x, y \in |\mathcal{A}|$,

$$x \ll y \iff (\exists i \in A)(\exists V \in \mathrm{Con}_i)\{i\} \cup V \subseteq y \wedge V \vdash_i x.$$

The characterization nicely reflects the intuition that $x \ll y$ if x is covered by a "finite part" of y.

Definition

$B \subseteq D$ is a basis of $\mathcal{D}$ if for each $x \in D$,

- $B_x = \{ u \in B \mid u \ll x \}$ is directed and
- $x = \bigsqcup B_x$.

Definition

$\mathcal{D}$ is an L-domain if

- $\mathcal{D}$ is directed-complete,
- has a basis, and
- for every $z \in D$ and all $x, y \sqsubseteq z$ there is some element $v \in D$ such that
  - $x, y \sqsubseteq v \sqsubseteq z$,
  - for all $w \in D$, if $x, y \sqsubseteq w \sqsubseteq z$, then $v \sqsubseteq w$,

  i.e., v is the least upper bound of x, y relative to $\downarrow z$. We write $x \sqcup^z y$.

Let us now conversely see how every L-domain determines an information frame.

Let $\mathcal{D}$ be an L-domain with basis B and least element $\bot$.

For $u \in B$ define,

$$\mathrm{Con}_u = \{ X \subseteq_f B \mid X \subseteq {\downarrow} u \}, \quad X \vdash_u v \iff v \ll \bigsqcup^u X$$

$$uRv \iff u \sqsubseteq v$$

Theorem

$\mathcal{B} = (B, R, (\mathrm{Con}_u)_{u \in B}, (\vdash_u)_{u \in B}, \bot)$ is an information frame such that $\mathcal{D}$ and $|\mathcal{B}|$ are isomorphic domains.

Let us finally consider some important special cases.

Definition

$\mathcal{D}$ is a continuous Scott domain if

- $\mathcal{D}$ is directed-complete,
- has a basis, and
- for all $x, y \in D$ such that $\{x, y\}$ is bounded from above the least upper bound $x \sqcup y$ exists in D.

So, a continuous Scott domain is an L-domain in which bounded finite sets have even global least upper bounds.

Let again B be a basis of $\mathcal{D}$ and

$$\mathrm{Con}_u = \{ X \subseteq_f B \mid X \subseteq \downarrow u \}$$

$$X \vdash_u v \iff v \ll \bigsqcup^u X$$

$$uRv \iff u \sqsubseteq v$$

Now, assume for $u, v \in B$ that

$$X \in \mathrm{Con}_u \cap \mathrm{Con}_v,$$

which means that both, u and v, are upper bounds of X. Then we have that

$$\bigsqcup^u X = \bigsqcup X = \bigsqcup^v X$$

which implies that

$$X \vdash_u w \iff X \vdash_v w$$

Theorem

Let $\mathcal{A} = (A, R, (\mathrm{Con}_i)_{i\in A}, (\vdash_i)_{i\in A}, \mathtt{true})$ *be an information frame. Then* $|\mathcal{A}|$ *is a continuous Scott domain if, and only if,* $\mathcal{A}$ *satisfies Condition (S) saying that for all* $X \subseteq_f A$ *and* $i, j \in A$,

$$X \in \mathrm{Con}_i \cap \mathrm{Con}_j \Rightarrow (\forall a \in A)[X \vdash_i a \Leftrightarrow X \vdash_j a]. \qquad \text{(S)}$$

In the presence of Condition (S) we even have a syntactic translation from information frames to continuous information systems known to characterize continuous Scott domains. The idea is to hide the explicit consistency witnesses i.

