Probabilistic Justification Logic

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Overview

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1. Introduction
   - Motivation
   - The Justification Logic J
2. Probabilistic Justification Logic
   - The Logics PJ and PPJ
   - Formalization of the Lottery Paradox
3. Epilogue

From Classical Logic to Justification Logic

c.p.l. $+$ "$\Box\alpha$" $\rightarrow$ (Propositional) Modal Logic

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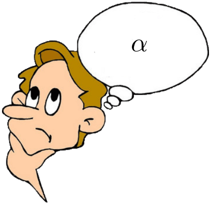
Modal Logic



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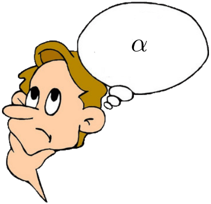

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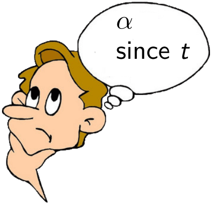
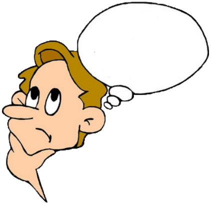
Justification Logic



$t : \alpha$

Justification Logic



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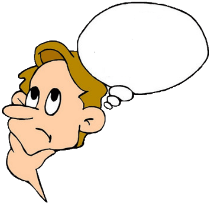

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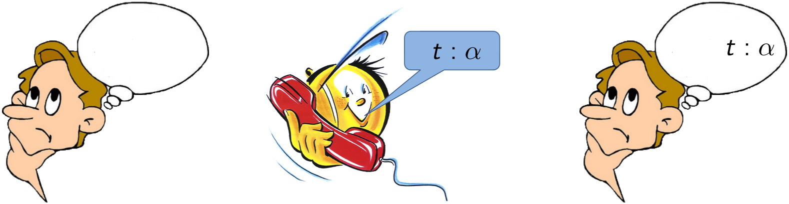

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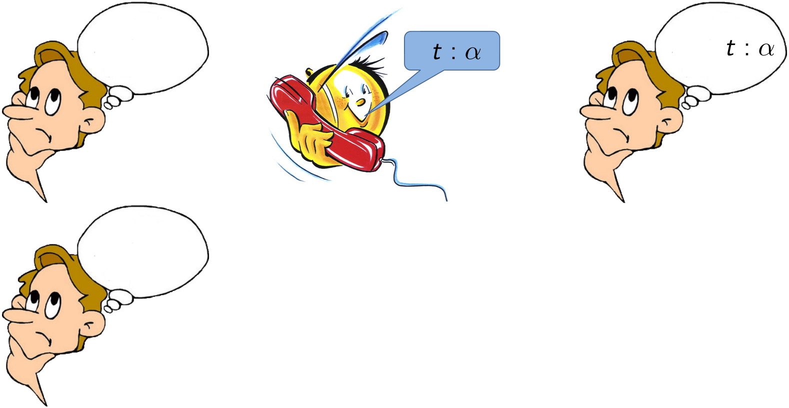

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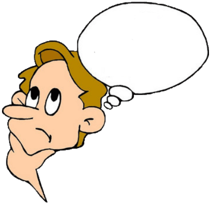

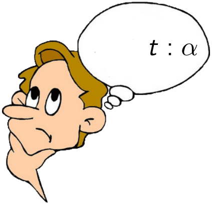

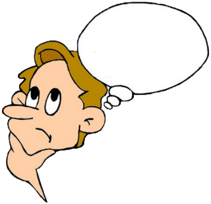

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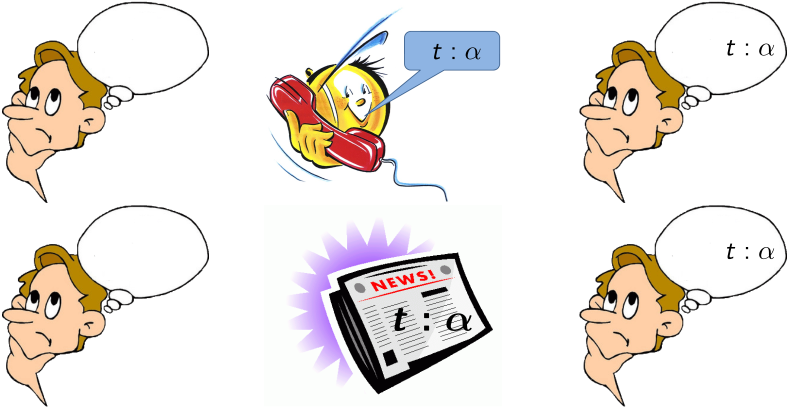
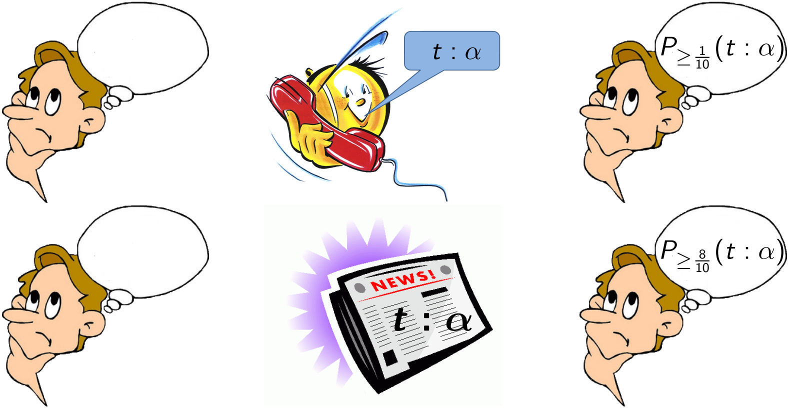


