Computational interpretations of the classical axiom of choice

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Background

Arithmetic

Heyting Arithmetic (HA)

Gödel (1941/1958) *Dialectica* interpretation using System T (higher-type primitive recursion)

Kleene (1945) Relizability using general recursion

Kreisel (1962) Modified realizability via System T

Arithmetic

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Peano Arithmetic (PA)

- Works for formulas implied by their own double negation translations
- Thanks to the fact that the induction axiom is one such formula

Analysis

What happens when the Axiom of Choice

$$\forall x \exists y A(x, y) \to \exists f \forall x A(x, f(x)), \tag{AC}$$

is added to Arithmetic?

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Computational interpretations still apply to HA+AC.

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But double-negation translation of AC is not provable from AC+HA, so interpretations not directly applicable to classical Analysis.

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But double-negation translation of AC is not provable from AC+HA, so interpretations not directly applicable to classical Analysis.

Digression

There are forms of AC that are **resistant** to double-negation translations: Raoult's Open Induction Principle:

$$\forall \alpha \, (\forall \beta < \alpha U(\beta) \to U(\alpha)) \to \forall \alpha U(\alpha),$$

where $\alpha \in \mathbb{N} \to \{0, 1\}$ or $\alpha \in \mathbb{N} \to \mathbb{N}$ and U is open (i.e. Σ^0_1).

Kuroda's Principle (1951)

If we add

$$\neg\neg\forall x(A(x) \vee \neg A(x)) \tag{KC}$$

to HA+AC, then the D-N translation of AC becomes provable!

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to HA+AC, then the D-N translation of AC becomes provable!

This was known to Gödel.
Kreisel gives credit in §2.43 of Spector's (1962) paper.

Double Negation Shift – intuitionistic equivalent of KC

$$\forall x\neg\neg B(x) \rightarrow \neg\neg\forall x B(x). \qquad \text{(DNS)}$$

Double Negation Shift

Double Negation Shift

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Can we interpret it computationally?

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Can we interpret it computationally?

Formal/False Church's Thesis

Already Gödel (1941) considers the special case of KC for

$$A(x) := \exists y\, \mathrm{T}(x, x, y).$$

That directly refutes:

$$\forall x^{\mathbb{N}} \exists y^{\mathbb{N}} A(x, y) \rightarrow \exists e^{\mathbb{N}} \forall x^{\mathbb{N}} \exists u^{\mathbb{N}} (T(e, x, u) \wedge A(x, U(u))). \qquad (\mathrm{CT}_0)$$

Ex. A form of CT_0 is used to prove soundness of Kleene's realizability.

Double Negation Shift

Bar Recursion

Kreisel and Spector gave a computational interpretation of DNS by extending the *primitive* recursive System T with a *general* recursive schema:

$$\mathrm{BR}(G, Y, H, s) =$$
$$= \begin{cases} G(s) & \text{if } Y(\lambda k.\, \texttt{if } k < |s| \texttt{ then } s_k \texttt{ else } 0) < |s| \\ H(s, \lambda x.\, \mathrm{BR}(G, Y, H, s * x)) & \text{otherwise} \end{cases}$$

- Soundness of BR is proven by an additional axiom like Bar Induction
- Improved in works of Coquand, Kohlenbach, Berger, Oliva, ...
- One of the rare applications of Proof Theory to Mathematics other than Logic itself – Kohlenbach's Proof Mining

Analysis

Interpretations based on computational side-effects

Griffin 1990 "A formulae-as-types notion of control"

Krivine 2003 "Dependent choice, 'quote' and the clock"

Herbelin 2011 "A constructive proof of the axiom of dependent choice, compatible with classical logic"

Questions

- Can one simplify the approach of side-effect and abstract machines?
  - Ex. Do `call/cc` and `quote` go beyond primitive recursion?
- Is full classical logic necessary to prove soundness?
  - Ex. DNS does not brake the Disjunction Property of intuitionistic predicate calculus

Do we need more than System T?

Schwichtenberg (1979)

System T is closed over bar recursion at types $\mathbb{N}$ and $\mathbb{N} \to \mathbb{N}$.

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Kreisel (§12.2 of Spector (1962))

Those low types are sufficient for interpreting the classical AC for formulas of the form

$$\exists \alpha^{\mathbb{N}\to\mathbb{N}} \forall x^{\mathbb{N}} A_0(\alpha, x),$$

where A_0 is quantifier-free.

