On the various definitions of cyclic operads

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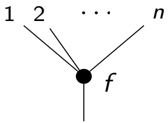
Overview: different definitions of (cyclic) operads

Symmetric operad = operations
+ (associative) compositions
+ permutation of variables (+ identities)



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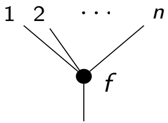

	Biased (individual compositions)			**Unbiased** (monad of trees)	**Algebraic** (microcosm principle)	
	Classical	**Partial**			**Classical**	**Partial**
Symmetric Operads	*Boardman, Vogt, May*	*Markl*		*Smirnov, May, Getzler, Jones*	*May, Smirnov, Kelly*	*Fiore*
		Exchangeable output	**Entries only**		**Exchangeable output**	**Entries only**
Cyclic Operads	*Getzler, Kapranov*	*Markl*	*Markl*	*Getzler, Kapranov*	??	??

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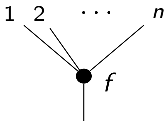

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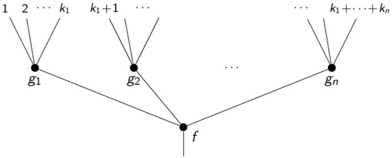
Plan:
- Examine these definitions
- Introduce a λ-calculus-style syntax: the μ-syntax
- Fill in the question marks

Symmetric operads

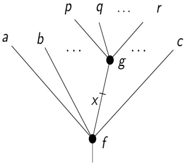
Classical + Skeletal

- $\mathcal{P} : \mathbf{\Sigma}^{op} \to \mathbf{C}$
- $\gamma : \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_n) \to \mathcal{P}(k_1 + \cdots + k_n)$
- $\eta : \mathbf{1} \to \mathcal{P}(1)$



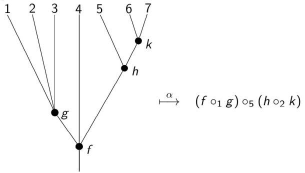
Partial + Non-skeletal

- $\mathcal{S} : \mathbf{Bij}^{op} \to \mathbf{C}$
- $\circ_x : \mathcal{S}(X) \times \mathcal{S}(Y) \to \mathcal{S}((X \cup Y) \backslash \{x\})$
- $id_x \in \mathcal{S}(\{x\})$



Unbiased

An operad is an algebra over the monad of rooted, decorated, labeled trees (which constitute the category $\mathbf{Tree}_n$).



Exchangeable output: from ordinary to cyclic operads

Ordinary operads $\longrightarrow$ **Cyclic operads**

an action of relabeling the leaves of a rooted tree

an action of interchanging the labels of **all** *leaves of a rooted tree, including the label given to the root*

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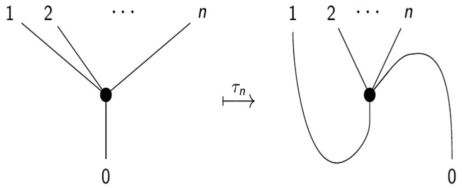
This is achieved by enriching the operad structure with the action of the cycle $\tau_n = (0, 1, \ldots, n)$:



The **distinction between inputs and the output** of an operation is **no longer visible**...

Cyclic operads: entries only

Definition 1 (Partial + non-skeletal)

A *cyclic operad* is a functor $\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{Set}$, together with a distinguished element $id_{x,y} \in \mathcal{C}(\{x,y\})$ for each two-element set $\{x,y\}$, and a partial composition operation

$${}_{x}\circ_{y} : \mathcal{C}(X) \times \mathcal{C}(Y) \to \mathcal{C}((X \cup Y)\backslash\{x,y\}).$$

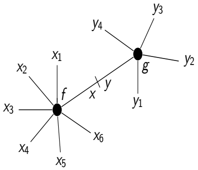
These data are required to satisfy the associativity, equivariance, unitality and commutativity equations.

