Lindström's theorem for interpretability logic

Logic and Applications 2014, IUC, Dubrovnik

Mladen Vuković
vukovic@math.hr
web.math.pmf.unizg.hr/∼vukovic/

Department of Mathematics,
Faculty of Science,
University of Zagreb

September, 2014

Contents

§ 1. Lindström's theorem for first–order logic (1)

Lindström's theorem characterizes logics in terms of model–theoretic conditions such as **Compactness** and **the Löwenheim–Skolem property**.

We repeat some definitions and emphasize some facts.

§ 1. FO (2) - abstract logic

A logic $\mathfrak{L}$ is a pair $(L, \models_{\mathfrak{L}})$, where L is a function that maps signatures σ to the sets $L(\sigma)$ of $\mathfrak{L}$–formulas over σ and $\models_{\mathfrak{L}}$ is a relation between structures and formulas of $\mathfrak{L}$.

Any logic $\mathfrak{L}$ is assumed to satisfy the following: **The Expansion Property**, **The Reduct Property** and **The Isomorphism Property**.

§ 1. FO (3) - abstract logic

1. **The Expansion Property**: if $\sigma \subseteq \sigma'$ then $L(\sigma) \subseteq L(\sigma')$;
2. **The Reduct Property**: If $\sigma \subseteq \sigma'$, $\phi \in L(\sigma)$ and $\mathfrak{M}$ a σ'–structure, then $\mathfrak{M} \restriction \sigma \models_{\mathfrak{L}} \phi$ iff $\mathfrak{M} \models_{\mathfrak{L}} \phi$;
3. **The Isomorphism Property**: if $\mathfrak{M}$ is isomorphic to $\mathfrak{N}$, then $\mathfrak{M} \models_{\mathfrak{L}} \phi$ iff $\mathfrak{N} \models_{\mathfrak{L}} \phi$.

§ 1. FO (3) - abstract logic

1. **The Expansion Property**: if $\sigma \subseteq \sigma'$ then $L(\sigma) \subseteq L(\sigma')$;
2. **The Reduct Property**: If $\sigma \subseteq \sigma'$, $\phi \in L(\sigma)$ and $\mathfrak{M}$ a σ'–structure, then $\mathfrak{M} \upharpoonright \sigma \models_{\mathfrak{L}} \phi$ iff $\mathfrak{M} \models_{\mathfrak{L}} \phi$;
3. **The Isomorphism Property**: if $\mathfrak{M}$ is isomorphic to $\mathfrak{N}$, then $\mathfrak{M} \models_{\mathfrak{L}} \phi$ iff $\mathfrak{N} \models_{\mathfrak{L}} \phi$.

§ 1. FO (3) - abstract logic

1. **The Expansion Property**: if $\sigma \subseteq \sigma'$ then $L(\sigma) \subseteq L(\sigma')$;
2. **The Reduct Property**: If $\sigma \subseteq \sigma'$, $\phi \in L(\sigma)$ and $\mathfrak{M}$ a σ'–structure, then $\mathfrak{M} \upharpoonright \sigma \models_{\mathfrak{L}} \phi$ iff $\mathfrak{M} \models_{\mathfrak{L}} \phi$;
3. **The Isomorphism Property**: if $\mathfrak{M}$ is isomorphic to $\mathfrak{N}$, then $\mathfrak{M} \models_{\mathfrak{L}} \phi$ iff $\mathfrak{N} \models_{\mathfrak{L}} \phi$.

§ 1. FO (4) - regular logic

Regular logic $\mathfrak{L}$

- **The Closure property**: if $\varphi, \psi \in L(\sigma)$ then so are $\phi \wedge \psi$, $\neg\phi$, and $\exists x \phi$.
- **The Relativization Property**: Let $\mathfrak{M}$ be a σ–structure. Let $\mathfrak{M}$ be a substructure of $\mathfrak{M}$ such that the universe of $\mathfrak{N}$ is defined by a first–order formula $\varphi(x)$. For any $\psi \in L(\sigma)$ there exists $\psi_\phi \in L(\sigma)$ such that $\mathfrak{M} \models_{\mathfrak{L}} \psi_\phi$ iff $\mathfrak{N} \models_{\mathfrak{L}} \phi$.
- **The Relational Property**: elimination of functional and constant symbols.