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$$\exists \alpha^{\mathbb{N}\to\mathbb{N}} \forall x^{\mathbb{N}} A_0(\alpha, x),$$

where A_0 is quantifier-free.

Can we ask for more than that?

There are classically true formulas that are not recursively realizable, ex.:

$$\forall x \exists y \forall z \exists u \big((u = 0 \to T(x, x, y)) \wedge (u \neq 0 \to T(x, x, z))\big)$$

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Conservative extension of System T with control operators

Goal: System T^+ and its properties

Theorem (Normalization)

*There is a normalization function $\downarrow [\![-]\!]$ s.t. for every term p of **System T**$^+$ of type $\gamma \vdash \tau$, the term $\downarrow [\![p]\!]$ is a normal form of **System T** of the same type ($\gamma \vdash_T \tau$).*

Proposition (Equations)

$$\downarrow [\![\mathit{wkn}\, p]\!]_{\alpha,\rho} = \downarrow [\![p]\!]_\rho \qquad \downarrow [\![\mathit{hyp}]\!]_{\alpha,\rho} = \downarrow \alpha$$

$$\downarrow [\![\mathit{fst}\, \mathit{pair}(p,q)]\!]_\rho = \downarrow [\![p]\!]_\rho \qquad \downarrow [\![\mathit{snd}\, \mathit{pair}(p,q)]\!]_\rho = \downarrow [\![q]\!]_\rho$$

$$\downarrow [\![\mathit{app}(\mathit{lam}\, p, q)]\!]_\rho = \downarrow [\![p]\!]_{[\![q]\!]_\rho,\rho} \qquad \downarrow [\![\mathit{rec}(\mathit{zero}, p, q)]\!]_\rho = \downarrow [\![p]\!]_\rho$$

$$\downarrow [\![\mathit{rec}(\mathit{succ}\, r, p, q)]\!]_\rho = \cdots$$

$$\downarrow^{\mathbb{N}} [\![\mathit{shift}\, p]\!]_\rho = \downarrow^{\mathbb{N}} [\![p]\!]_{\phi,\rho} \qquad \downarrow^{\mathbb{N}} [\![\mathit{app}(\mathit{app}(\mathit{hyp}, x), y)]\!]_{\phi,\rho} = \downarrow^{\mathbb{N}} [\![y]\!]_{\phi,\rho}$$

$$\phi := \eta(\geq_2 \nu \mapsto \eta(\geq_3 \alpha \mapsto \eta(\mu\alpha)))$$

T^+ = T + composable continuations
Danvy-Filinski's `shift` in call-by-name

Types: $\mathcal{T} \ni \sigma, \tau ::= \mathbb{N} \mid \sigma \to \tau \mid \sigma * \tau$

Terms:

$$\text{hyp}\ \frac{}{(\sigma;\gamma) \vdash \sigma} \qquad \text{wkn}\ \frac{\gamma \vdash \sigma}{(\tau;\gamma) \vdash \sigma} \qquad \text{lam}\ \frac{(\sigma;\gamma) \vdash \tau}{\gamma \vdash \sigma \to \tau}$$

$$\text{app}\ \frac{\gamma \vdash \sigma \to \tau \quad \gamma \vdash \sigma}{\gamma \vdash \tau} \qquad \text{pair}\ \frac{\gamma \vdash \sigma \quad \gamma \vdash \tau}{\gamma \vdash \sigma * \tau} \qquad \text{fst}\ \frac{\gamma \vdash \sigma * \tau}{\gamma \vdash \sigma}$$

$$\text{snd}\ \frac{\gamma \vdash \sigma * \tau}{\gamma \vdash \tau} \qquad \text{zero}\ \frac{}{\gamma \vdash \mathbb{N}} \qquad \text{succ}\ \frac{\gamma \vdash \mathbb{N}}{\gamma \vdash \mathbb{N}}$$

$$\text{rec}\ \frac{\gamma \vdash \mathbb{N} \quad \gamma \vdash \sigma \quad \gamma \vdash \mathbb{N} \to \sigma \to \sigma}{\gamma \vdash \sigma} \qquad \text{shift}\ \frac{(\mathbb{N} \to \sigma \to \mathbb{N};\gamma) \vdash \mathbb{N}}{\gamma \vdash \sigma}$$

$$\mathrm{A} := \lambda m.\, \mathsf{R}\, m(\lambda n.n+1)(\lambda m'.\lambda u.\lambda n.\, \mathsf{R}\, n(u1)(\lambda n'.\lambda w.uw)),$$

is represented by

```
lam
  (rec hyp(lam(succ hyp))
    (lam
      (lam
        (lam
          (rec hyp(app(wkn hyp)(succ zero))
            (lam(lam(app(wkn(wkn(wkn hyp))) hyp))))))))
```

i.e. a 1st-order representation with de Bruijn indices 0 := hyp, 1 := wkn hyp, ...