Associativity.
$(f\ {}_{x}\circ_{y}\ g)\ {}_{u}\circ_{z}\ h = f\ {}_{x}\circ_{y}\ (g\ {}_{u}\circ_{z}\ h)$
$(f\ {}_{x}\circ_{y}\ g)\ {}_{u}\circ_{z}\ h = (f\ {}_{u}\circ_{z}\ h)\ {}_{x}\circ_{y}\ g$

Equivariance.
$f^{\sigma_1}\ {}_{x}\circ_{y}\ g^{\sigma_2} = (f\ {}_{\sigma_1(x)}\circ_{\sigma_2(y)}\ g)^{\sigma}$

Unitality.
$f\ {}_{x}\circ_{y}\ id_{y,z} = f^{\sigma}$
$id_{y,z}\ {}_{y}\circ_{x}\ f = f^{\sigma}$

Commutativity.
$f\ {}_{x}\circ_{y}\ g = g\ {}_{y}\circ_{x}\ f.$



This definition induces a natural *combinator syntax*.

Cyclic operads: exchangeable output

Definition 2 (Partial + non-skeletal)

A cyclic operad is an ordinary operad $\mathcal{S}$, augmented with actions

$$D_{xy} : S(X) \to S((X \backslash \{x\}) \cup \{y\}),$$

indexed by variables $x \in X$ and $y \notin X \backslash \{x\}$, and subject to the following list of axioms:

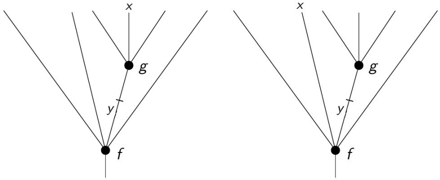
Identity.
$D_{xx}(f) = f$

Coherence.
$D_{zx}(D_{xy}(f)) = D_{zy}(f)$

α-conversion.
$D_{xa}(f) = D_{x'a}(f^{\sigma})$,
where $\sigma(x) = x'$, and $\sigma = id$ elsewhere

Equivariance.
$D_{\sigma(x)\sigma(y)}(f^{\sigma}) = D_{xy}(f)^{\sigma}$

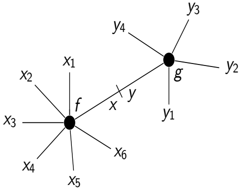
Compatibility with compositions.
$D_{xz}(f \circ_y g) = D_{xu}(g) \circ_u D_{yz}(f)$
$D_{xz}(f \circ_y g) = D_{xz}(f) \circ_y g$



Cyclic operads: unbiased definition

The *entries-only* characterization of cyclic operads reflects the ability to carry out the (partial) composition of two operations along *any* edge.

The pasting shemes for cyclic operads are unrooted, decorated, labeled trees.



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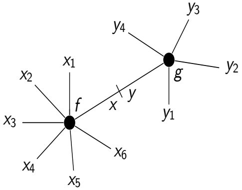

Given a functor $\mathcal{P} : \mathbf{Bij}^{op} \to \mathbf{C}$, we build the *free operad* $F(\mathcal{P})$ by grafting of such trees. The free operad functor F and the forgetiful functor U constitute a monad $\Gamma = UF$ in $\mathbf{C}^{\mathbf{Bij}^{op}}$, called the *monad of unrooted trees*.

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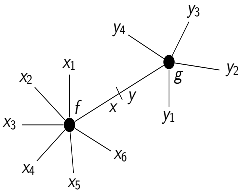

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Definition 3

A cyclic operad is an algebra over this monad.