§ 1. FO (4) - regular logic

Regular logic $\mathfrak{L}$

- **The Closure property**: if $\varphi, \psi \in L(\sigma)$ then so are $\phi \wedge \psi$, $\neg\phi$, and $\exists x\phi$.
- **The Relativization Property**: Let $\mathfrak{M}$ be a σ–structure. Let $\mathfrak{M}$ be a substructure of $\mathfrak{M}$ such that the universe of $\mathfrak{N}$ is defined by a first–order formula $\varphi(x)$. For any $\psi \in L(\sigma)$ there exists $\psi_\phi \in L(\sigma)$ such that $\mathfrak{M} \models_{\mathfrak{L}} \psi_\phi$ iff $\mathfrak{N} \models_{\mathfrak{L}} \phi$.
- **The Relational Property**: elimination of functional and constant symbols.

§ 1. FO (4) - regular logic

Regular logic $\mathfrak{L}$

- **The Closure property**: if $\varphi, \psi \in L(\sigma)$ then so are $\phi \wedge \psi$, $\neg\phi$, and $\exists x \phi$.
- **The Relativization Property**: Let $\mathfrak{M}$ be a σ–structure. Let $\mathfrak{M}$ be a substructure of $\mathfrak{M}$ such that the universe of $\mathfrak{N}$ is defined by a first–order formula $\varphi(x)$. For any $\psi \in L(\sigma)$ there exists $\psi_\phi \in L(\sigma)$ such that $\mathfrak{M} \models_{\mathfrak{L}} \psi_\phi$ iff $\mathfrak{N} \models_{\mathfrak{L}} \phi$.
- **The Relational Property**: elimination of functional and constant symbols.

§ 1. FO (5)

If $\mathfrak{L}$ is a logic and $\varphi \in L(\sigma)$, let

$$\mathrm{Mod}^{\sigma}_{\mathfrak{L}}(\varphi) = \{\mathfrak{M} \ : \ \mathfrak{M} \ \text{ is a } \ \sigma\text{–structure and } \ \mathfrak{M} \models_{\mathfrak{L}} \varphi\}$$

We say that a logic $\mathfrak{L}_2$ is **at least strong as a logic** $\mathfrak{L}_1$ (written: $\mathfrak{L}_1 \leq \mathfrak{L}_2$) if for any σ and $\varphi \in L_1(\sigma)$ there is a sentence $\psi \in L_2(\sigma)$ such that

$$\mathrm{Mod}^{\sigma}_{\mathfrak{L}_1}(\varphi) = \mathrm{Mod}^{\sigma}_{\mathfrak{L}_2}(\psi)$$

Logics $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are **equally strong** (written: $\mathfrak{L}_1 \sim \mathfrak{L}_2$) if $\mathfrak{L}_1 \leq \mathfrak{L}_2$ and $\mathfrak{L}_2 \leq \mathfrak{L}_1$.

§ 1. FO (6)

We introduce the following abbreviations:

Comp($\mathfrak{L}$) – The Compactness theorem holds for logic $\mathfrak{L}$, i.e. if Φ is a set of sentences of $\mathfrak{L}$ such that every finite subset of Φ is satisfiable, then Φ is satisfiable.

LöSko($\mathfrak{L}$) – The Löwenheim–Skolem theorem holds for logic $\mathfrak{L}$, i.e. every satisfiable sentence of $\mathfrak{L}$ has an at most countable model.

§ 1. FO (7)

Lindström's theorem for first–order logic (1969)

Let $\mathfrak{L}$ be a regular logic such that $\mathtt{FO} \leq \mathfrak{L}$, Comp$(\mathfrak{L})$ and LöSko$(\mathfrak{L})$.
Then $\mathtt{FO} \sim \mathfrak{L}$.

§ 1. FO (8) – proof

We point out some parts of a proof of Lindström's theorem for FO:

- **The main lemma**. Let $\mathfrak{L}$ be a regular logic such that Comp($\mathfrak{L}$) and FO$\leq \mathfrak{L}$. If $\mathfrak{M} \equiv_{\mathrm{FO}} \mathfrak{N}$ implies $\mathfrak{M} \equiv_{\mathfrak{L}} \mathfrak{N}$, then FO$\sim \mathfrak{L}$.
- **Fraïsse Theorem**. We have $\mathfrak{M} \equiv_{\mathrm{FO}} \mathfrak{N}$ if and only if there is a sequence of sets of partial isomorphisms (I_n) such that $(I_n) : \mathfrak{M} \simeq_{fin} \mathfrak{N}$.
- (**Ehrenfeucht games**)