The Agda formalization really computes ex. A(3,2) to be $\underbrace{\text{succ} \cdots \text{succ}}_{\text{29 times}}$ zero.

(If one has enough RAM available)

System T (*not* T$^+$)

Normal terms ($\vdash_r$),

$$\text{e}\ \frac{\gamma \vdash_e \sigma}{\gamma \vdash_r \sigma} \qquad \text{lam}\ \frac{(\sigma;\gamma) \vdash_r \tau}{\gamma \vdash_r \sigma \to \tau} \qquad \text{pair}\ \frac{\gamma \vdash_r \sigma \quad \gamma \vdash_r \tau}{\gamma \vdash_r \sigma * \tau}$$

$$\text{zero}\ \frac{}{\gamma \vdash_r \mathbb{N}} \qquad \text{succ}\ \frac{\gamma \vdash_r \mathbb{N}}{\gamma \vdash_r \mathbb{N}}$$

and **neutral** terms ($\vdash_e$),

$$\text{hyp}\ \frac{}{(\sigma;\gamma) \vdash_e \sigma} \qquad \text{wkn}\ \frac{\gamma \vdash_r \sigma}{(\tau;\gamma) \vdash_e \sigma} \qquad \text{app}\ \frac{\gamma \vdash_e \sigma \to \tau \quad \gamma \vdash_r \sigma}{\gamma \vdash_e \tau}$$

$$\text{fst}\ \frac{\gamma \vdash_e \sigma * \tau}{\gamma \vdash_e \sigma} \qquad \text{snd}\ \frac{\gamma \vdash_e \sigma * \tau}{\gamma \vdash_e \tau} \qquad \text{rec}\ \frac{\gamma \vdash_e \mathbb{N} \quad \gamma \vdash_r \sigma \quad \gamma \vdash_r \mathbb{N} \to \sigma \to \sigma}{\gamma \vdash_e \sigma}.$$

Normalization-by-evaluation

Normalization is proven using a constructive normalization-by-evaluation proof in continuation-passing style (CPS). System T^+ is evaluated into the following continuation monad:

$$\gamma \Vdash \sigma = \forall \gamma_1 \geq \gamma \, (\forall \gamma_2 \geq \gamma_1 \, (\gamma_2 \Vdash_{\mathsf{s}} \sigma \Rightarrow \gamma_2 \vDash_{\mathsf{r}} \mathbb{N}) \Rightarrow \gamma_1 \vDash_{\mathsf{r}} \mathbb{N})$$

$$\begin{aligned} \gamma \Vdash_{\mathsf{s}} \mathbb{N} &= \gamma \vDash_{\mathsf{r}} \mathbb{N} \\ \gamma \Vdash_{\mathsf{s}} (\sigma \to \tau) &= \forall \gamma' \geq \gamma (\gamma' \Vdash \sigma \Rightarrow \gamma' \Vdash \tau) \\ \gamma \Vdash_{\mathsf{s}} (\sigma * \tau) &= \gamma \Vdash \sigma \times \gamma \Vdash \tau \end{aligned}$$

where

$$\geq_{\text{refl}} \frac{}{\gamma \geq \gamma} \qquad \geq_{\text{cons}} \frac{\gamma_2 \geq \gamma_1}{(\sigma; \gamma_2) \geq \gamma_1}$$

Normalization-by-evaluation

The ‘return’ and ‘run’ operations:

$$\eta(-) : \gamma \Vdash_{\mathsf{s}} \sigma \Rightarrow \gamma \Vdash \sigma \qquad \mu(-) : \gamma \Vdash \mathbb{N} \Rightarrow \gamma \Vdash_{\mathsf{s}} \mathbb{N}$$

$$\eta H = {\geq_1}\ \kappa \mapsto \kappa \geq_{\mathsf{refl}} \lfloor H \rfloor_{\geq_1} \qquad \mu H = H \geq_{\mathsf{refl}} ({\geq_1}\ \alpha \mapsto \alpha)$$

