Justification Logic – A Short Introduction

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Modal Logic

Modal logic adds a new connective $\Box$ to the language of logic.

Two traditions:

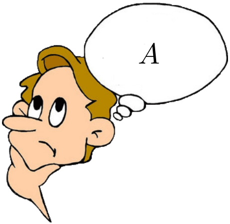
Epistemic logic:

$\Box A$ means A is known / believed

Proof theory:

$\Box A$ means A is provable in system S

Modal Logic: How It Works



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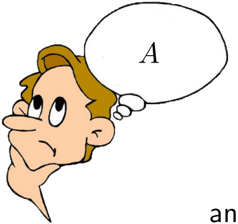

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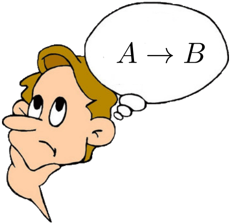

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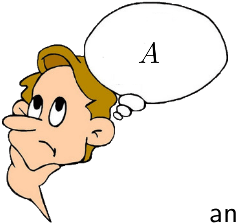

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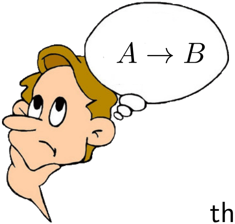

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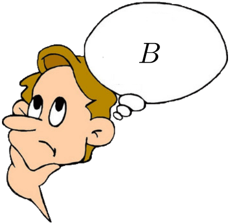

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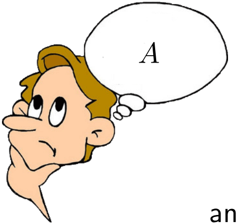

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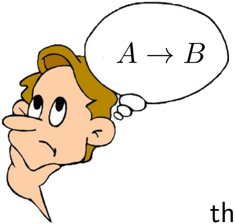

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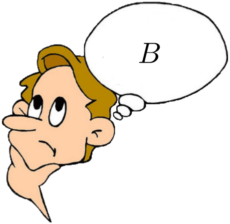

$$\Box A \quad \wedge \quad \Box(A \to B) \quad \to \quad \Box B$$

Problems: Epistemic Tradition

Knowledge is justified true belief

True belief is modeled by $\Box A \to A$ but

where are the justifications in modal logic?

Modal logic is logically omniscient

If an agent knows A, then she also knows all logical consequences of A.

This is follows from $\Box(A \to B) \to (\Box A \to \Box B)$ and the property that if $A \to B$ is provable, so is $\Box(A \to B)$.

Problems: Proof-Theoretic Tradition

$\Box\bot \rightarrow \bot$ is an axiom.

$\neg\Box\bot$ is provable.

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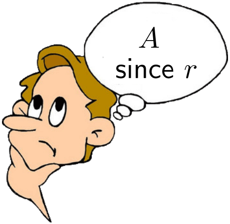
Gödel: if S has a certain strength, it cannot prove its own consistency.

Justification Logic



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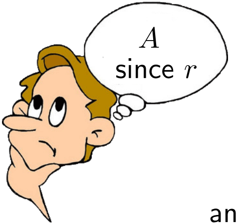

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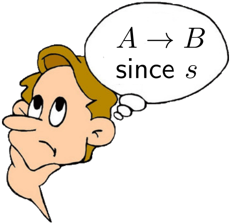

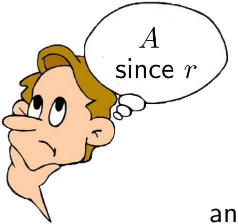

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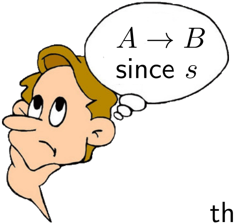

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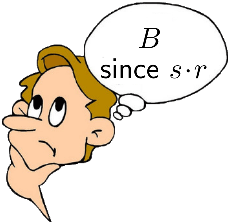