Probabilistic Justification Logic



Overview

Justification Terms and Formulas

Definition (Justification Terms)

$$t ::= c \mid x \mid (t \cdot t) \mid (t + t) \mid\ !t$$

Definition (Justification Formulas-Language $\mathsf{L_J}$)

$$\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid t : \alpha$$

where t is a term and p is an atomic proposition.

Axiomatization for J

Axioms of J:

(P)	finitely many axiom schemes for c.p.l. in the language $\mathsf{L_J}$
(J)	$\vdash u : (\alpha \to \beta) \to (v : \alpha \to u \cdot v : \beta)$
(+)	$\vdash (u : \alpha \vee v : \alpha) \to u + v : \alpha$

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$\mathsf{CS} \subseteq \{(c, \alpha) \mid c \text{ is a constant and } \alpha \text{ is an instance of some axiom of J}\}$

For some CS the system J_{CS} is:

	axioms of J
	+
(AN!)	$\vdash \underbrace{!! \cdots !!}_{n \text{ times}} c : \cdots : !c : c : \alpha$, where $(c, \alpha) \in \mathsf{CS}$ and $n \in \mathbb{N}$
(MP)	if $T \vdash \alpha$ and $T \vdash \alpha \to \beta$ then $T \vdash \beta$

Semantics for J

For a given CS, we define the function $*$ (basic CS-evaluation):

$$* : \text{Prop} \rightarrow \{\text{T}, \text{F}\} \qquad [\text{T} = \text{ true and F} = \text{false}]$$

$$* : \text{Tm} \rightarrow \mathcal{P}(\text{L}_\text{J}) \qquad [\mathcal{P} \text{ stands for powerset}]$$

such that:

1. $\big(\alpha \rightarrow \beta \in u^* \text{ and } \alpha \in v^*\big) \Longrightarrow \beta \in (u \cdot v)^*$
2. $u^* \cup v^* \subseteq (u + v)^*$
3. for $(c, \alpha) \in \text{CS}$:

$$\alpha \in c^*$$

$$c : \alpha \in (!c)^*$$

$$!c : c : \alpha \in (!!c)^*$$

$$\vdots$$

Truth under a Basic CS-Evaluation

Let $*$ be a basic CS-evaluation. We have:

- $* \Vdash p \Longleftrightarrow p^* = \mathsf{T}$
- $* \Vdash t : \alpha \Longleftrightarrow \alpha \in t^*$
- $* \Vdash \neg\alpha \Longleftrightarrow * \nVdash \alpha$
- $* \Vdash \alpha \wedge \beta \Longleftrightarrow \left(* \Vdash \alpha \text{ and } * \Vdash \beta \right)$

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Probabilistic Formulas

Definition (Probabilistic Formulas-Language $\mathsf{L_P}$)

$$A ::= P_{\geq s}\alpha \mid \neg A \mid A \wedge A$$

for $s \in \mathbb{Q} \cap [0,1]$ and $\alpha \in \mathsf{L_J}$.

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$$P_{<s}\alpha \equiv \neg P_{\geq s}\alpha$$
$$P_{\leq s}\alpha \equiv P_{\geq 1-s}\neg\alpha$$
$$P_{>s}\alpha \equiv \neg P_{\leq s}\alpha$$
$$P_{=s}\alpha \equiv P_{\geq s}\alpha \wedge P_{\leq s}\alpha$$