§ 1. FO (8) – proof

We point out some parts of a proof of Lindström's theorem for FO:

- **The main lemma**. Let $\mathfrak{L}$ be a regular logic such that Comp($\mathfrak{L}$) and FO$\leq \mathfrak{L}$. If $\mathfrak{M} \equiv_{\mathtt{FO}} \mathfrak{N}$ implies $\mathfrak{M} \equiv_{\mathfrak{L}} \mathfrak{N}$, then FO$\sim \mathfrak{L}$.
- **Fraïsse Theorem**. We have $\mathfrak{M} \equiv_{\mathtt{FO}} \mathfrak{N}$ if and only if there is a sequence of sets of partial isomorphisms (I_n) such that $(I_n) : \mathfrak{M} \simeq_{fin} \mathfrak{N}$.
- (**Ehrenfeucht games**)

§ 1. FO (8) – proof

We point out some parts of a proof of Lindström's theorem for FO:

- **The main lemma**. Let $\mathfrak{L}$ be a regular logic such that Comp($\mathfrak{L}$) and FO$\leq \mathfrak{L}$. If $\mathfrak{M} \equiv_{\mathtt{FO}} \mathfrak{N}$ implies $\mathfrak{M} \equiv_{\mathfrak{L}} \mathfrak{N}$, then FO$\sim \mathfrak{L}$.
- **Fraïsse Theorem**. We have $\mathfrak{M} \equiv_{\mathtt{FO}} \mathfrak{N}$ if and only if there is a sequence of sets of partial isomorphisms (I_n) such that $(I_n) : \mathfrak{M} \simeq_{fin} \mathfrak{N}$.
- (**Ehrenfeucht games**)

§ 1. FO (9)

Most existing Lindström's theorems concern extensions of first-order logic (Barwise, Flum, Väänänen, ...)

On the other hand, many logics relevant to computer science are fragments or extensions of fragments of first-order logic, e.g., k–variable logics and various modal logics. (van Benthem, ten Cate, Väänänen, Log. Meth. Com. Sci. 2009)

§ 2. Lindström's theorem for modal logic (1)

To prove an analogous characterization result for modal logic we need to agree on a number of things.

- What will be the distinguishing property of the logic that we want to characterize (on top of its invariance for bisimulations)?
- What is a suitable notion of an abstract modal logic?
  Many authors introduce some bookkeeping properties from the formulation of the original Lindström's Theorem for first–order logic, and add a further property having to do with invariance under bisimulations.

§ 2. Lindström's theorem for modal logic (1)

To prove an analogous characterization result for modal logic we need to agree on a number of things.

- What will be the distinguishing property of the logic that we want to characterize (on top of its invariance for bisimulations)?
- What is a suitable notion of an abstract modal logic?
  Many authors introduce some bookkeeping properties from the formulation of the original Lindström's Theorem for first–order logic, and add a further property having to do with invariance under bisimulations.

§ 2. Modal logic (2) - basic definitions

A similarity type Σ is any non–empty set, the elements of which are called **modal operators**.

Similarly, **a propositional signature** is any non–empty set P, the elements of which are called **propositional variables**.

A frame of similarity type Σ is defined as a structure $\langle W, \{R_i\}_{\Diamond_i \in \Sigma} \rangle$ with a binary accessibility relation assigned to each operator in Σ.

A model of the same similarity type and propositional signature P is a frame together with a valuation $V : P \times W \to \{0, 1\}$, and **pointed model** of similarity type Σ and propositional signature P is a pair (M, u) where M is a model and $u \in W$.

§ 2. Modal logic (3)

Elements in the domain of a model will henceforth be referred to as **nodes** or **worlds**.

The basic modal language $L_{BML}(P)$ for a similarity type Σ and a propositional signature P is defined as:

$$L_{BML} \ : \ p \mid \neg(A) \mid (A) \wedge (B) \mid \Diamond_i(A),$$

where $\Diamond_i \in \Sigma$ and $p \in P$.