Theorem

Let $\mathcal{A} = (A, R, (\mathrm{Con}_i)_{i \in A}, (\vdash_i)_{i \in A}, \mathtt{true})$ *be an information frame satisfying Condition (S) and define*

$$\mathrm{Con} = \bigcup \{ \mathrm{Con}_i \mid i \in A \} \quad \textit{and} \quad \vdash = \bigcup \{ \vdash_i \mid i \in A \}.$$

Then $(A, \mathrm{Con}, \vdash)$ *is a continuous information system, i.e., for all* $a \in A$ *and all finite subsets* X, Y *of* A *the following conditions hold:*

- $\emptyset \in \mathrm{Con}$
- $X \subseteq Y \in \mathrm{Con} \Rightarrow X \in Con$
- $\{a\} \in \mathrm{Con}$
- $X \vdash Y \Rightarrow Y \in \mathrm{Con}$
- $X \vdash a \wedge X \subseteq Y \Rightarrow Y \vdash a$
- $X \vdash Y \wedge Y \vdash a \Rightarrow X \vdash a$
- $X \vdash a \Rightarrow (\exists Y \in \mathrm{Con}) X \vdash Y \wedge Y \vdash a.$

Definition

An L-domain $\mathcal{D}$ is algebraic if the set

$$K_D = \{\, x \in D \mid x \ll x \,\}$$

is a basis of $\mathcal{D}$.

Definition

An element $j \in A$ is reflexive, if $\{j\} \vdash_j j$.

Let A_{refl} denote the set of reflexive elements of A.

Theorem

Let $\mathcal{A} = (A, R, (\mathrm{Con}_i)_{i \in A}, (\vdash_i)_{i \in A}, \mathtt{true})$ be an information frame. Then $|\mathcal{A}|$ is an algebraic L-domain if, and only if, $\mathcal{A}$ satisfies Condition (ALG) saying that for all $X \subseteq_f A$ and $i, j \in A$,

$$X \vdash_i F \Rightarrow (\exists j \in A_{\mathrm{refl}}) X \vdash_i j \wedge \{j\} \vdash_j F. \qquad \text{(ALG)}$$

In the presence of both Conditions (S) and (ALG) we even have a syntactic translation from information frames to algebraic information systems known to characterize algebraic Scott domains.

Theorem

Let $\mathcal{A} = (A, R, (\mathrm{Con}_i)_{i \in A}, (\vdash_i)_{i \in A}, \mathtt{true})$ *be an information frame satisfying Conditions (S) as well as (ALG), and define*

$$\mathrm{Con}_{\mathrm{refl}} = \{\, X \subseteq_f A_{\mathrm{refl}} \mid (\exists i \in A_{\mathrm{refl}}) A \in \mathrm{Con}_i \,\}$$

and

$$X \vdash_{\mathrm{refl}} a \iff (\exists i \in A_{\mathrm{refl}}) X \vdash_i a.$$

Then $(A_{\mathrm{refl}}, \mathrm{Con}_{\mathrm{refl}}, \vdash_{\mathrm{refl}})$ *is an algebraic information system, i.e., for all* $a \in A_{\mathrm{refl}}$ *and all finite subsets* X, Y *of* A_{refl} *the following conditions hold:*

- $\emptyset \in \mathrm{Con}_{\mathrm{refl}}$
- $X \subseteq Y \in \mathrm{Con}_{\mathrm{refl}} \Rightarrow X \in Con_{\mathrm{refl}}$
- $\{a\} \in \mathrm{Con}_{\mathrm{refl}}$
- $X \vdash_{\mathrm{refl}} Y \Rightarrow Y \in \mathrm{Con}_{\mathrm{refl}}$
- $X \vdash_{\mathrm{refl}} a \wedge X \subseteq Y \Rightarrow Y \vdash_{\mathrm{refl}} a$
- $X \vdash_{\mathrm{refl}} Y \wedge Y \vdash_{\mathrm{refl}} a \Rightarrow X \vdash_{\mathrm{refl}} a$
- $a \in X \Rightarrow X \vdash a.$

In the case of Scott's information systems morphisms between two such systems

$$(A, \mathrm{Con}_A, \vdash_A) \quad \text{and} \quad (B, \mathrm{Con}_B, \vdash_B)$$

are certain relations

$$H \subseteq \mathrm{Con}_A \times B,$$

called approximable mappings.

In the case of information frames they are families $(H_i)_{i \in A}$ of such relations

$$H_i \subseteq \mathrm{Con}_i^A \times B.$$

Further results:

- With approximable mappings as morphisms information frames form a Cartesian closed category equivalent to the category of L-domains with Scott continuous functions.
- The exponent between two information frames can explicitly be constructed.