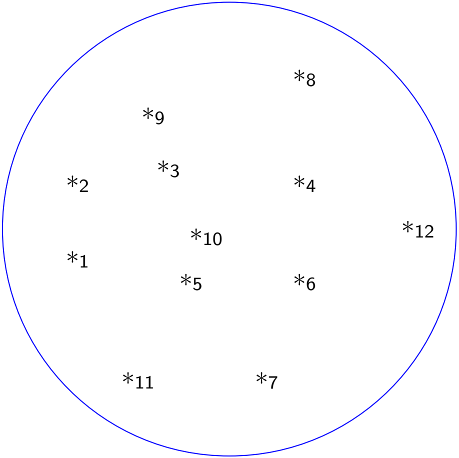
Semantics

$$\models P_{\geq s}\alpha \Longleftrightarrow ?$$

Semantics

Probability Space: $M = \langle W, H, \mu \rangle$

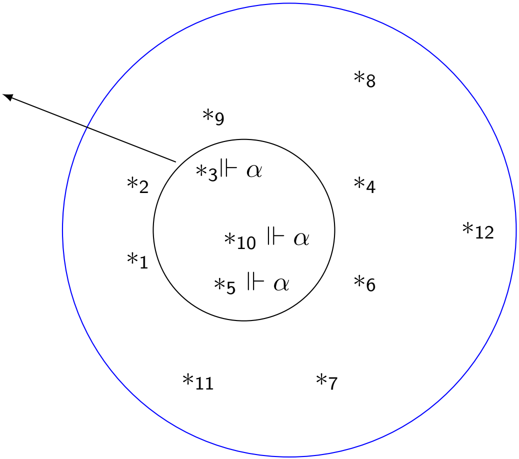
$\models P_{\geq s}\alpha \iff ?$



Semantics

Probability Space: $M = \langle W, H, \mu \rangle$

$M \models P_{\geq s}\alpha \iff [\alpha]$



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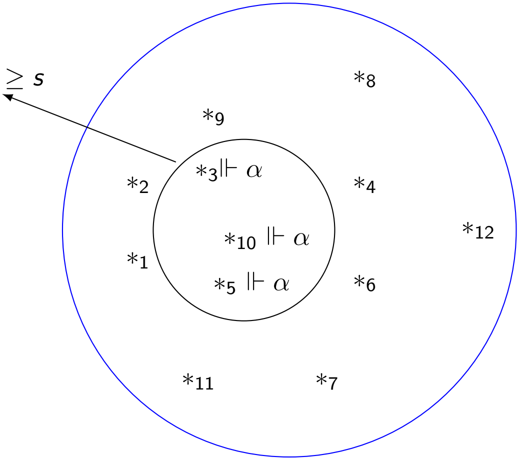
$$M \models P_{\geq s}\alpha \iff \mu([\alpha]) \geq s$$



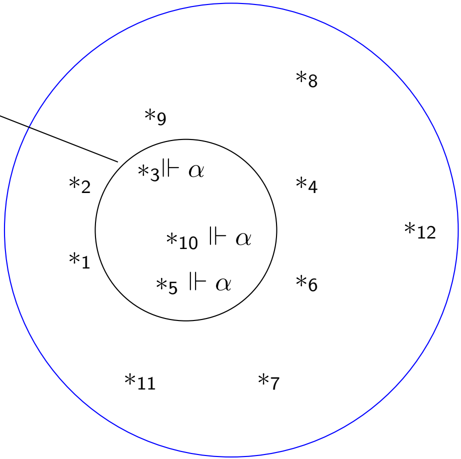
Semantics

Probability Space: $M = \langle W, H, \mu \rangle$

$$M \models P_{\geq s}\alpha \iff \mu([\alpha]) \geq s$$

$$M \models \neg A \iff M \not\models A$$

$$M \models A \wedge B \iff$$
$$\big(M \models A \text{ and } M \models B\big)$$



Axiomatization

Axioms of PJ:

(P)	finitely many axiom schemes for c.p.l. in the language L_P
(PI)	$\vdash P_{\geq 0}\alpha$
(WE)	$\vdash P_{\leq r}\alpha \rightarrow P_{<s}\alpha$, where $s > r$
(LE)	$\vdash P_{<s}\alpha \rightarrow P_{\leq s}\alpha$
(DIS)	$\vdash P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge P_{\geq 1}\neg(\alpha \wedge \beta) \rightarrow P_{\geq \min(1,r+s)}(\alpha \vee \beta)$
(UN)	$\vdash P_{\leq r}\alpha \wedge P_{<s}\beta \rightarrow P_{<r+s}(\alpha \vee \beta)$, where $r + s \leq 1$

For some CS the system PJ_{CS} is:

	Axioms of PJ +
(MP)	if $T \vdash A$ and $T \vdash A \rightarrow B$ then $T \vdash B$
(CE)	if $\vdash_{J_{CS}} \alpha$ then $\vdash_{PJ_{CS}} P_{\geq 1}\alpha$
(ST)	if $T \vdash A \rightarrow P_{\geq s - \frac{1}{k}}\alpha$ for every integer $k \geq \frac{1}{s}$ and $s > 0$ then $T \vdash A \rightarrow P_{\geq s}\alpha$

Soundness and Strong Completeness

Theorem

Any $\mathsf{PJ}_{\mathsf{CS}}$ *is sound and strongly complete with respect to* $\mathsf{PJ}_{\mathsf{CS}}$*-models, i.e.:*

$$T \vdash_{\mathsf{PJ}_{\mathsf{CS}}} A \Longleftrightarrow T \Vdash_{\mathsf{PJ}_{\mathsf{CS}}} A$$

Complexity

It has been proved that, under some restrictions on the CS, the J_{CS}-satisfiability problem belongs to Σ_2^p.