Monotonicity properties:

$$\ulcorner - \urcorner^{(-)} : \gamma_2 \geq \gamma_1 \Rightarrow \gamma_1 \vDash_{\mathsf{r}} \sigma \Rightarrow \gamma_2 \vDash_{\mathsf{r}} \sigma$$

$$\llcorner - \lrcorner_{(-)} : \gamma_2 \geq \gamma_1 \Rightarrow \gamma_1 \vDash_{\mathsf{e}} \sigma \Rightarrow \gamma_2 \vDash_{\mathsf{e}} \sigma$$

$$\lceil - \rceil^{(-)} : \gamma_2 \geq \gamma_1 \Rightarrow \gamma_1 \Vdash \sigma \Rightarrow \gamma_2 \Vdash \sigma$$

$$\lfloor - \rfloor_{(-)} : \gamma_2 \geq \gamma_1 \Rightarrow \gamma_1 \Vdash_{\mathsf{s}} \sigma \Rightarrow \gamma_2 \Vdash_{\mathsf{s}} \sigma$$

$$[\![-]\!]^{(-)} : \gamma_2 \geq \gamma_1 \Rightarrow \gamma_1 \Vvdash \gamma \Rightarrow \gamma_2 \Vvdash \gamma$$

Normalization-by-evaluation

$$
\begin{aligned}
{}^{\gamma}{\downarrow}^{\sigma}\ (-) &: \gamma \Vdash \sigma \Rightarrow \gamma \vdash_{\mathsf{r}} \sigma \\
{}^{\gamma}{\downarrow}^{\mathbb{N}}\ H &= \mu H \\
{}^{\gamma}{\downarrow}^{\sigma\rightarrow\tau}\ H &= \mathsf{lam}({}^{\gamma}{\downarrow}^{\tau} \\
&\qquad (\geq_1\ \kappa \mapsto \\
&\qquad\quad H(\geq_1 \cdot \geq_{\mathsf{cons}} \geq_{\mathsf{refl}}) \\
&\qquad\quad (\geq_2\ \phi \mapsto \\
&\qquad\qquad \phi \geq_{\mathsf{refl}} (\lceil {}^{\sigma;\gamma}{\uparrow}^{\sigma}\ \mathit{hyp} \rceil^{\geq_2 \cdot \geq_1}) \geq_{\mathsf{refl}} \\
&\qquad\qquad (\geq_3 \mapsto \kappa(\geq_3 \cdot \geq_2))))) \\
{}^{\gamma}{\downarrow}^{\sigma*\tau}\ H &= \mathsf{pair} \\
&\qquad {}^{\alpha}{\downarrow}^{\sigma}\ (\geq_1\ \kappa \mapsto \\
&\qquad\qquad H \geq_1 (\geq_2\ \alpha \mapsto \mathsf{proj}_1\ \alpha \geq_{\mathsf{refl}} (\geq_3 \mapsto \kappa(\geq_3 \cdot \geq_2)))) \\
&\qquad {}^{\alpha}{\downarrow}^{\tau}\ (\geq_1\ \kappa \mapsto \\
&\qquad\qquad H \geq_1 (\geq_2\ \alpha \mapsto \mathsf{proj}_2\ \alpha \geq_{\mathsf{refl}} (\geq_3 \mapsto \kappa(\geq_3 \cdot \geq_2))))
\end{aligned}
$$

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&\qquad (\geq_1\ \kappa \mapsto \\
&\qquad\qquad H(\geq_1 \cdot \geq_{\mathsf{cons}} \geq_{\mathsf{refl}}) \\
&\qquad\qquad (\geq_2\ \phi \mapsto \\
&\qquad\qquad\qquad \phi \geq_{\mathsf{refl}} (\ulcorner^{\sigma;\gamma}{\uparrow}^{\sigma}\ \mathit{hyp}\urcorner^{\geq_2\cdot\geq_1}) \geq_{\mathsf{refl}} \\
&\qquad\qquad\qquad (\geq_3 \mapsto \kappa(\geq_3 \cdot \geq_2))))) \\
&{}^{\gamma}{\downarrow}^{\sigma*\tau}\ H = \mathsf{pair} \\
&\qquad {}^{\alpha}{\downarrow}^{\sigma}\ (\geq_1\ \kappa \mapsto \\
&\qquad\qquad\qquad H \geq_1 (\geq_2\ \alpha \mapsto \mathsf{proj}_1\ \alpha \geq_{\mathsf{refl}} (\geq_3 \mapsto \kappa(\geq_3 \cdot \geq_2)))) \\
&\qquad {}^{\alpha}{\downarrow}^{\tau}\ (\geq_1\ \kappa \mapsto \\
&\qquad\qquad\qquad H \geq_1 (\geq_2\ \alpha \mapsto \mathsf{proj}_2\ \alpha \geq_{\mathsf{refl}} (\geq_3 \mapsto \kappa(\geq_3 \cdot \geq_2))))
\end{aligned}
$$

