On the various definitions of cyclic operads

Logic and Applications 2015, Dubrovnik

Pierre-Louis Curien and Jovana Obradović*

πr² team, PPS Laboratory, CNRS, Université Paris Diderot and Inria, France
*Joint PhD studies with Universiy of Novi Sad, Serbia

September 22, 2015
Overview: different definitions of (cyclic) operads

**Symmetric operad** = operations
+ (associative) compositions
+ permutation of variables (+ identities)
Overview: different definitions of (cyclic) operads

**Symmetric operad** = operations
+ (associative) compositions
+ permutation of variables (+ identities)

<table>
<thead>
<tr>
<th><strong>Biased</strong> (individual compositions)</th>
<th><strong>Unbiased</strong> (monad of trees)</th>
<th><strong>Algebraic</strong> (microcosm principle)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Symmetric Operads</strong></td>
<td><strong>Classical</strong></td>
<td><strong>Partial</strong></td>
</tr>
<tr>
<td>Boardman, Vogt, May</td>
<td>Markl</td>
<td>Smirnov, May, Getzler, Jones</td>
</tr>
<tr>
<td><strong>Cyclic Operads</strong></td>
<td>Getzler, Kapranov</td>
<td>Getzler, Kapranov</td>
</tr>
<tr>
<td>Exchangeable output</td>
<td>Entries only</td>
<td>Exchangable output</td>
</tr>
<tr>
<td>Markl</td>
<td>Markl</td>
<td>Markl</td>
</tr>
</tbody>
</table>

+ Two flavours: **skeletal** and **non-skeletal**
Overview: different definitions of (cyclic) operads

**Symmetric operad** = operations
+ (associative) compositions
+ permutation of variables (+ identities)

<table>
<thead>
<tr>
<th>Biased (individual compositions)</th>
<th>Unbiased (monad of trees)</th>
<th>Algebraic (microcosm principle)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classical</td>
<td>Partial</td>
<td>Classical</td>
</tr>
<tr>
<td>Boardman, Vogt, May</td>
<td>Markl</td>
<td>Markl</td>
</tr>
<tr>
<td>Smirnov, May, Getzler, Jones</td>
<td></td>
<td>May, Smirnov, Kelly</td>
</tr>
<tr>
<td>Getzler, Kapranov</td>
<td>Exchangeable output</td>
<td>Exchangeable output</td>
</tr>
<tr>
<td></td>
<td>Entries only</td>
<td>Entries only</td>
</tr>
<tr>
<td></td>
<td>Markl</td>
<td>Markl</td>
</tr>
<tr>
<td></td>
<td>Getzler, Kapranov</td>
<td></td>
</tr>
<tr>
<td>+ Two flavours: skeletal and non-skeletal</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Plan:
- Examine these definitions
- Introduce a $\lambda$-calculus-style syntax: the $\mu$-syntax
- Fill in the question marks
Cyclic operads revisited

The $\mu$-syntax

Microcosm principle for cyclic operads

Symmetric operads

### Classical + Skeletal

- $\mathcal{P} : \Sigma^{op} \to \mathcal{C}$
- $\gamma : \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \cdots \otimes \mathcal{P}(k_n) \to \mathcal{P}(k_1 + \cdots + k_n)$
- $\eta : 1 \to \mathcal{P}(1)$

### Partial + Non-skeletal

- $\mathcal{S} : \text{Bij}^{op} \to \mathcal{C}$
- $\circ \mathcal{S}(X) \times \mathcal{S}(Y) \to \mathcal{S}((X \cup Y) \setminus \{x\})$
- $id_x \in \mathcal{S}(\{x\})$

### Unbiased

An operad is an algebra over the monad of rooted, decorated, labeled trees (which constitute the category $\text{Tree}_n$).
Exchangeable output: from ordinary to cyclic operads

Ordinary operads

an action of relabeling the leaves of a rooted tree

Cyclic operads

an action of interchanging the labels of all leaves of a rooted tree, including the label given to the root
Exchangeable output: from ordinary to cyclic operads

Ordinary operads → Cyclic operads

an action of relabeling the leaves
of a rooted tree → an action of interchanging the
labels of all leaves of a rooted
tree, including the label given to
the root

This is achieved by enriching the operad structure with the
action of the cycle $\tau_n = (0, 1, \ldots, n)$:

The distinction between inputs and the output of an operation is no
longer visible...
Definition 1 (Partial + non-skeletal)

A cyclic operad is a functor $\mathcal{C} : \text{Bij}^{op} \to \text{Set}$, together with a distinguished element $id_{x,y} \in \mathcal{C}(\{x, y\})$ for each two-element set $\{x, y\}$, and a partial composition operation

$$x^o y : \mathcal{C}(X) \times \mathcal{C}(Y) \to \mathcal{C}((X \cup Y) \setminus \{x, y\}).$$

These data are required to satisfy the associativity, equivariance, unitality and commutativity equations.