We can prove that, under the same restrictions on the CS, the $\mathsf{PJ}_{\mathsf{CS}}$-satisfiability problem belongs to the same complexity class.

Thus, probability operators do not increase the complexity of the justification logic J.

Iterations of the Probability Operator

Can we have formulas like $P_{\geq s} P_{\geq r} \alpha$ or $t : P_{\geq r} \alpha$?

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The logic PPJ is defined over the following language:

$$\alpha ::= p \mid \alpha \wedge \alpha \mid \neg \alpha \mid t : \alpha \mid P_{\geq s} \alpha$$

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$\mathsf{J} + \mathsf{PJ} \rightarrow \mathsf{PPJ}$

- soundness: ✓
- completeness: ✓
- decidability : ✓
- complexity: open

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The Lottery Paradox

Fair Lottery:

- with 1000 tickets
- exactly 1 winning ticket

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Postulates of Rational Belief:

1. $\text{Deg}_{\text{bel}}(\alpha) > 0.99 \iff \text{Bel}(\alpha)$ [*The Lockean Thesis*]
2. $\big(\text{Bel}(\alpha) \text{ and } \text{Bel}(\beta)\big) \implies \text{Bel}(\alpha \wedge \beta)$

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Paradox:

- $(\forall 1 \leq i \leq 1000)\big[\mathsf{Deg}_{\mathsf{bel}}(\neg w_i) = 0.999\big]$

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- $\text{Deg}_{\text{bel}}(w_1 \vee \ldots \vee w_{1000}) = 1$

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  $(\forall 1 \leq i \leq 1000)[\text{Bel}(\neg w_i)] \Longrightarrow \text{Bel}(\neg w_1 \wedge \ldots \wedge \neg w_{1000})$
- $\text{Deg}_{\text{bel}}(w_1 \vee \ldots \vee w_{1000}) = 1 \Longrightarrow \text{Bel}(w_1 \vee \ldots \vee w_{1000})$

Formalization of the Lottery Paradox in PPJ

Formalization of the Postulates of Rational Belief in PPJ:

1. For every term t we add to our assumptions:

$$t : P_{>0.99}(\alpha) \to \mathsf{pb}(t) : \alpha \qquad \text{[The Lockean Thesis]}$$

2. For some CS we have:

$$\vdash_{\mathsf{PPJ}_{\mathsf{CS}}} s_1 : \alpha \wedge s_2 : \beta \to \mathsf{con}(s_1, s_2) : \alpha \wedge \beta$$

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Let $1 \leq i \leq 1000$. There is a term t_i:

$$t_i : (P_{=\frac{999}{1000}} \neg w_i)$$

and by the Lockean Thesis:

$$\mathrm{pb}(t_i) : \neg w_i$$

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Thus there is a term t:

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But there also exists a term u such that:

$$u : (P_{=1}(w_1 \vee \ldots \vee w_{1000}))$$

thus, by the Lockean Thesis:

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Avoiding the paradox in PPJ:
Restrict the CS such that:

$$\vdash_{\mathsf{PPJ}_{\mathsf{CS}}} s_1 : \alpha \wedge s_2 : \beta \rightarrow \mathrm{con}(s_1, s_2) : \alpha \wedge \beta$$

holds only if $\mathrm{con}(s_1, s_2)$ does not contain two different subterms of the form $\mathrm{pb}(t)$. [*Formalization of an idea by Leitgeb (2014)*]

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Further Work: Statistical Evidence

Obtain justifications from formulas?

$\alpha \mapsto h(\alpha)$

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$$CP(\beta \mid \alpha) \geq s$$

interprets to something like:

$$P_{\geq s}(h(\alpha) : \beta)$$

Summary

- Probabilistic justification logic can be used to model the idea

  different kinds of evidence for α lead to
  different degrees of belief in α

- $\mathsf{PJ} = P_{\geq s} + \mathsf{J}$
- PJ is sound and strongly complete
- complexity of PJ is no worse than the complexity of J
- sound, complete and decidable PPJ
- Complexity of PPJ? Statistical evidence?

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Thank you for your attention!