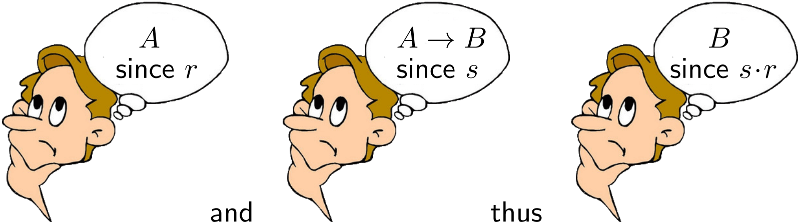

$$r{:}A \quad \wedge \quad s{:}(A \to B) \quad \to \quad s \cdot r{:}B$$

Syntax of the Logic of Proofs

Logic

The logic of proofs $\mathsf{LP}_{\mathsf{CS}}$ is the justification counterpart of the modal logic S4.

Justification terms Tm

$$t ::= x \mid c \mid (t \cdot t) \mid (t + t) \mid !t$$

Formulas $\mathcal{L}_j$

$$A ::= p \mid \neg A \mid (A \to A) \mid t{:}A$$

Axioms for LP

- all propositional tautologies
- $t{:}(A \to B) \to (s{:}A \to (t \cdot s){:}B)$ (J)
- $t{:}A \to (t + s){:}A, \quad s{:}A \to (t + s){:}A$ (+)
- $t{:}A \to A$ (jt)
- $t{:}A \to !t{:}t{:}A$ (j4)

Deductive System

Constant specification

A constant specification CS is any subset

$$\mathsf{CS} \subseteq \{(c, A) \mid c \text{ is a constant and } A \text{ is an axiom}\}.$$

The deductive system $\mathsf{LP}_{\mathsf{CS}}$ consists of the above axioms and the rules of modus ponens and axiom necessitation.

$$\frac{A \quad A \to B}{B} \qquad\qquad \frac{(c, A) \in \mathsf{CS}}{c{:}A}$$

A Justified Version of $\Box A \vee \Box B \to \Box(A \vee B)$

Assume we are given $\mathsf{LP}_{\mathsf{CS}}$ with

$$(a, A \to (A \vee B)) \in \mathsf{CS} \quad \text{and} \quad (b, B \to (A \vee B)) \in \mathsf{CS} \ .$$

By axiom necessitation we get

$$\mathsf{LP}_{\mathsf{CS}} \vdash a{:}(A \to (A \vee B)) \quad \text{and} \quad \mathsf{LP}_{\mathsf{CS}} \vdash b{:}(B \to (A \vee B)) \ .$$

Using (J) and (MP) we obtain

$$\mathsf{LP}_{\mathsf{CS}} \vdash x{:}A \to (a \cdot x){:}(A \vee B) \quad \text{and} \quad \mathsf{LP}_{\mathsf{CS}} \vdash y{:}B \to (b \cdot y){:}(A \vee B) \ .$$

Finally, from $(+)$ we have

$$\mathsf{LP}_{\mathsf{CS}} \vdash (a \cdot x){:}(A \vee B) \to (a \cdot x + b \cdot y){:}(A \vee B) \text{ and}$$
$$\mathsf{LP}_{\mathsf{CS}} \vdash (b \cdot y){:}(A \vee B) \to (a \cdot x + b \cdot y){:}(A \vee B) \ .$$

Using propositional reasoning, we obtain

$$\mathsf{LP}_{\mathsf{CS}} \vdash (x{:}A \vee y{:}B) \to (a \cdot x + b \cdot y){:}(A \vee B) \ .$$

Internalization

Definition

A constant specification CS for LP is called *axiomatically appropriate* if for each axiom F of LP, there is a constant c such that $(c, F) \in \mathsf{CS}$.

Lemma (Internalization)

Let CS be an axiomatically appropriate constant specification. For arbitrary formulas $A, B_1, \ldots, B_n$*, if*

$$B_1, \ldots, B_n \vdash_{\mathsf{LP}_{\mathsf{CS}}} A \ ,$$

then there is a term t *such that*

$$x_1{:}B_1, \ldots, x_n{:}B_n \vdash_{\mathsf{LP}_{\mathsf{CS}}} t{:}A$$

for fresh variables $x_1, \ldots, x_n$.

