Conditions for doing research

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Conditions

1. Attitude
not for gain

2. Genuine interest
Finding a topic: reading, listening
Liking it!
Creating a new world OR exploring a known world (or both)
Having a *problem*

3. Disicpline
Concentration

4. Relax

Theory of combinators CL $\boxed{\texttt{terms ::= x | I | K | S | term term}}$

$$\boxed{\begin{array}{rcl} \mathsf{I}\,P & = & P \\ \mathsf{K}\,PQ & = & P \\ \mathsf{S}\,PQR & = & PR(QR) \end{array}}$$

Simulating λ-abstraction

$$\begin{array}{rcl} [x]x & \triangleq & \mathsf{I} \\ [x]y & \triangleq & \mathsf{K}\,y \\ [x]\mathsf{C} & \triangleq & \mathsf{K}\,\mathsf{C} \\ [x](PQ) & \triangleq & \mathsf{S}([x]P)([x]Q) \end{array}$$

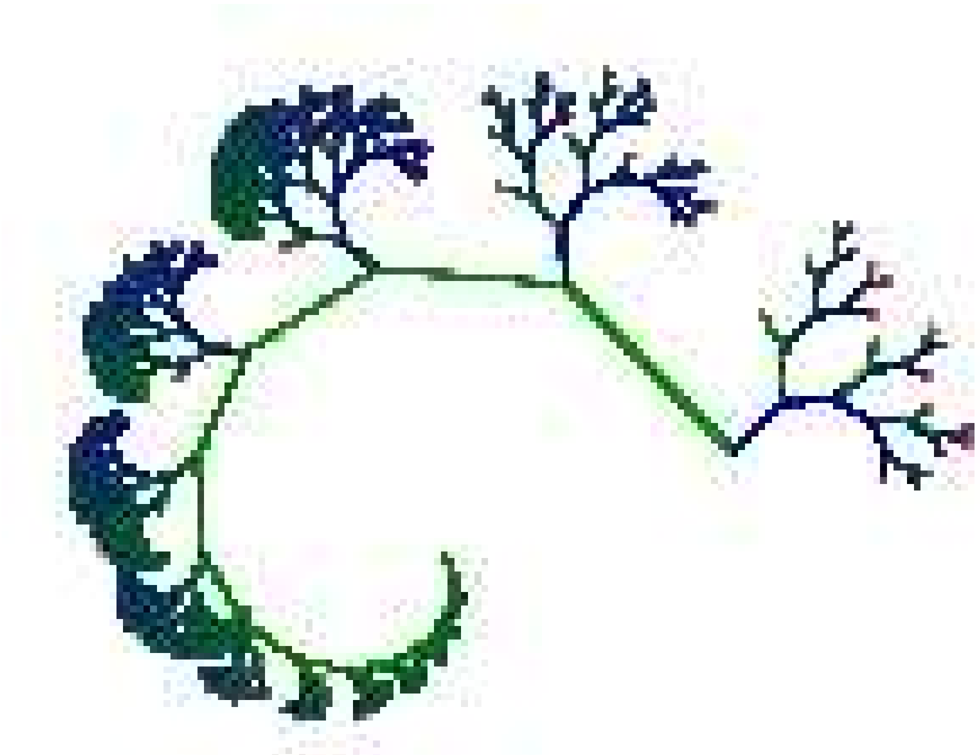
Simulating λ-calculus in CL

$$\begin{array}{rcl} (x)_\lambda & \triangleq & x \\ (PQ)_\lambda & \triangleq & (P)_\lambda(Q)_\lambda \\ (\lambda x.M)_\lambda & \triangleq & [x](M)_\lambda \end{array}$$

Playing with terms

$\mathsf{SSSSSSSSS} \twoheadrightarrow \mathsf{SS(SS(SS(SSS)))}$

$\mathsf{SSS(SSS)(SSS)} \twoheadrightarrow$



Picture: Joerg Endrullis

Barendregt-Endrullis-Klop-Waldmann [2015]
In Festschrift for logician (still under embargo)

What does it mean that a term doesn't have a nf?

Church: no normal form, no meaning.

Problem: make a recursion theoretic model of the combinators such that

$$P \text{ doesn't have a nf} \iff [\![P]\!]\uparrow$$

On $\mathbb{N}_* \triangleq \mathbb{N} \cup \{*\}$ define

$$\begin{array}{rcll} xy & \triangleq & \varphi_x(y) & \text{if defined} \\ & \triangleq & * & \text{else} \end{array}$$

In particular $x* = *x = **$

Using the S^m_n theorem one can construct $i, k, s \in \mathbb{N}$ such that for all $p, q, r \in \mathbb{N}$

$$\boxed{\begin{array}{rcl} ip & = & p \\ kpq & = & p \\ spqr & = & pr(qr) \end{array}}$$

This also holds for almost all $p, q, r \in \mathbb{N}_*$, except $kp* = * \neq p$ in general :(