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	Classical	**Partial**		(monad of trees)
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		Exchangable output	**Entries only**	
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μ-syntax

μ-syntax for cyclic operads

The μ-syntax consists of two kinds of **expressions**:

commands $c : X$ (no entry selected) *terms* $X \,|\, s$ (one entry selected)

$$c ::= \langle s \,|\, t \rangle \;\mid\; \underline{f}\{t_x \,|\, x \in X\} \qquad s, t ::= x \;\mid\; \mu x.c$$

The **typing rules** are as follows:

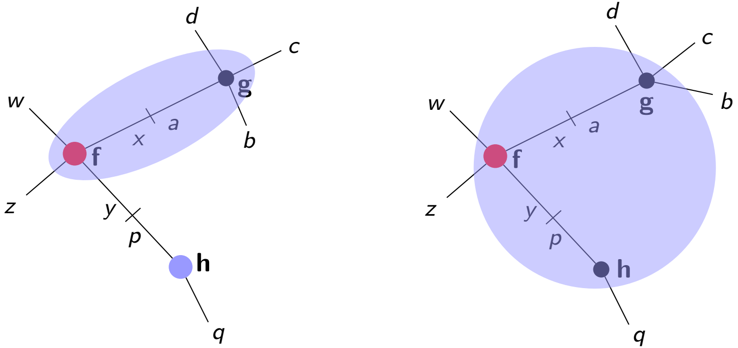
$$\frac{}{\{x\} \,|\, x} \qquad \frac{f \in \mathcal{S}(X) \qquad \ldots Y_x \,|\, t_x \ldots}{\underline{f}\{t_x \,|\, x \in X\} : \bigcup Y_x} \qquad \frac{X \,|\, s \quad Y \,|\, t}{\langle s \,|\, t \rangle : X \cup Y} \qquad \frac{c : X}{X \backslash \{x\} \,|\, \mu x.c}$$

The **equations** are $\langle s \,|\, t \rangle = \langle t \,|\, s \rangle$ and (oriented from left to right):

$$\langle \mu x.c \,|\, s \rangle = c[s/x] \qquad (\mu\text{-reduction})$$

μ-syntax: intuition

$\langle \mu x.c \,|\, s\rangle$ and $c[s/x]$ describe two ways to build the same underlying tree!



$$\begin{aligned}\langle \mu y.\underline{f}\{\mu a.\underline{g}\{a,b,c,d\},y,z,w\} \,|\, \mu p.\underline{h}\{p,q\}\rangle &= \underline{f}\{\mu a.\underline{g}\{a,b,c,d\},y,z,w\}[\mu p.\underline{h}\{p,q\}/y] \\ &= \underline{f}\{\mu a.\underline{g}\{a,b,c,d\},\mu p.\underline{h}\{p,q\},z,w\}\end{aligned}$$

μ-syntax as a rewriting system

Non-confluent - critical pairs arise from the second equation viewed as a rewriting rule:

$$c_2[\mu x.c_1/y] \longleftarrow \langle \mu x.c_1 \mid \mu y.c_2 \rangle \longrightarrow c_1[\mu y.c_2/x]$$

Terminating (modulo the commutativity of $\langle s \mid t \rangle$) - the set NF of normal forms consists of terms produced only with the following rules:

$x \in NF$
if $f \in \mathcal{C}(X)$ and $t_x \in NF$ for all $x \in X$, then $\underline{f}\{t_x \mid x \in X\} \in NF$
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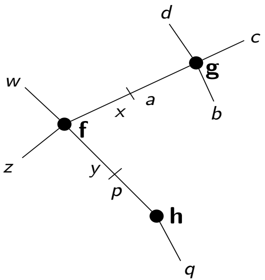

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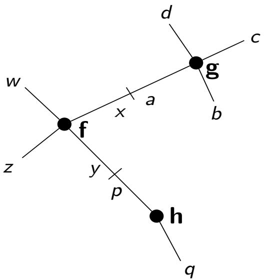

$\underline{f}\{\mu a.\underline{g}\{a,b,c,d\}, \mu p.\underline{h}\{p,q\}, z, w\}$
$\underline{g}\{\mu x.\underline{f}\{x, \mu p.\underline{h}\{p,q\}, z, w\}, b, c, d\}$
$\underline{g}\{\mu x.\underline{h}\{\mu y.\underline{f}\{x,y,z,w\}, q\}, b, c, d\}$
$\underline{h}\{\mu y.\underline{f}\{\mu a.\underline{g}\{a,b,c,d\}, y, z, w\}, p\}$
$\underline{h}\{\mu y.\underline{g}\{\mu x.\underline{f}\{x,y,z,w\}, b, c, d\}, p\}$

The list shows different **parallel tree traversals** of the tree.