Forgetful Projection

Definition (Forgetful projection)

The mapping $\circ$ from justified formulas to modal formulas is defined as follows

1. $P^\circ := P$ for P atomic;
2. $(\neg A)^\circ := \neg A^\circ$;
3. $(A \to B)^\circ := A^\circ \to B^\circ$;
4. $(t{:}A)^\circ := \Box A^\circ$.

Lemma (Forgetful projection)

For any constant specification CS *and any formula* F *we have*

$$\mathsf{LP}_{\mathsf{CS}} \vdash F \quad \textit{implies} \quad \mathsf{S4} \vdash F^\circ \ .$$

Realization

Definition (Realization)

A *realization* is a mapping r from modal formulas to justified formulas such that $(r(A))^\circ = A$.

Definition

We say a justification logic $\mathsf{LP}_{\mathsf{CS}}$ *realizes* S4 if there is a realization r such that for any formula A we have

$$\mathsf{S4} \vdash A \quad \text{implies} \quad \mathsf{LP}_{\mathsf{CS}} \vdash r(A) \ .$$

Realization Theorem

Definition (Schematic CS)

We say that a constant specification is *schematic* if it satisfies the following: for each constant c, the set of axioms $\{A \mid (c, A) \in \mathsf{CS}\}$ consists of all instances of one or several (possibly zero) axiom schemes of LP.

Theorem (Realization)

Let CS be an axiomatically appropriate and schematic constant specification. There exists a realization r such that for all formulas A

$$\mathsf{S4} \vdash A \quad \Longrightarrow \quad \mathsf{LP}_{\mathsf{CS}} \vdash r(A) \ .$$

Arithmetical Semantics

Originally, $\mathsf{LP}_{\mathsf{CS}}$ was developed to provide classical provability semantics for intuitionistic logic.

Arithmetical Semantics for $\mathsf{LP}_{\mathsf{CS}}$: Justification terms are interpreted as proofs in Peano arithmetic and operations on terms correspond to computable operations on proofs in PA.

$$\mathsf{Int} \quad \xrightarrow{\text{Gödel}} \quad \mathsf{S4} \quad \xrightarrow{\text{Realization}} \quad \mathsf{JL} \quad \xrightarrow{\text{Arithm. sem.}} \quad \mathsf{CL} + \text{proofs}$$

Self-referentiality

Definition (Self-referential CS)

A constant specification CS is called *self-referential* if $(c, A) \in \mathsf{CS}$ for some axiom A that contains at least one occurrence of the constant c.

S4 and $\mathsf{LP}_{\mathsf{CS}}$ describe self-referential knowledge. That means if $\mathsf{LP}_{\mathsf{CS}}$ realizes S4 for some constant specification CS, then that constant specification must be self-referential.

Lemma

Consider the S4*-theorem* $G := \neg\Box((P \to \Box P) \to \bot)$ *and let* F *be any realization of* G*.*
If $\mathsf{LP}_{\mathsf{CS}} \vdash F$*, then* CS *must be self-referential.*

Basic evaluation

Definition (Basic Evaluation)

A *basic evaluation* $*$ for $\mathsf{LP}_{\mathsf{CS}}$ is a function:

$$*: \mathsf{Prop} \to \{0,1\} \text{ and } *: \mathsf{Tm} \to \mathcal{P}(\mathcal{L}_j), \text{ such that}$$

1. $F \in (s \cdot t)^*$ if $(G \to F) \in s^*$ and $G \in t^*$ for some G
2. $F \in (s + t)^*$ if $F \in s^*$ or $F \in t^*$
3. $F \in t^*$ if $(t, F) \in \mathsf{CS}$
4. $s{:}F \in (!s)^*$ if $F \in s^*$

Quasimodel

Definition (Quasimodel)

A *quasimodel* is a tuple $\mathcal{M} = (W, R, *)$ where $W \neq \varnothing$, $R \subseteq W \times W$, and the *evaluation* $*$ maps each world $w \in W$ to a basic evaluation $*_w$.

