Galois Connections in Clone Theory

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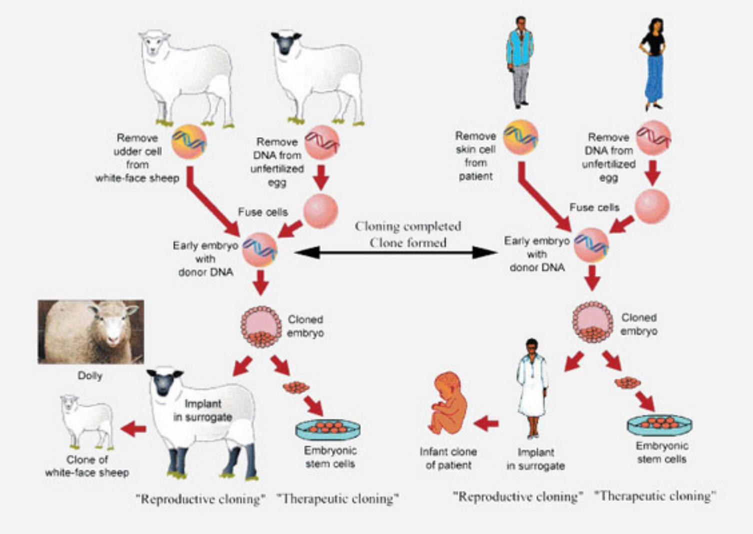
"Galois connections provide the structure-preserving passage between two worlds of our imagination."

K. Denecke, M. Erné, S. L. Wismath
from the preface to *Galois connections and applications*

Some applications:

- Logic (syntax $\longleftrightarrow$ semantics)
- Program construction (abstraction $\longleftrightarrow$ concretisation)
- Fixed point calculus
- Formal concept analysis (objects $\longleftrightarrow$ attributes)
- Constraint satisfaction problems (relations $\longleftrightarrow$ functions)

Biological clones



Mathematical clones

Clone is an operational structure *closed* with respect to composition of operations.

CLONE = CLosed ONE

CLONE = CLosed Operation NEtwork

GALOIS CONNECTIONS

Where does the name come from?

Galois theory:

Connection between subfields of some field and subgroups of the automorphism group of that field.

Galois connection (general case)

Definition

$\alpha : \mathcal{P}(A) \to \mathcal{P}(B)$, $\beta : \mathcal{P}(B) \to \mathcal{P}(A)$,
$X, X_1, X_2 \in \mathcal{P}(A)$, $Y, Y_1, Y_2 \in \mathcal{P}(B)$

(i) $X_1 \subseteq X_2 \Rightarrow \alpha(X_1) \supseteq \alpha(X_2)$, $Y_1 \subseteq Y_2 \Rightarrow \beta(Y_1) \supseteq \beta(Y_2)$;

(ii) $X \subseteq \beta(\alpha(X))$, $Y \subseteq \alpha(\beta(Y))$.

Pair (α, β) is a **Galois connection** between sets A and B.

Let (α, β) be a Galois connection between sets A and B. Then $\alpha\beta$ and $\beta\alpha$ are closure operators on B and A, respectfully.

Construction of a Galois connection

Theorem

For nonempty sets A and B, and $R \subseteq A \times B$ we define mappings

$$\overrightarrow{R} : \mathcal{P}(A) \to \mathcal{P}(B) \quad \textit{and} \quad \overleftarrow{R} : \mathcal{P}(B) \to \mathcal{P}(A)$$

by

$$\overrightarrow{R}(X) = \{y \in B : (\forall x \in X)\,(x, y) \in R\}, \;\; X \subseteq A,$$
$$\overleftarrow{R}(Y) = \{x \in A : (\forall y \in Y)\,(x, y) \in R\}, \;\; Y \subseteq B.$$

Then the pair $(\overrightarrow{R}, \overleftarrow{R})$ is a Galois connection between sets A and B.

Example from Galois theory

Example

F - field

G - group of automorphisms of F

$R \subseteq F \times G$: $(a, g) \in R$ iff g fixes a

$$\begin{aligned} \overrightarrow{R}(A) &= \{g \in G : (\forall a \in A)\, g(a) = a\},\ A \subseteq F \\ \overleftarrow{R}(H) &= \{a \in F : (\forall g \in H)\, g(a) = a\},\ H \subseteq G \end{aligned}$$

CLONES

Operations

A is a finite set with at least two elements.

$$f : A^n \to A, \qquad n\text{-ary operation on } A$$

Example

f	0	1
0	1	0
1	0	0

$O_A^{(n)}$ - set of all n-ary operations on A

$O_A = \bigcup_{n\geq 1} O_A^{(n)}$ - set of all finitary operations on A

Composition of operations

$e_i^{n,A}(x_1, \ldots, x_i, \ldots, x_n) = x_i$ is n-ary projection

J_A is the set of all projections on A.