This function shows that we can actually run the monad at *any* type

Normalization-by-evaluation

$$
\begin{aligned}
{}^{\gamma}\!\uparrow^{\sigma} (-) &: \gamma \vdash_{\mathsf{e}} \sigma \Rightarrow \gamma \Vdash \sigma \\
{}^{\gamma}\!\uparrow^{\mathbb{N}} p &= \eta(\mathsf{e}\, p) \\
{}^{\gamma}\!\uparrow^{\sigma \to \tau} p &= \eta(\geq_2 \alpha \mapsto {}^{\gamma}\!\uparrow^{\tau} \mathsf{app}(\llcorner p \lrcorner_{\geq_2}, {}^{\gamma}\!\downarrow^{\sigma} \alpha)) \\
{}^{\gamma}\!\uparrow^{\sigma * \tau} p &= \eta({}^{\gamma}\!\uparrow^{\sigma} \mathsf{fst}\, p, {}^{\gamma}\!\uparrow^{\tau} \mathsf{snd}\, p)
\end{aligned}
$$

This function is needed *only* by the $\sigma \to \tau$-case of reify. Morally, it only performs η-expansion.

Normalization-by-evaluation

The *evaluation* function

$$
\begin{aligned}
{}^{\gamma}[\![-]\!]^{\sigma}_{(-)} &: \gamma \vdash \sigma \Rightarrow \forall \gamma' \Vdash \gamma (\gamma' \Vdash \sigma) \\
[\![\mathrm{hyp}]\!]_\rho &= \mathrm{proj}_1\ \rho \\
[\![\mathrm{wkn}\ p]\!]_\rho &= [\![p]\!]_{\mathrm{proj}_2\ \rho} \\
[\![\mathrm{lam}\ p]\!]_\rho &= \eta(\geq_1\ \alpha \mapsto [\![p]\!]_{(\alpha, [\![\rho]\!]^{\geq_1})}) \\
&\vdots \\
[\![\mathrm{pair}(p,q)]\!]_\rho &= \eta([\![p]\!]_\rho, [\![q]\!]_\rho) \\
[\![\mathrm{fst}\ p]\!]_\rho &= \geq_1\ \kappa \mapsto [\![p]\!]_\rho \geq_1 (\geq_2\ \alpha \mapsto \mathrm{proj}_1\ \alpha \geq_{\mathrm{refl}} (\geq_3 \mapsto \kappa(\geq_3 \cdot \geq_2))) \\
&\vdots \\
[\![\mathrm{shift}\ p]\!]_\rho &= \geq_1\ \kappa \mapsto \mu [\![p]\!]_{\eta(\geq_2 \nu \mapsto \eta(\geq_3 \alpha \mapsto \eta(\alpha \geq_{\mathrm{refl}} (\geq_4 \mapsto \kappa(\geq_4 \cdot \geq_3 \cdot \geq_2))))), [\![\rho]\!]^{\geq_1}} \\
&\vdots
\end{aligned}
$$

Normalization-by-evaluation

Theorem (Normalization)

*There is a normalization function $\downarrow [\![-]\!]$ s.t. for every term p of **System T$^+$** of type $\gamma \vdash \tau$, the term $\downarrow [\![p]\!]$ is a normal form of **System T** of the same type ($\gamma \vDash_{\mathsf{r}} \tau$).*

Proof.

Compose the defined functions:

$$
\begin{aligned}
{}^{\gamma}[\![-]\!]^{\sigma}_{(-)} &: \gamma \vdash \sigma \Rightarrow \forall \gamma' \Vvdash \gamma (\gamma' \Vdash \sigma) \\
{}^{\gamma}\uparrow^{\sigma} (-) &: \gamma \vDash_{\mathsf{e}} \sigma \Rightarrow \gamma \Vdash \sigma \\
{}^{\gamma}\downarrow^{\sigma} (-) &: \gamma \Vdash \sigma \Rightarrow \gamma \vDash_{\mathsf{r}} \sigma
\end{aligned}
$$