§ 2. Modal logic (4)

The satisfaction relation $\models_{BML}$ between pointed models and modal formulas is defined by the following clauses:

- $(M, u) \models_{BML} p$ iff $V(p, u) = 1$
- $(M, u) \models_{BML} A \wedge B$ iff $(M, u) \models_{BML} A$ and $(M, u) \models_{BML} B$
- $(M, u) \models_{BML} \neg A$ iff $(M, u) \not\models_{BML} A$
- $(M, u) \models_{BML} \Diamond_i A$ iff there is some v with $uR_i v$ and $(M, v) \models_{BML} A$

The basic modal logic for similarity type Σ will be identified with the pair $(L_{BML}, \models_{BML})$ and denoted by BML.

§ 2. Modal logic (4)

The satisfaction relation $\models_{BML}$ between pointed models and modal formulas is defined by the following clauses:

- $(M, u) \models_{BML} p$ iff $V(p, u) = 1$
- $(M, u) \models_{BML} A \wedge B$ iff $(M, u) \models_{BML} A$ and $(M, u) \models_{BML} B$
- $(M, u) \models_{BML} \neg A$ iff $(M, u) \not\models_{BML} A$
- $(M, u) \models_{BML} \Diamond_i A$ iff there is some v with $uR_i v$ and $(M, v) \models_{BML} A$

The basic modal logic for similarity type Σ will be identified with the pair $(L_{BML}, \models_{BML})$ and denoted by $\mathtt{BML}$.

§ 2. Modal logic (5) - abstract modal logic

Let Σ be a given similarity type. **An abstract modal logic** $\mathfrak{L}$ is a pair $(L_{\mathfrak{L}}, \models_{\mathfrak{L}})$, where $L_{\mathfrak{L}}$ is a function that maps a propositional signature P to the sets $L_{\mathfrak{L}}(P)$ of $\mathfrak{L}$–formulas.
$\models_{\mathfrak{L}}$ is a relation between structures and formulas of $\mathfrak{L}$ such that:

- **"The logic $\mathfrak{L}$ is at least strong as BML"**, i.e. $L_{BML}(P) \subseteq L_{\mathfrak{L}}(P)$ for each propositional signature P. Furthermore, for any pointed model (M, u) and $\varphi \in L_{BML}(P)$, we have $(M, u) \models_{BML} \varphi$ iff $(M, u) \models_{\mathfrak{L}} \varphi$.

§ 2. Modal logic (6) - abstract modal logic

- $L_{\mathfrak{L}}(P)$ is closed under Boolean and modal operators.
- if P and Q are propositional signatures with $P \subseteq Q$, then $L_{\mathfrak{L}}(P) \subseteq L_{\mathfrak{L}}(Q)$.
- if $A \in L_{\mathfrak{L}}(P)$ and (M, u) is a pointed model for the propositional signature Q, then $(M, u) \models_{\mathfrak{L}} \varphi$ iff $(M \upharpoonright P, u) \models_{\mathfrak{L}} \varphi$.
  Here, $M \upharpoonright P$ denotes the reduct of the model M to the propositional signature $P \subseteq Q$.
  If $M = (W, \{R_i\}_{i \in I}, V)$, then $M \upharpoonright P$ is the unique model for signature P of the form $M \upharpoonright P = (W, \{R_i\}_{i \in I}, V^\star)$, where $V^\star(p, w) = V(p, w)$ for each $p \in P$ and $w \in W$.

§ 2. Modal logic (7) – isomorphism

We would like to notice that there is no assumption to the effect that satisfaction of formulas is **invariant under isomorphism** in this definition.

This is simply because we will not need it in practice.

Since all the characterizations we shall consider will involve a condition of invariance under bisimulation (to be defined later), and since such a condition obviously entails invariance under isomorphism, there is no need to mention it explicitly in the definition of an abstract modal logic.

§ 2. Modal logic (8) – bisimulation

Let (M, R) and (N, R') be Kripke models. **Bisimulation** is a relation $Z \subseteq M \times N$ such that:

(at) if wZw' then $w \models p$ iff $w' \models p$, for any $p \in P$.

(zig) if wZw' and wRu then there is a node $u' \in N$ such that uZu' and $w'R'u'$.

(zag) if wZw' and $w'R'u'$ then there is a node $u \in M$ such that uZu' and wRu.

(We consider here the similarity type $\Sigma = \{\Diamond\}$, where $\Diamond$ is unary modal operator.)

If there is a bisimulation Z such that wZw' then we write $w \underline{\leftrightarrow} w'$.