μ-syntax does the job!

A cyclic operad can equivalently be characterised via the monad of

trees with half-edges (Markl) $\overset{\text{T(a)}}{\Longleftrightarrow}$ Vernon trees $\overset{\text{T(b)}}{\Longleftrightarrow}$ μ-syntax

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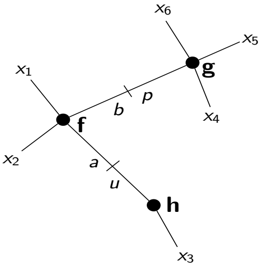

$T = (\{a,b,c,d,p,q,r,s,u,v\},\lambda,\sigma)$,
$\lambda = \{\{a,b,c,d\},\{p,q,r,s\},\{u,v\}\}$, $\sigma = (a\ p)(b\ u)$

$+$ labeling $l : Leaves(T) \to X$

$+\ f \in \mathcal{C}(\{a,b,c,d\}),\ g \in \mathcal{C}(\{p,q,r,s\}),\ h \in \mathcal{C}(\{u,v\})$

$\mathcal{T} = (f(a,b,x_1,x_2), g(p,x_4,x_5,x_6), h(u,x_3);\sigma)$,
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$\underline{f}\{\mu u.\underline{h}\{u,x_3\},\mu p.\underline{g}\{p,x_4,x_5,x_6\},x_1,x_2\}$

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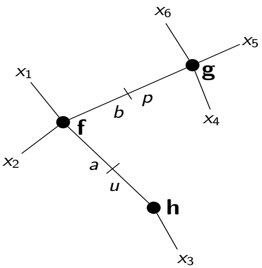

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Theorem

a) $\mathtt{Tree}^{\mathrm{he}}_{\mathcal{C}}(X)/_{\sim} \simeq \mathtt{VT}_{\mathcal{C}}(X)/_{\alpha}$

b) $\mathtt{Comm}_{\mu}(X)/_{=_\mu} \simeq \mathtt{VT}_{\mathcal{C}}(X)/_{\alpha}$ $\longleftarrow$ discussed in the next slide

An insight in the proof of $\mathtt{Comm}_\mu(X)/_{=_\mu} \simeq \mathtt{VT}_{\mathcal{C}}(X)/_\alpha$

After defining an *interpretation* $[\![\_]\!] : \mathtt{Exp}_\mu \to \mathtt{Term}_c$ *of the* μ*-syntax into an arbitrary cyclic operad*, and exibiting the *cyclic operad structure of Vernon trees*, we associate with every command c a Vernon tree $VT(c)$, as an instance of this interpretation.

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The equality relation $=_\mu$ lives in the set of *all* commands.
We introduce an equality $='$ that relates *normal forms only*:

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$$c_1 =_\mu c_2 \;\Rightarrow\; VT(c_1) = VT(c_2) \;\Rightarrow\; c_1 =' c_2 \;\Rightarrow\; c_1 =_\mu c_2$$

Algebraic

(microcosm principle)

Classical	Partial
May, Smirnov, Kelly	*Fiore*
Exchangable output	**Entries only**
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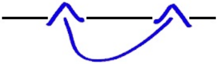

Algebraic environment: Species of structures (Joyal)

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$$\text{The category of species } \mathbf{Spec} := \mathbf{Set}^{\mathbf{Bij}^{op}}$$

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The *substitution product of species S and T* (with $S(\emptyset) = \emptyset$):

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The *derivative of a species S*:

$$\partial S(X) = S(X + \{*\})$$

Symmetric operads: classical, algebraic

How does an element of $(T \circ S)(X)$ *look like?*

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- It has the species of singletons I as neutral element.