□

Equations holding of the normalization function

Proposition

The following definitional equalities hold,

$$\downarrow [\![\mathit{wkn}\, p]\!]_{\alpha,\rho} = \downarrow [\![p]\!]_{\rho} \tag{1}$$

$$\downarrow [\![\mathit{hyp}]\!]_{\alpha,\rho} = \downarrow \alpha \tag{2}$$

$$\downarrow [\![\mathit{fst}\, \mathit{pair}(p, q)]\!]_{\rho} = \downarrow [\![p]\!]_{\rho} \tag{3}$$

$$\downarrow [\![\mathit{snd}\, \mathit{pair}(p, q)]\!]_{\rho} = \downarrow [\![q]\!]_{\rho} \tag{4}$$

$$\downarrow [\![\mathit{app}(\mathit{lam}\, p, q)]\!]_{\rho} = \downarrow [\![p]\!]_{[\![q]\!]_{\rho},\rho} \tag{5}$$

$$\downarrow [\![\mathit{rec}(\mathit{zero}, p, q)]\!]_{\rho} = \downarrow [\![p]\!]_{\rho} \tag{6}$$

$$\downarrow [\![\mathit{rec}(\mathit{succ}\, r, p, q)]\!]_{\rho} = \downarrow [\![\mathit{app}(\mathit{app}(q, r), \mathit{rec}(r, p, q))]\!]_{\rho} \tag{7}$$

$$\downarrow^{\mathbb{N}} [\![\mathit{shift}\, p]\!]_{\rho} = \downarrow^{\mathbb{N}} [\![p]\!]_{\phi,\rho} \tag{8}$$

$$\downarrow^{\mathbb{N}} [\![\mathit{app}(\mathit{app}(\mathit{hyp}, x), y)]\!]_{\phi,\rho} = \downarrow^{\mathbb{N}} [\![y]\!]_{\phi,\rho} \tag{9}$$

where for the last two equations,

$$\phi := \eta(\geq_2 \nu \mapsto \eta(\geq_3 \alpha \mapsto \eta(\mu\alpha))).$$

Equations holding of the normalization function

Proof of the proposition.

Equations (1)–(7) follow from the ones that hold already of the $[\![-]\!]_{(-)}$ function. Equations (8)–(9) also follow by definition, this time reification being applied for only one concrete type, $\mathbb{N}$. $\square$

Equations holding of the normalization function

Proof of the proposition.

Equations (1)–(7) follow from the ones that hold already of the $[\![-]\!]_{(-)}$ function. Equations (8)–(9) also follow by definition, this time reification being applied for only one concrete type, $\mathbb{N}$. □

This is easy to say but difficult to prove: one needs to find the right formulation of defining equations for $[\![-]\!]$.

3

A modified realizability interpretation

Optimized modified realizability translation

Berger-Schwichtenberg-Buchholz (2002); Seisenberger (2008)

Computationally irrelevant formulas

$$N ::= P \mid N \wedge N \mid \forall x^{\tau} N \mid A \to N$$

Σ_2-formulas

$$S ::= N \mid \exists x^{\mathbb{N}} N \mid N \to S \mid N \wedge S \mid S \wedge N$$

Forgetful map of formulas to types

$$\begin{aligned} |N \wedge B| &:= |B| \\ |N \to B| &:= |B| \\ |\exists x^{\tau} N| &:= \tau \end{aligned} \qquad \begin{aligned} |A \wedge N| &:= |A| \\ |A \to B| &:= |A| \to |B| \\ |\exists x^{\tau} A| &:= \tau * |A| \end{aligned} \qquad \begin{aligned} |A \wedge B| &:= |A| * |B| \\ |\forall x^{\tau} A| &:= \tau \to |A| \\ |N| &:= \mathbb{N} \end{aligned}$$

Σ_2 are exactly those A for which $|A| = \mathbb{N}$

Goal: Extract programs from proofs of Σ_2-formulas

Definition (Modified realizability interpretation "p **mr** A" of a formula A by a term p of type $|\Gamma| \Vdash_r |A|$ of System T)