§ 2. Modal logic (9) – bisimulation invariance

A modal logic $\mathfrak{L}$ is **bisimulation invariant** if $w \underline{\leftrightarrow} w'$ implies $w \equiv w'$, i.e. $w \models_{\mathfrak{L}} \varphi$ iff $w' \models_{\mathfrak{L}} \varphi$, for any formula φ.

It is easy to see that the logic $\mathtt{BML}$ is bisimulation invariant.

§ 2. Modal logic (10)

There are several known Lindström–style characterization results for basic modal logic.

J. van Benthem (2006) showed in that no logic that is compact, bisimulation invariant and has **the Relativization Property** can properly extend basic modal logic.

This characterization itself may be seen as a methodological improvement on the characterization by **M. de Rijke** (1995) (see also P. Blackburn, M. de Rijke, Y. Venema, Modal Logic, 2001), which explicitly stipulated **the Finite Depth Property** as a crucial criterion.

§ 2. Modal logic (11)

M. Otto and **R. Piro** (2008) established a Lindström type characterization of **the extension of basic modal logic** by a global modality as maximal among compact logics with the corresponding bisimulation invariance and **the Tarski Union Property**.

S. Enqvist (2013) proved a generalization of Lindström's theorem that covers any normal modal logic corresponding to a class of Kripke frames **definable by a set of formulas called strict universal Horn formulas**.

He also proved a negative result showing that the result cannot be strengthened to cover every first–order elementary class of frames.

§ 2. Modal logic (12) – finite depth property

We say that a logic $\mathfrak{L}$ satisfies **the Finite Depth Property** if for any formula φ, there is a natural number k such that, for all pointed models (M, u) we have:

$$(M, u) \models_{\mathfrak{L}} \varphi \qquad \text{iff} \qquad (M|k, u) \models_{\mathfrak{L}} \varphi,$$

where $M|k$ is the model M with its domain restricted to just those points that can be reached from u in k or fewer successive R-steps.

§ 2. Modal logic (13) – relativization

We say that a logic $\mathfrak{L}$ satisfies **the Relativization Property** if for any $\mathfrak{L}$–formula φ and new proposition letter p, there exists an $\mathfrak{L}$–formula $Rel(\varphi, p)$ such that for every pointed model (M, u) we have

$$(M, u) \models_{\mathfrak{L}} Rel(\varphi, p) \qquad \text{iff} \qquad (M|p, u) \models_{\mathfrak{L}} \varphi,$$

where $M|p$ is the submodel of M with just the points in M satisfying p for its domain.

§ 2. Modal logic (14) – union of models

We now define the Tarski Union Property. Firstly, we define union of a countable set of Veltman models.

Let $\{M_n\}_{n\in\omega}$ be a countable set of models of similarity type Σ, where $M_n = \langle W_n, \{R_i^n\}_{\Diamond_i\in\Sigma}, V_n\rangle$. Then **the union**

$$\bigcup_{n\in\omega} M_n = \langle W^*, \{R_i^*\}_{\Diamond_i\in\Sigma}, V^*\rangle$$

is defined by setting:

- $W^* = \bigcup_{n\in\omega} W_n$
- $R_i^* = \bigcup_{n\in\omega} R_i^n$ for each $\Diamond_i \in \Sigma$ and
- $V^*(p,w) = 1$ iff $V_n(p,w) = 1$ for all n with $w \in M_n$, for each propositional variable p.

§ 2. Modal logic (15) – Tarski Union Property

Let M and N be models of similarity type Σ, and let $\mathfrak{L}$ be an abstract modal logic.

We say that model M is an **$\mathfrak{L}$–elementary submodel of model N**, written $M \preceq_{\mathfrak{L}} N$, if M is a submodel of N and, for every $u \in M$ and every $\mathfrak{L}$–formula φ, we have $(M, u) \models_{\mathfrak{L}} \varphi$ iff $(N, u) \models_{\mathfrak{L}} \varphi$.

An $\mathfrak{L}$–elementary chain is a sequence of models $\{M_n\}_{n\in\omega}$ such that $M_n \preceq_{\mathfrak{L}} M_{n+1}$ for all $n \in \omega$.