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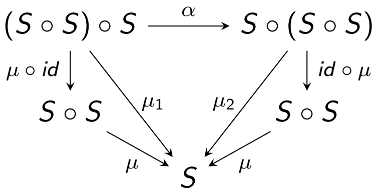
Definition 1

An operad is a monoid $(S,\, \mu : S \circ S \to S)$ in the category of species.



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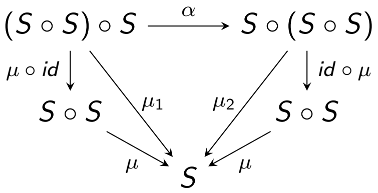

Specifying a *monoid in a monoidal category* is a typical instance of the *microcosm principle* of higher algebra (Baez-Dolan).

Symmetric operads: partial, algebraic (Fiore)

The product on species needed to formulate partial composition algebraically is

$$T * S := (\partial T) \cdot S.$$

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Definition 2

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From ordinary to cyclic operads

Our cornerstone is an ordinary operad $(S, \nu : S * S \to S)$.

How to enrich this structure, so that it encompasses the actions

$$D_{xz} : S(X) \to S((X \backslash \{x\}) \cup \{z\})?$$

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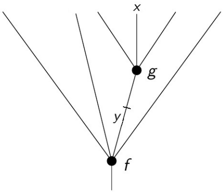
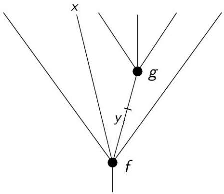


$\nu_3 : \partial S \cdot \partial S \to \partial S$

and



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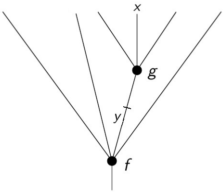

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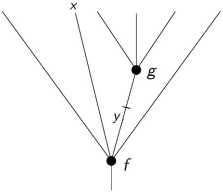

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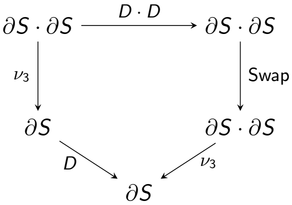
This suggests to mimick the actions of D_{xz} by a natural transformation $D : \partial S \to \partial S$, which will commute with ν_3 and ν_4 in the appropriate way.

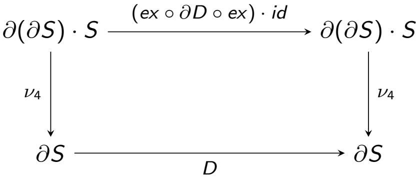
Cyclic operads: exchangeable output

Definition 3

A cyclic operad is a triple $(S, \nu : S * S \to S, D : \partial S \to \partial S)$, such that:

- $(S, \nu : S * S \to S)$ is an operad, and
- the morphism D is satisfies the laws
$$D^2 = id \text{ and } (\partial D \circ ex)^3 = id,$$
where $ex : \partial(\partial S) \to \partial(\partial S)$ exchanges the two distinguished elements in $(X + \{*\}) + \{\diamond\}$,
- as well as the laws given by the following commuting diagrams:





Cyclic operads: entries only

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Following the microcosm principle, we examine its properties:

- It is commutative
- It satisfies the identity given by the isomorphism

$$\gamma : (S \blacktriangle T) \blacktriangle U + T \blacktriangle (S \blacktriangle U) + (T \blacktriangle U) \blacktriangle S \rightarrow S \blacktriangle (T \blacktriangle U) + (S \blacktriangle U) \blacktriangle T + U \blacktriangle (S \blacktriangle T)$$

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Definition 4

A cyclic operad is a pair $(S, \rho : S\blacktriangle S \rightarrow S)$, such that $\rho_2 \circ \gamma = \rho_1$, where ρ_1 and ρ_2 are induced by ρ.