Composition of operations $f \in O_A^{(n)}$ and $g_1, \ldots, g_n \in O_A^{(m)}$ is $f(g_1, \ldots, g_n) \in O_A^{(m)}$, such that

$$f(g_1, \ldots, g_n)(x_1, \ldots, x_m) = f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m)).$$

Example

$$\begin{array}{c|cc} f & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 0 \end{array} \qquad \begin{array}{c|c|c} & g_1 & g_2 \\ \hline 0 & 1 & 1 \\ 1 & 0 & 1 \end{array} \qquad \Rightarrow \qquad \begin{array}{c|c} & f(g_1, g_2) \\ \hline 0 & 0 \\ 1 & 0 \end{array}$$

$$f(g_1, g_2)(0) = f(g_1(0), g_2(0)) = f(1, 1) = 0$$

First definition of a clone

Definition

Set of operations is a **clone** iff it

- contains all projections
- is closed with respect to composition of operations.

Example

The following sets of operations on set A are clones:

- O_A - set of all operations;
- J_A - set of all projections;
- set of all idempotent operations on A (f is idempotent if it holds $f(x, \ldots, x) = x$, for all $x \in A$).

Clone lattice

For $F \subseteq O_A$

$$\langle F \rangle = \bigcap \{C : F \subseteq C \text{ and } C \text{ is a clone}\}$$

is the least clone that contains F.

Theorem

The set of all clones on A, ordered by set inclusion, is an algebraic lattice $\mathcal{L}_A$.

Relations

$\rho \subseteq A^\ell$, $\ell \geq 1$, is ℓ-ary relation on A

$R_A^{(\ell)} = \mathcal{P}(A^\ell)$ - set of all ℓ-ary relations on A

$R_A = \bigcup_{\ell \geq 1} R_A^{(\ell)}$ - set of all finitary relations on A

Instead of

$$(a_{11}, a_{21}, \ldots, a_{\ell 1}), (a_{12}, a_{22}, \ldots, a_{\ell 2}), \ldots, (a_{1n}, a_{2n}, \ldots, a_{\ell n}) \in \rho,$$

we write

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\ell 1} & a_{\ell 2} & \ldots & a_{\ell n} \end{pmatrix} \in \rho^*.$$

Preservation relation

$f \in O_A^{(n)}, \ M = [a_{ij}]_{\ell \times n}$

$$f(M) = \begin{pmatrix} f(a_{11}, a_{12}, \ldots, a_{1n}) \\ f(a_{21}, a_{22}, \ldots, a_{2n}) \\ \vdots \\ f(a_{\ell 1}, a_{\ell 2}, \ldots, a_{\ell n}) \end{pmatrix}.$$

Definition

An operation f **preserves** a relation ρ (or ρ is an **invariant** relation of f) if

$$M \in \rho^* \Rightarrow f(M) \in \rho.$$

Preservation relation (examples)

Example

1) Every operation on A preserves the relation $\delta^2_{A;\{1,2\}}$.
2) Arbitrary projection $e^n_i \in J_A$ preserves all relations on A.
3) If operation f preserves order relation $\leq$, then f is monotonous with respect to $\leq$.
4) Equivalence relation ε on A which is preserved by operation f is a congruence of algebra (A, f).
5) Idempotent operations preserve all unary relations.

Galois connection (*Pol*, *Inv*) - Second definition of a clone

$$Pol : \mathcal{P}(R_A) \to \mathcal{P}(O_A), \qquad Inv : \mathcal{P}(O_A) \to \mathcal{P}(R_A)$$

$$\begin{aligned} Pol\, Q &= \{f \in O_A : f \text{ preserves every } \rho \in Q\},\ Q \subseteq R_A, \\ Inv\, F &= \{\rho \in R_A : \text{every } f \in F \text{ preserves } \rho\},\ F \subseteq O_A. \end{aligned}$$

Theorem (Bondarčuk, Kalužnin, Kotov, Romov 1969)

$$C \subseteq O_A \ \textit{ is a clone} \quad \Leftrightarrow \quad C = Pol\, Inv\, C.$$