An abstract modal logic $\mathfrak{L}$ is said to have **the Tarski Union Property (TUP)** if, for every $\mathfrak{L}$–elementary chain $\{M_n\}_{n\in\omega}$, we have

$$M_k \preceq_{\mathfrak{L}} \bigcup_{n\in\omega} M_n, \qquad \text{for each } \ k \in \omega.$$

§ 2. Modal logic (16) – equivalent logics

Given an abstract modal logic $\mathfrak{L}$ for similarity type Σ, and given a propositional signature P, we say that a class X of pointed models of signature P is **$\mathfrak{L}$–definable** if X is of the form $\{(M, u) \ : \ (M, u) \models_{\mathfrak{L}} \varphi\}$ for some $\varphi \in L_{\mathfrak{L}}(P)$.

Abstract modal logics $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are said **to be equivalent** if for any propositional signature P, the $\mathfrak{L}_1$–definable classes and the $\mathfrak{L}_2$–definable classes are the same.

§ 2. Modal logic (17) – Lindström's theorem

Lindström's theorem for basic modal logic

An abstract modal logic $\mathfrak{L}$ is equivalent to $\mathtt{BML}$ if and only if logic $\mathfrak{L}$ satisfies the following:

(1) bisimulation invariance

(2) compactness

(3) Finite Depth Property (M. de Rijke, 1995)

(3) Relativization Property (J. van Benthem, 2006)

(3) Tarski Union Property (M. Otto, R. Piro, 2008)

§ 3. Lindström's theorem for interpretability logic (1)

Interpretability logic, i.e. IL, is a nonstandard modal logic.

IL is an extension of provability logic GL (Gödel–Löb).

We study purely modal properties of IL. So, we do not consider arithmetical interpretation of IL.

§ 3. IL (2) – relative interpretability

Let T be a first–order theory sufficiently strong for coding its syntax.

Relative interpretability is a binary relation between extensions of T which are like $T + A$, where A is a formula.

We say "$T + A$ is interpretable in $T + B$" if we can "translate" formulas theory $T + A$ such that all theorems of $T + A$ will be theorems theory $T + B$.

For many theories, such as PA and its extensions in the same language, the notion of relative interpretability coincides with that of Π_1–conservativity.

§ 3. IL (3) – history

Modal logics for interpretability were first studied by **P. Hájek** (1981) and **V. Švejdar** (1983).

A. Visser (1988) introduced the modal system *IL*.

The interpretability logic *IL* results from the provability logic *GL* (Gödel–Löb) by adding the binary modal operator $\rhd$.

§ 3. IL (4) – language

The language of the interpretability logic contains:

- propositional letters $p_0, \ p_1, \ldots,$
- logical connectives $\wedge, \ \vee, \rightarrow$ and $\neg$,
- unary modal operator $\Box$
  (we use modal operator $\Diamond$ for abbreviation $\neg\Box\neg$)
- binary modal operator $\rhd$.

§ 3. IL (5) – Veltman models

The basic semantics are **Veltman models**.

An ordered quadruple $(W, R, \{S_w : w \in W\}, \Vdash)$ is called **Veltman model**, if it satisfies the following conditions:

a) (W, R) is a GL–frame, i.e. W is a non empty set, and the relation R is transitive and reverse well–founded relation on W;

b) For every $w \in W$ is $S_w \subseteq W[w] \times W[w]$, where $W[w] = \{x : wRx\}$;

c) The relation S_w is reflexive and transitive, for every $w \in W$;

d) If $wRvRu$ then $vS_w u$;

e) $\Vdash$ is a forcing relation. We emphasize only the definition: $w \Vdash A \rhd B$ if and only if

$$\forall v((wRv \ \& \ v \Vdash A) \Rightarrow \exists u(vS_w u \ \& \ u \Vdash B)).$$

To keep the notation simple we identify a Veltman model with its universe W.

§ 3. IL (6) – axioms

Here are the axioms of the system *IL* (**interpretability logic**).

(L0)–(L3) *axioms of the system GL*

(J1) $\Box(A \to B) \to (A \rhd B)$

(J2) $((A \rhd B) \wedge (B \rhd C)) \to (A \rhd C)$

(J3) $((A \rhd C) \wedge (B \rhd C)) \to ((A \vee B) \rhd C)$

(J4) $(A \rhd B) \to (\Diamond A \to \Diamond B)$

(J5) $\Diamond A \rhd A$

The deduction rules of *IL* are modus ponens and necessitation.