$$
\begin{aligned}
p \;\mathbf{mr}\; N &:= N \\
p \;\mathbf{mr}\; N \wedge B &:= N \wedge (p \;\mathbf{mr}\; B) \\
p \;\mathbf{mr}\; A \wedge N &:= (p \;\mathbf{mr}\; A) \wedge N \\
p \;\mathbf{mr}\; A \wedge B &:= (\downarrow [\![\mathrm{fst}\, p]\!]_\rho \;\mathbf{mr}\; A) \wedge (\downarrow [\![\mathrm{snd}\, p]\!]_\rho \;\mathbf{mr}\; B) \\
p \;\mathbf{mr}\; N \to B &:= N \to (p \;\mathbf{mr}\; B) \\
p \;\mathbf{mr}\; A \to B &:= \forall x([\downarrow [\![x]\!]_\rho \;\mathbf{mr}\; A] \to [\downarrow [\![\mathrm{app}(p, x)]\!]_\rho \;\mathbf{mr}\; B]) \\
p \;\mathbf{mr}\; \forall x^\tau A(x) &:= \forall x^\tau (\downarrow [\![\mathrm{app}(p, x)]\!]_\rho \;\mathbf{mr}\; A(x)) \\
p \;\mathbf{mr}\; \exists x^\tau N(x) &:= N(p) \\
p \;\mathbf{mr}\; \exists x^\tau A(x) &:= \downarrow [\![\mathrm{snd}\, p]\!]_\rho \;\mathbf{mr}\; A(\downarrow [\![\mathrm{fst}\, p]\!]_\rho),
\end{aligned}
$$

where $\downarrow [\![-]\!]$ is normalization and we assume an interpretation $\rho : |\Gamma| \Vdash |\Gamma|$.

HA$^{\omega+}$+AC

$$\frac{}{A, \Gamma \vdash A}\ \textsc{Ax} \qquad \frac{\Gamma \vdash A}{B, \Gamma \vdash A}\ \textsc{Wkn}$$

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A \to B}\ \to_{\mathrm{I}} \qquad \frac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B}\ \to_{\mathrm{E}}$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ \wedge^1_{\mathrm{E}} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ \wedge^2_{\mathrm{E}}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge_{\mathrm{I}} \qquad \frac{\Gamma \vdash A(r^\tau)}{\Gamma \vdash \exists x^\tau A(x)}\ \exists_{\mathrm{I}}$$

$$\frac{\Gamma \vdash \exists x^\tau A(x) \qquad \Gamma \vdash \forall x^\tau (A(x) \to B) \qquad x \notin \mathrm{FV}(B)}{\Gamma \vdash B}\ \exists_{\mathrm{E}}$$

$$\frac{\Gamma \vdash A(x^\tau) \qquad x \notin \mathrm{FV}(\Gamma)}{\Gamma \vdash \forall x^\tau A(x)}\ \forall_{\mathrm{I}} \qquad \frac{\Gamma \vdash \forall x^\tau A(x)}{\Gamma \vdash A(r^\tau)}\ \forall_{\mathrm{E}}$$

$$\ldots$$

HA$^{\omega+}$+AC

$$\dots$$

$$\frac{\Gamma \vdash A(\text{zero}) \qquad \Gamma \vdash \forall x^{\mathbb{N}}(A(x) \to A(\text{succ}\, x))}{\Gamma \vdash \forall x^{\mathbb{N}} A(x)} \ \textsc{Ind}$$

$$\frac{\forall x^{\mathbb{N}}(A(x) \to S(x)), \Gamma \vdash S(r)}{\Gamma \vdash A(r)} \ \textsc{Shift} \qquad (A, S \in \Sigma_2)$$

+ the full Axiom of Choice

$$\forall x^{\sigma} \exists y^{\tau} A(x, y) \to \exists^{\sigma \to \tau} f \forall x^{\sigma} A(x, f(x)). \qquad (\mathrm{AC}^{\sigma\tau})$$

Soundness of modified realizability

Theorem (Soundness)

If $HA^{\omega+}$+AC proves $C_1, C_2, \ldots, C_n \vdash A$, and A is computationally relevant, then there exists a term p of System T^+ such that $HA^{\omega+}$ alone proves that, for every $\rho : |C_1|, |C_2|, \ldots, |C_n| \Vdash |C_1|, |C_2|, \ldots, |C_n|$,

$$\downarrow [\![hyp]\!]_\rho \ \mathbf{mr}\ C_1, \downarrow [\![wkn\, hyp]\!]_\rho \ \mathbf{mr}\ C_2, \ldots, \downarrow [\![wkn^n\ hyp]\!]_\rho \ \mathbf{mr}\ C_n \vdash \downarrow [\![p]\!]_\rho \ \mathbf{mr}\ A.$$

Proof.