Definition (Truth in quasimodels)

$\mathcal{M}, w \Vdash p$ if and only if $p^*_w = 1$ for $p \in \mathsf{Prop}$;

$\mathcal{M}, w \Vdash F \to G$ if and only if $\mathcal{M}, w \nVdash F$ or $\mathcal{M}, w \Vdash G$;

$\mathcal{M}, w \Vdash \neg F$ if and only if $\mathcal{M}, w \nVdash F$;

$\mathcal{M}, w \Vdash t{:}F$ if and only if $F \in t^*_w$.

Model

Given $\mathcal{M} = (W, R, *)$ and $w \in W$, we define

$$\Box_w := \{F \in \mathcal{L}_j \mid \mathcal{M}, v \Vdash F \text{ whenever } R(w, v)\} \ .$$

Definition (Modular Model)

A *modular model* $\mathcal{M} = (W, R, *)$ is a quasimodel with

1. $t^*_w \subseteq \Box_w$ for all $t \in \mathsf{Tm}$ and $w \in W$; (JYB)
2. R is reflexive;
3. R is transitive.

Theorem (Soundness and Completeness)

For all formulas $F \in \mathcal{L}_j$,

$$\mathsf{LP}_{\mathsf{CS}} \vdash F \quad \Longleftrightarrow \quad \mathcal{M} \Vdash F \text{ for all modular models } \mathcal{M}.$$

Decidability

In modal logic, decidability is a consequence of the finite model property. For $\mathsf{LP}_{\mathsf{CS}}$ the situation is more complicated since CS usually is infinite.

Theorem

$\mathsf{LP}_{\mathsf{CS}}$ *is decidable for decidable schematic constant specifications* CS*.*

A decidable CS is not enough:

Theorem

There exists a decidable constant specification CS *such that* $\mathsf{LP}_{\mathsf{CS}}$ *is undecidable.*

Complexity

Theorem

Let CS *be a schematic constant specification.*
The problem whether $\mathsf{LP}_{\mathsf{CS}} \vdash t{:}B$ *belongs to NP.*

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Definition

A constant specification is called *schematically injective* if it is schematic and each constant justifies no more that one axiom scheme.

Theorem

Let CS *be a schematically injective and axiomatically appropriate constant specification.*
The derivability problem for $\mathsf{LP}_{\mathsf{CS}}$ *is* Π^p_2*-complete.*

Logical Omniscience

Modal logic of knowledge contains the epistemic closure principle in the form of axiom

$$\Box(A \to B) \to (\Box A \to \Box B) \ ,$$

which yields an unrealistic feature called *logical omniscience* whereby an agent knows all logical consequences of her assumptions.

Definition

A *proof system* for a logic L is a binary relation $E \subset \Sigma^\star \times \mathsf{L}$ between words in some alphabet, called proofs, and theorems of L such that

1. E is computable in polynomial time and
2. for all formulas F, $\mathsf{L} \vdash F$ if and only if there exists y with $E(y, F)$.

Logical Omniscience II

Knowledge assertion A is a provable formula of the form

$$\Box B \text{ for S4} \qquad \text{or} \qquad t{:}B \text{ for } \mathsf{LP}_{\mathsf{CS}}$$

with the object of knowledge function $\mathsf{OK}(A) := B$.

Definition (Logical Omniscience Test (LOT))

An proof system E for an epistemic logic L is *not logically omniscient*, or *passes LOT*, if there exists a polynomial P such that for any knowledge assertion A, there is a proof of $\mathsf{OK}(A)$ in E with the size bounded by $P(\mathsf{size}(A))$.

Logical Omniscience III

Theorem (S4 is logically omniscient)

There is no proof system for S4 that passes LOT unless NP=PSPACE.