Clone lattice

$\mathcal{L}_A = (\{C \subseteq O_A : C \text{ is a clone }\}, \subseteq)$ *is an algebraic lattice.*

- *The least element of* $\mathcal{L}_A$ *is* J_A, *and the greatest element is* O_A.
- *Lattice operations are*

$$C_1 \wedge C_2 = C_1 \cap C_2 \quad i \quad C_1 \vee C_2 = \langle C_1 \cup C_2 \rangle.$$

- *Atoms of the lattice* $\mathcal{L}_A$ *are called minimal clones, while coatoms are called maximal clones.*

Cardinality of the clone lattice

Theorem (Post 1920)

Clone lattice on a two-element set is countable.

Theorem (Janov, Mučnik 1959)

Clone lattice on at least three-element set has the cardinality of continuum.

Maximal clones

Theorem (Rosenberg 1970)

Clone is maximal iff it is of the form Polρ, where ρ is a relation belonging to one of the following classes:

(R1) bounded partial orders;

(R2) prime permutations;

(R3) prime-affine relations;

(R4) non-trivial equivalence relations;

(R5) central relations;

(R6) h-regular relations.

HYPERCLONES

Hyperoperations

$P_A^* := \mathcal{P}(A) \setminus \{\emptyset\}$

$$f : A^n \to P_A^*, \quad n\text{-ary hyperoperation on } A$$

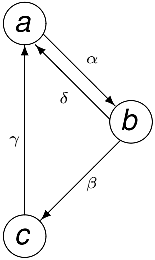
Example

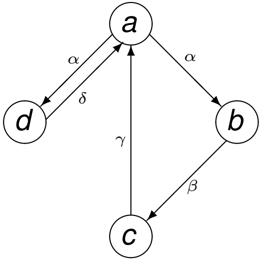
f	0	1
0	$\{1\}$	$\{0\}$
1	$\{0\}$	$\{0, 1\}$

$H_A^{(n)}$ - set of all n-ary hyperoperations on A

$H_A = \bigcup_{n \geq 1} H_A^{(n)}$ - set of all finitary hyperoperations on A

Deterministic vs. non-deterministic automaton





$f(a, \alpha) = b$

$f(a, \alpha) = \{b, d\}$

Composition of hyperoperations

$e_i^{n,A}(x_1, \ldots, x_i, \ldots, x_n) = \{x_i\}$ is an *n-ary (hiper)projection*

Composition of hyperoperations $f \in H_A^{(n)}$ and $g_1, \ldots, g_n \in H_A^{(m)}$ is $f[g_1, \ldots, g_n] \in H_A^{(m)}$, where

$$f[g_1, \ldots, g_n](x_1, \ldots, x_m) = \bigcup_{\substack{(y_1, \ldots, y_n) \in A^n \\ y_i \in g_i(x_1, \ldots, x_m) \\ 1 \le i \le n}} f(y_1, \ldots, y_n).$$

Example

f	0	1
0	$\{1\}$	$\{0\}$
1	$\{0\}$	$\{0,1\}$

	g_1	g_2
0	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$

$\Rightarrow$

	$f[g_1, g_2]$
0	$\{0,1\}$
1	$\{0,1\}$

$$f[g_1, g_2](0) = f(0,0) \cup f(1,0) = \{0,1\}$$

Definition of a hyperclone

Definition

Set of hyperoperations is a **hyperclone** iff it

- contains all hyperprojections
- is closed with respect to composition of hyperoperations.

Example

The following sets of hyperoperations on set A are hyperclones:

- H_A - set of all hyperoperations;
- J_A - set of all hyperprojections;
- O_A - set of all operations.

Hyperclone lattice

For $F \subseteq H_A$

$$\langle F \rangle_h = \bigcap \{C : F \subseteq C \text{ and } C \text{ is a hyperclone}\}$$

is the least hyperclone that contains F.

$\mathcal{L}_A^h = (\{C \subseteq O_A : C$ is a hyperclone $\}, \subseteq)$ is an algebraic lattice.

- *The least element of $\mathcal{L}_A^h$ is J_A, and the greatest element is H_A.*
- *Lattice operations are*

$$C_1 \wedge_h C_2 = C_1 \cap C_2 \quad i \quad C_1 \vee_h C_2 = \langle C_1 \cup C_2 \rangle_h.$$

- *Atoms of the lattice $\mathcal{L}_A^h$ are called minimal hyperclones, while coatoms are called maximal hyperclones.*

Hyperclone lattice on $\{0, 1\}$

Theorem (Machida 2002)

Hyperclone lattice on a two-element set has the cardinality of continuum.