§ 3. IL (7) – completeness

D. de Jongh and F. Veltman proved in 1988 the **modal completeness** of system *IL* with respect Veltman semantics.

Theorem. For each formula F of interpretability logic we have:

$$\vdash_{IL} F \quad \text{if and only if} \quad (\forall W) W \models F$$

§ 3. IL (8) – reference

We introduce our notation and some basic facts, following the article:

A. Visser, *An overview of interpretability logic*, In: Kracht, Marcus (ed.) et al., Advances in modal logic. Vol. 1. Selected papers from the 1st international workshop (AiML'96), Berlin, Germany, October 1996, Stanford, CA: CSLI Publications, CSLI Lect. Notes. 87, pp. 307–359 (1998)

http://www.phil.uu.nl/preprints/lgps/authors/visser/

§ 3. IL (9) – bisimulation

A. Visser (1988) defined bisimulation of Veltman models and proved that every Veltman model with some special property can be bisimulated by a finite Friedman model.

A. Berarducci (1990) used a bisimulation for the proof of completeness of system *IL* w.r.t. simplified Veltman models.

By using a bisimulation **A. Visser** proved that Craig interpolation lemma is not valid for systems between ILM_0 and ILM.

§ 3. IL (10) – bisimulation

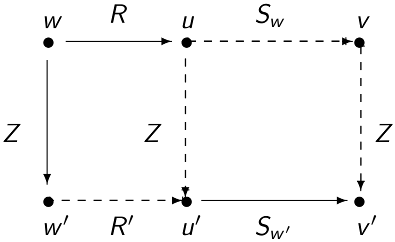
A bisimulation between two Veltman models W and W' is a nonempty binary relation $Z \subseteq W \times W'$ such that the following conditions hold:

(at) If wZw' then $W, w \Vdash p$ if and only if $W', w' \Vdash p$, for all propositional variables p;

(forth) If wZw' and wRu, then there exists $u' \in W'$ with $w'R'u'$, uZu' and for all $v' \in W'$ if $u'S_{w'}v'$ there is $v \in W$ such that uS_wv;

(back) If wZw' and $w'Ru'$, then there exists $u \in W$ with wRu, uZu' and for all $v \in W$ if uS_wv there is $v' \in W'$ such that $u'S_{w'}v'$.

§ 3. IL (11) – (forth)



$\mathtt{IL}$ (12) – invariance for bisimulation

It is easy to prove that logic $\mathtt{IL}$ is **bisimulation invariant** (see D. Vrgoč, M. Vuković, Logic Jou. IGPL, 2011), i.e.

$$\mathfrak{M}, w \underline{\leftrightarrow} \mathfrak{N}, w' \qquad \text{implies} \qquad \mathfrak{M}, w \equiv \mathfrak{N}, w',$$

for all Veltman models $\mathfrak{M}, \mathfrak{N}$, and all nodes $w \in \mathfrak{M}$ and $w' \in \mathfrak{N}$.

IL (13) – non–compactness

The interpretability logic is **not compact** w.r.t. (generalized) Veltman models.

So, there exists a set of formulas Γ such that each finite subset of Γ is satisfiable, but the set Γ is not satisfiable.

We prove (Math. Communications, 2004) non–compactness of $\mathtt{IL}$ by modifying Fine's and Rautenberg's proof of non–compactness of provability logic.

IL (14) – TUP

There are a two problems when we consider the Tarski Union Property for IL.

Let $\{W_n\}_{n\in\omega}$ be an elementary chain of Veltman models.

- Does $\cup_{n\in\omega} W_n$ have to be a Veltman model?
  Is the accessability relation R of the union necessary reverse well–founded?
  Can we give a counterexample by using games (Čacić, Vrgoč, 2013)?
- Let us assume that $\cup_{n\in\omega} W_n$ is Veltman model. Is then valid $W_k \preceq \cup_{n\in\omega} W_n$, for each $k \in \omega$?

IL (14) – TUP

There are a two problems when we consider the Tarski Union Property for IL.

Let $\{W_n\}_{n\in\omega}$ be an elementary chain of Veltman models.

- Does $\cup_{n\in\omega} W_n$ have to be a Veltman model?
  Is the accessability relation R of the union necessary reverse well–founded?
  Can we give a counterexample by using games (Čacić, Vrgoč, 2013)?

- Let us assume that $\cup_{n\in\omega} W_n$ is Veltman model. Is then valid $W_k \preceq \cup_{n\in\omega} W_n$, for each $k \in \omega$?