Descent theory for species (Lamarche)

Question:

Can we reconstruct a species T, given ∂T?

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The answer is **no**, but it becomes **yes** if additional data is provided.

Lamarche (private communication) defines a descent data as a pair

$$(S, D : \partial S \to \partial S),$$

such that $D^2 = id$ and $(\partial D \circ ex)^3 = id$.

Lamarche has also shown that the functor

$$T \mapsto (\partial T, ex)$$

is an equivalence of categories when restricted to species which are empty on the empty set.

“exchangeable output” $\Leftrightarrow$ “entries only”

Theorem 2

This equivalence carries over to an equivalence between the previous two definitions of cyclic operads.

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The steps of the proof are as follows:

- Given $\rho : T\blacktriangle T \to T$, we define $\nu : \partial T * \partial T \to \partial T$ as follows

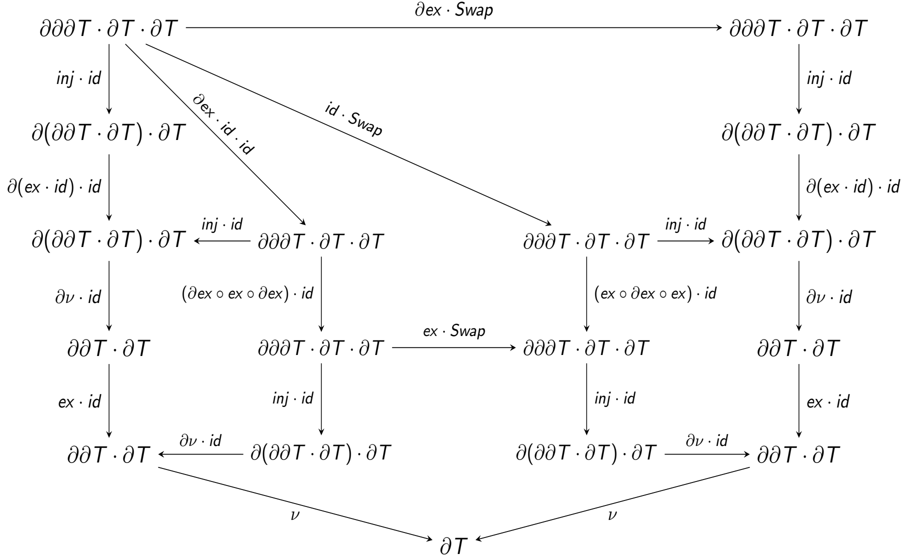
$$\nu : \partial\partial T \cdot \partial T \xrightarrow{ex\cdot id} \partial\partial T \cdot \partial T \xrightarrow{inj} \partial(\partial T \cdot \partial T) \xrightarrow{\partial\rho} \partial T$$

- Given $\nu : \partial T * \partial T \to \partial T$, to define $\rho : T\blacktriangle T \to T$ amounts to define $\rho' : \partial(\partial T \cdot \partial T) \to \partial T$ (ρ is then uniquely defined via $\partial\rho = \rho'$):

$$\rho'_1 : \partial\partial T \cdot \partial T \xrightarrow{ex\cdot id} \partial\partial T \cdot \partial T \xrightarrow{\nu} \partial T$$
$$\rho'_2 : \partial T \cdot \partial\partial T \xrightarrow{Swap} \partial\partial T \cdot \partial T \xrightarrow{\rho'_1} \partial T$$
$$\rho' = [\rho'_1, \rho'_2]$$

- And then we have to check that everything commutes in the appropriate way...

...for example: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ν
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Some directions of investigations:

- Figure out the microcosm principle for modular and wheeled operads
- Study the “up to homotopy” versions of all these (infinity operads, properads, wheeled properads,...)

Hope that you enjoyed the promenade !