We can interpret combinators $P \longmapsto [\![P]\!] \in \mathbb{N}_*$

I showed P is a normal form $\Rightarrow$ $[\![P]\!] \neq *$ (by smallprint S^m_n theorem)
P has no nf $\Rightarrow$ $[\![P]\!] = *$ (non-trivial, via length of computation)

This seems to imply Con($\{P = Q \mid P, Q$ do not have a nf$\}$)

But $[\Omega, \mathsf{I}] = [\Omega, \mathsf{S}] \vdash \mathsf{I} = \mathsf{S}$, giving a contradiction! Memo: $[R, S] \triangleq \lambda z.zRS$

For the combinator Ω one has $[\![\Omega]\!] = *$

$\Omega = \omega\omega$ with $\omega \triangleq (\lambda x.xx)_\lambda =$ **SII**

Now $[\![\omega]\!] = sii = e$, say, with and $ex = xx$

So $ex\downarrow \;\Leftrightarrow\; xx\downarrow$

Do we have $ee\downarrow$? [Problem of Henkin!]

We have $ee\downarrow$ if and only if $ee\downarrow$ in a non-trivial way

We can fiddle with the S^m_n-theorem and make $ee = 17$

but for natural choices of i, k, s one has $ee\uparrow$, using 'length of computation'

Analyzing the contradiction

Although Ω and hence $[\Omega, \mathsf{I}]$ do not have normal forms, one has

$$\begin{aligned} [\Omega, \mathsf{I}]\mathsf{K} &= \mathsf{I} \\ [\Omega, \mathsf{S}]\mathsf{K} &= \mathsf{S} \end{aligned}$$

So $[\Omega, \mathsf{I}]$ and $[\Omega, \mathsf{S}]$ cannot be equated

What is at stake is:

although $[\Omega, \mathsf{I}]$ isn't a normal form, it can be *solved* to a nf: $[\Omega, \mathsf{I}]\mathsf{K} = \mathsf{I}$

From this it is a small step to

DEFINITION. P is *solvable* iff for some $Q_1 \ldots Q_n$ one has $PQ_1 \ldots Q_n = \mathsf{I}$

With some work I could show

THEOREM [1971]. Con($\{P = Q \mid P, Q$ are unsolvable$\}$)

Needed: LEMMA. If U is unsolvable and $FU = \mathsf{I}$, then $\forall P.FP = \mathsf{I}$

Not much later Hyland and Wadsworth proved

THEOREM [1973]. P is unsolvable iff $[\![P]\!]^{\mathcal{D}_\infty} = \bot$

Kleene’s first model is what later became a pca, ‘partial combinatory algebra’

Proving the recursion theorem one first constructs a PR function g such that

$$\begin{array}{lcll} g(n,m) & \sim & nm & \text{if } nm\downarrow \\ & \sim & * & \text{else} \end{array}$$

where $x \sim y$ iff $\forall z.xz \simeq yz$.

Attempt 1.

$\mathbb{S} = \langle \mathbb{N}_*/\sim, .\rangle$ with $n.m = g(n,m)$

Doesn’t work:

$$\mathbb{S} \not\models P = Q \Rightarrow FP = FQ$$

Attempt 2

Definition

$$x \; E_0 \; y \quad \Leftrightarrow \quad x = y$$

$$x \; E_{n+1} \; y \quad \Leftrightarrow \quad \forall z{\in}\mathbb{N}_*.[xzE_nyz]$$

$$\mathbb{S}_0 \quad = \quad \mathbb{N}_*$$

$$\mathbb{S}_{n+1} \quad = \quad \{p{\in}\mathbb{S}_n \mid xE_ny \Rightarrow pxE_npy\}$$

Eventually this stabilizes $\mathbb{S}_\infty, E_\infty$ and we consider

$$\mathbb{S} = \langle \mathbb{S}_\infty / E_\infty \rangle$$

But it is not clear whether $i, k, s{\in}\mathbb{S}$

Definition

$$\mathbb{N}^{\mathsf{CL}} = \mathbb{N}^{\mathsf{CL}}_{*}$$

$$\begin{aligned} x\ E_0\ y &\quad\Leftrightarrow\quad x = y \\ x\ E_{n+1}\ y &\quad\Leftrightarrow\quad \forall z{\in}\mathbb{N}^{\mathsf{CL}}.[xzE_nyz] \end{aligned}$$

For the limit $\mathbb{S}$ it is not clear whether it is not degenerate in the sense that $i = k = s$

Omega rule

$$\frac{PZ = QZ \text{ for all closed } Z}{P = Q}$$

Consistency of it: proved in 1971

Universal generators: expect for a possible exception the ω-rule is valid:

P should'nt be a universal generator

Plotkin [1973] ω-incompleteness

λ-algebras vs λ-models

Categorical model in category with not enough points, Koymans [1982]

Hyland [2015] avoids the syntactical definition of λ-algebra

Hyland [2015] In Böhm Festschrift