IL (15) – TUP

The cause of these difficulties lies, of course, in the fact that reverse well–foundedness is not a first–order definable property.

For this reason Dawar and Otto (APAL, 2009) used bisimulation games to prove their characterization theorem over GL.

T. Perkov and M. Vuković (Log. Jou. IGPL, 2014) modify their techniques in order to prove a generalization of their result to IL.

IL (16) – games

Vedran Čačić and **Domagoj Vrgoč** (Studia Logica, 2013) defined a bisimulation game between Veltman models.

A **bisimulation game** between Veltman models $(W_1, R_1, \{S_w^{(1)} : w \in W_1\}, \Vdash_1)$ and $(W_2, R_2, \{S_w^{(2)} : w \in W_2\}, \Vdash_2)$ is played by two players, *Challenger* and *Defender*, who play each round starting from the current configuration, which is a pair (w_1, w_2), where $w_1 \in W_1$ and $w_2 \in W_2$, as follows:

IL (17) – games

1. Challenger chooses one of the models W_i. Let W_j denote the other model.
2. Challenger picks a world $u_i \in W_i$ such that $w_i R_i u_i$. If there is no such world, Defender wins.
3. Defender picks a world $u_j \in W_j$ such that $w_j R_j u_j$. If there is no such world, Challenger wins.
4. Challenger picks $v_j \in W_j$ such that $u_j S^{(j)}_{w_j} v_j$.
5. Defender chooses $v_i \in W_i$ such that $u_i S^{(i)}_{w_i} v_i$.

IL (18) – games

The next round is played from the configuration (u_1, u_2) or (v_1, v_2), as chosen by Challenger.

At the beginning of a game, it is checked if w_1 and w_2 are propositionally equivalent (i.e. satisfy the same propositional variables). Also, after each round it is checked if u_1 and u_2 are propositionally equivalent, and if v_1 and v_2 are propositionally equivalent. If any of these checks fail, Challenger wins.

IL (19) – bisimulation games

Čačić and Vrgoč show that nodes w_1 and w_2 of a Veltman model are bisimilar if and only if Defender has a winning strategy in the bisimulation game with the starting configuration (w_1, w_2).

They also define the notion of ***n*–bisimulation** between Veltman models as a finite approximation of bisimulation game.

Parkov and Vuković proved (Log. Jou. IGPL, 2014) the following:

Assume that the set of propositional variables is finite and let $(W, R, \{S_w : w \in W\}, \Vdash)$ and $(W', R', \{S'_{w'} : w' \in W'\}, \Vdash')$ be Veltman models. Then for any $w \in W$ and $w' \in W'$ the following are equivalent:

1. w and w' are n–bisimilar,
2. w and w' are are modally n–equivalent,
3. $w' \Vdash' \chi^n_w$, where χ^n_w is a characteristic formula.

§ 3. IL (20)

Questions 1

What is an abstract non–standard modal logic?

Some examples:

logic of Π^0_1–conservativity

logic of Σ^0_n–interpolability (Ignatiev)

logic of arithmetical hierarchy (Japaridze)

polymodal provability logic GLP

§ 3. IL (21)

Questions 2

Is there a connection between "Lindström's theorem for IL" and the modal invariance theorem for IL?

J. van Benthem proved (2006) that **Lindström's theorem for modal logic implies modal invariance theorem**, i.e. up to logical equivalence, the basic modal formulas are precisely those first-order formulas which are invariant for bisimulation.

T. Perkov and M. Vuković proved (Log. Jou. IGPL, to appear) the following **modal invariance theorem for IL**:
A first–order formula is equivalent to standard translation of some formula of interpretability logic with respect to Veltman models if and only if it is invariant under bisimulations between Veltman models.

§ 3. IL (21)

Questions 2

Is there a connection between "Lindström's theorem for IL" and the modal invariance theorem for IL?

J. van Benthem proved (2006) that **Lindström's theorem for modal logic implies modal invariance theorem**, i.e. up to logical equivalence, the basic modal formulas are precisely those first-order formulas which are invariant for bisimulation.

T. Perkov and M. Vuković proved (Log. Jou. IGPL, to appear) the following **modal invariance theorem for IL**:
A first–order formula is equivalent to standard translation of some formula of interpretability logic with respect to Veltman models if and only if it is invariant under bisimulations between Veltman models.