Induction on the derivation, with realizing terms as usual. One further analyses the components of A to give optimized realizers. For example, in general, the axiom $\mathrm{AC}^{\sigma\tau}$ is realized by the term

$$\mathrm{lam\, pair(lam\, app(fst\, wkn\, hyp, hyp), lam\, app(snd\, wkn\, hyp, hyp))},$$

but when $A(x, y)$ is computationally irrelevant the realizer is the term

$$\mathrm{lam\, hyp}.$$

□

...

Soundness of modified realizability

$$\frac{\forall x^{\mathbb{N}}(A(x) \to S(x)), \Gamma \vdash S(r)}{\Gamma \vdash A(r)} \textsc{Shift} \qquad (A, S \in \Sigma_2)$$

Proof for the SHIFT case.

The goal is to prove

$$\downarrow [\![\mathrm{hyp}]\!]_\rho \ \mathbf{mr}\ C_1, \ldots, \downarrow [\![\mathrm{wkn}^n\, \mathrm{hyp}]\!]_\rho \ \mathbf{mr}\ C_n \vdash \downarrow [\![\mathrm{shift}\, p]\!]_\rho \ \mathbf{mr}\ A(r).$$

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Using equation (8), we obtain ϕ and the goal becomes

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We can now use the induction hypothesis with $\rho := (\phi, \rho)$,

$$\downarrow [\![\mathsf{hyp}]\!]_{\phi,\rho} \ \mathbf{mr}\ \forall x^{\mathbb{N}}(A(x) \to S(x)), \downarrow [\![\mathsf{wkn}\,\mathsf{hyp}]\!]_{\phi,\rho} \ \mathbf{mr}\ C_1, \ldots,$$
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Thanks to equation (1), the induction hypothesis becomes $\cdots$ □

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Finally, thanks to equation (9), we can finish the proof by applying the SHIFT rule for:

$$S'(x, y) := \downarrow [\![y]\!]_{\phi,\rho} \ \mathbf{mr}\ S(x)$$
$$A'(x, y) := \downarrow [\![y]\!]_{\phi,\rho} \ \mathbf{mr}\ A(x).$$

□

Soundness of modified realizability

The limitations to A, S of the SHIFT rule are not strict. We can actually extract a program of System T for *full* classical Analysis.

The *catch* is that not always is such a program correct.

Way forward

Although full classical Analysis is not *uniformly* realizable it may well be realizable for *concrete* non-Σ_2 statements — such that are sound w.r.t. *some* SHIFT rule.

Σ_2-Analysis refutes "Church's Thesis" but satisfies Church's Rule

Corollary

The Σ_2-fragment of classical Analysis satisfies the Existence Property,

> *Given a derivation of $\Gamma \vdash \exists x^\tau A(x)$, there exists a term p of type τ of System T such that $\Gamma \vdash A(p)$.*

and, consequently, the Weak Church's Rule,

> *Given a (closed) derivation of $\emptyset \vdash \forall x^{\mathbb{N}} \exists y^{\mathbb{N}} A(x, y)$, there exists a total recursive function $\mathbf{f} : \mathbb{N} \to \mathbb{N}$ such that, for all $\mathbf{n} \in \mathbb{N}$, we have that $\emptyset \vdash A(\overline{\mathbf{n}}, \overline{\mathbf{fn}})$, where $\overline{\mathbf{m}}$ denotes the term $\underbrace{succ \cdots succ}_{\mathbf{m}\ times}$ zero.*

Σ_2-Analysis satisfies Church's Rule

Example Application

Principles like

$$\neg\neg\exists x^{\mathbb{N}} N \to \exists x^{\mathbb{N}} N \tag{MP}$$

$$\forall x^{\mathbb{N}} \neg\neg A \to \neg\neg\forall x^{\mathbb{N}} A, \tag{DNS}$$

where

$$\neg B := B \to M \qquad M, N - \text{comp. irrelevant} \qquad A - \text{any}$$

are constructive even in presence of AC and Induction, solely because

$$\text{MP}, \text{DNS} \in \Sigma_2.$$

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Conclusion

One can:

1. avoid bar recursion (viz. supplement Schwichtenberg (1979))
2. replace control operators at *run-time* with partial evaluation at *compile-time*

Further details

- An interpretation of the Sigma-2 fragment of classical Analysis in System T, ArXiV:1301.5089
- Agda script: `http://www.lix.polytechnique.fr/~danko`
- A Direct Version of Veldman's Proof of Open Induction on Cantor Space via Delimited Control Operators (with **Keiko Nakata**), LIPIcs 26, 2014
- Delimited control operators prove Double-negation Shift, in APAL 163, 2012

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Thank you!