Theorem (Pantović, Vojvodić 2004)

There exist 13 minimal hyperclones on $\{0, 1\}$.

Theorem (Tarasov 1974)

There exist 9 maximal hyperclones on $\{0, 1\}$.

Three relations on P_A^*

Let $\rho \in R_A^{(\ell)}$. We define ρ_d, ρ_m, $\rho_h \in (R_A^*)^{(\ell)}$ as follows:

$$\begin{aligned}
\rho_d &= \{(A_1, \ldots, A_\ell) : A_1 \times \ldots \times A_\ell \subseteq \rho\}, \\
\rho_m &= \{(A_1, \ldots, A_\ell) : (\forall\, i \in \{1, \ldots, \ell\})\ (\forall\, a \in A_i) \\
&\quad\ (\exists\, \boldsymbol{a} \in (A_1 \times \cdots \times A_\ell) \cap \rho)\ \ e_i^{\ell,A}(\boldsymbol{a}) = a\} \quad \text{and} \\
\rho_h &= \{(A_1, \ldots, A_\ell) : (A_1 \times \cdots \times A_\ell) \cap \rho \neq \emptyset\}.
\end{aligned}$$

$$\boxed{\rho_d \subseteq \rho_m \subseteq \rho_h}$$

Three relations on P_A^* (example)

Example

Let $\rho = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Then

$$\rho_d = \begin{pmatrix} \{0\} & \{1\} & \{1\} & \{1\} & \{0,1\} \\ \{0\} & \{0\} & \{1\} & \{0,1\} & \{0\} \end{pmatrix}$$

$$\rho_m = \begin{pmatrix} \{0\} & \{1\} & \{1\} & \{1\} & \{0,1\} & \{0,1\} \\ \{0\} & \{0\} & \{1\} & \{0,1\} & \{0\} & \{0,1\} \end{pmatrix}$$

$$\rho_h = \begin{pmatrix} \{0\} & \{0\} & \{1\} & \{1\} & \{1\} & \{0,1\} & \{0,1\} & \{0,1\} \\ \{0\} & \{0,1\} & \{0\} & \{1\} & \{0,1\} & \{0\} & \{1\} & \{0,1\} \end{pmatrix}$$

Galois connections $(xPol, xInv)$, $x \in \{d, m, h\}$

Definition

Let $x \in \{d, m, h\}$.
$f \in H_A^{(n)}$ x-preserve $\rho \subseteq A^\ell$ if for all $a_{11}, \ldots, a_{\ell n} \in A$ it holds

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{\ell 1} & a_{\ell 2} & \cdots & a_{\ell n} \end{pmatrix} \in \rho^* \quad \Rightarrow \quad \begin{pmatrix} f(a_{11}, \ldots, a_{1n}) \\ \cdots \\ f(a_{\ell 1}, \ldots, a_{\ell n}) \end{pmatrix} \in \rho_x.$$

$$xPol\, Q = \{f : f \text{ } x\text{-preserves each } \rho \in Q\}, \quad Q \subseteq R_A,$$

$$xInv\, F = \{\rho : \text{each } f \in F \text{ } x\text{-preserves } \rho\}, \quad F \subseteq H_A.$$

$$\boxed{dPol\rho \subseteq mPol\rho \subseteq hPol\rho}$$

Four classes of maximal hyperclones

Theorem

Let $Pol\rho$ be a maximal clone on A such that

$$(\forall f \in H_A \setminus hPol\rho)\,(\exists f' \in O_A \setminus Pol\rho)\ f' \in \langle Pol\rho \cup \{f\}\rangle_h.$$

Then $hPol\rho$ is a maximal hyperclone.

Theorem (Machida, Pantović 2012)

$hPol\rho$ is a maximal hyperclone if $\rho \in R_4 \cup R_5 \cup R_6$.

Theorem (Čolić, Machida, Pantović 2015)

$hPol\rho$ is a maximal hyperclone if $\rho \in R_1$.

Open problems

- Additional properties of Galois connections $(xPol, xInv)$, $x \in \{d, m, h\}$.
- Is there a Galois connection such that its closed sets are exactly all hyperclones?
- Is $hPol\rho$ maximal hyperclone if ρ is a relation from (R_2) and (R_3)?
- Description of all maximal hyperclones.

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Thank you for your attention!

The people who know
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