Theorem ($\mathsf{LP}_{\mathsf{CS}}$ is not logically omniscient)

Let CS be a schematic constant specification. There is a proof system for $\mathsf{LP}_{\mathsf{CS}}$ that passes LOT.

Dynamic Epistemic Logic

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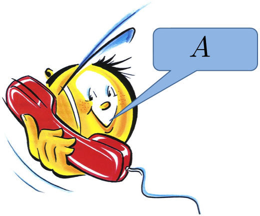

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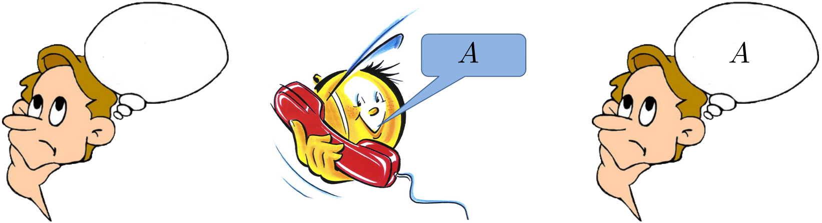

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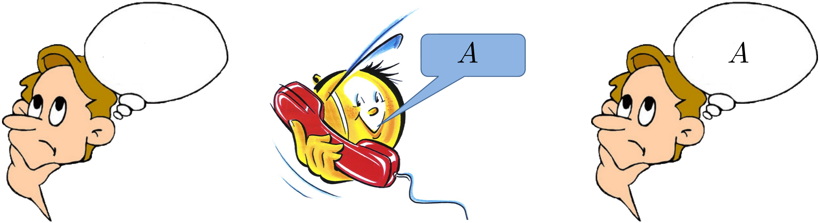

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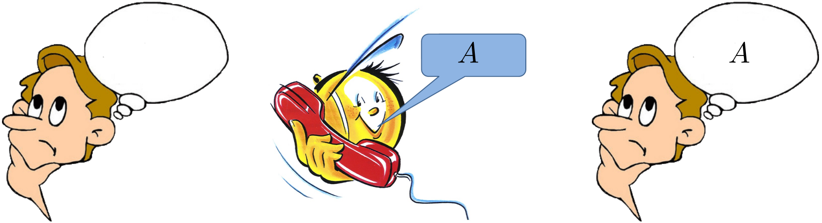
Dynamic Epistemic Logic



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Problem

The $\Box$-operator does not tell us whether A is believed because of the announcement or whether A is believed independent of it.

Update as Evidence: the Logic $\mathsf{JUP}_{\mathsf{CS}}$ for Belief Expansion

Fundamental principle

After the announcement of A,

the announcement itself justifies the agent's belief in A.

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For each formula A we add a new justification term $\mathsf{up}(A)$.

Some axioms of JUP:

- Success: $[A]\,\mathsf{up}(A){:}A$

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Some axioms of JUP:

- Success: $[A]\,\mathsf{up}(A){:}A$
- Persistence: $t{:}B \to [A]t{:}B$.
- Reduction axioms
- Minimal change
- Iterated updates

Basic Properties of $\mathsf{JUP}_{\mathsf{CS}}$

Lemma (Minimal change)

Let t *be a term that does not contain* $\mathsf{up}(A)$ *as a subterm. Then*

$$\mathsf{JUP}_{\mathsf{CS}} \vdash [A]t{:}B \leftrightarrow t{:}B \quad .$$

Lemma (Minimal change)

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Lemma (Ramsey I)

$\mathsf{JUP}_{\mathsf{CS}} \vdash t{:}(A \to B) \to [A](t \cdot \mathsf{up}(A)){:}B.$

Lemma (Ramsey II)

Let CS be an axiomatically appropriate constant specification. For each term t there exists a term s such that

$$\mathsf{JUP}_{\mathsf{CS}} \vdash [A]t{:}B \to s{:}(A \to B) \ .$$

Thank you!