**Associativity.**

$$(f \circ_y g) \circ_z h = f \circ_y (g \circ_z h)$$

$$(f \circ_y g) \circ_z h = (f \circ_z h) \circ_y g$$

**Equivariance.**

$$f^\sigma_1 \circ_y g^\sigma_2 = (f^\sigma_1(x) \circ \sigma_2(y) \circ g)^\sigma$$

**Unitality.**

$$f \circ_y id_{y,z} = f^\sigma$$

$$id_{y,z} \circ_x f = f^\sigma$$

**Commutativity.**

$$f \circ_y g = g \circ_x f.$$ 

This definition induces a natural combinator syntax.
Cyclic operads revisited

Cyclic operads: exchangeable output

Definition 2 (Partial + non-skeletal)

A cyclic operad is an ordinary operad \( S \), augmented with actions

\[
D_{xy} : S(X) \rightarrow S((X\backslash\{x\}) \cup \{y\}),
\]

indexed by variables \( x \in X \) and \( y \notin X\backslash\{x\} \), and subject to the following list of axioms:

**Identity.**
\[
D_{xx}(f) = f
\]

**Coherence.**
\[
D_{zx}(D_{xy}(f)) = D_{zy}(f)
\]

**\( \alpha \)-conversion.**
\[
D_{xa}(f) = D_{x’a}(f^\sigma),
\]

where \( \sigma(x) = x’ \), and \( \sigma = id \) elsewhere

**Equivariance.**
\[
D_{\sigma(x)\sigma(y)}(f^\sigma) = D_{xy}(f)^\sigma
\]

**Compatibility with compositions.**
\[
D_{xz}(f \circ_y g) = D_{xz}(f) \circ_y D_{yz}(g)
\]
\[
D_{xz}(f \circ_y g) = D_{xz}(f \circ_y g)
\]
Cyclic operads revisited

Cyclic operads: unbiased definition

The *entries-only* characterization of cyclic operads reflects the ability to carry out the (partial) composition of two operations along *any* edge.

The pasting schemes for cyclic operads are unrooted, decorated, labeled trees.
Cyclic operads revisited

The entries-only characterization of cyclic operads reflects the ability to carry out the (partial) composition of two operations along any edge.

The pasting schemes for cyclic operads are unrooted, decorated, labeled trees.

Given a functor \( P : \text{Bij}^{\text{op}} \to C \), we build the free operad \( F(P) \) by grafting of such trees. The free operad functor \( F \) and the forgetful functor \( U \) constitute a monad \( \Gamma = UF \) in \( C^{\text{Bij}^{\text{op}}} \), called the monad of unrooted trees.
Cyclic operads revisited

The entries-only characterization of cyclic operads reflects the ability to carry out the (partial) composition of two operations along any edge.

The pasting schemes for cyclic operads are unrooted, decorated, labeled trees.

Given a functor $\mathcal{P} : \text{Bij}^{op} \to \mathbf{C}$, we build the free operad $F(\mathcal{P})$ by grafting of such trees. The free operad functor $F$ and the forgetful functor $U$ constitute a monad $\Gamma = UF$ in $\mathbf{C}^{\text{Bij}^{op}}$, called the monad of unrooted trees.

**Definition 3**

A cyclic operad is an algebra over this monad.
<table>
<thead>
<tr>
<th></th>
<th>Biased (individual compositions)</th>
<th>Unbiased (monad of trees)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Classical</td>
<td>Partial</td>
</tr>
<tr>
<td>Symmetric Operads</td>
<td>Boardman, Vogt, May</td>
<td>Markl</td>
</tr>
<tr>
<td></td>
<td>Smirnov, May, Getzler, Jones</td>
<td></td>
</tr>
<tr>
<td>Cyclic Operads</td>
<td>Getzler, Kapranov</td>
<td>Exchangable output</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Entries only</td>
</tr>
<tr>
<td></td>
<td>Markl</td>
<td>Markl</td>
</tr>
<tr>
<td></td>
<td>Getzler, Kapranov</td>
<td></td>
</tr>
</tbody>
</table>

$\mu$-syntax
The \(\mu\)-syntax consists of two kinds of expressions:

- **commands** \(c : X\) (no entry selected)
  
  \[
  c ::= \langle s \mid t \rangle \mid f\{t_x \mid x \in X\}
  \]

- **terms** \(X \mid s\) (one entry selected)
  
  \[
  s, t ::= x \mid \mu x.c
  \]

The **typing rules** are as follows:

\[
\begin{align*}
\{x\} \mid x & \quad \frac{f \in S(X)}{f\{t_x \mid x \in X\} : \bigcup Y_x} \quad \frac{X \mid s \quad Y \mid t}{\langle s \mid t \rangle : X \cup Y} \\
\end{align*}
\]

The **equations** are \(\langle s \mid t \rangle = \langle t \mid s \rangle\) and (oriented from left to right):

\[
\langle \mu x.c \mid s \rangle = c[s/x] \quad (\mu\text{-reduction})
\]
Cyclic operads revisited

The $\mu$-syntax

$\mu$-syntax: intuition

$\langle \mu x.c \mid s \rangle$ and $c[s/x]$ describe two ways to build the same underlying tree!

$\langle \mu y.f\{\mu a.g\{a, b, c, d\}, y, z, w\} \mid \mu p.h\{p, q\} \rangle = f\{\mu a.g\{a, b, c, d\}, y, z, w\}[\mu p.h\{p, q\}/y]$
$= f\{\mu a.g\{a, b, c, d\}, \mu p.h\{p, q\}, z, w\}$
\[\mu\text{-syntax as a rewriting system}\]

**Non-confluent** - critical pairs arise from the second equation viewed as a rewriting rule:

\[
c_2[\mu x. c_1/y] \longleftarrow \langle \mu x. c_1 | \mu y. c_2 \rangle \longrightarrow c_1[\mu y. c_2/x]
\]

**Terminating** (modulo the commutativity of \(\langle s \mid t \rangle\)) - the set \(NF\) of normal forms consists of terms produced only with the following rules:

- \(x \in NF\)
- if \(f \in C(X)\) and \(t_x \in NF\) for all \(x \in X\), then \(f\{t_x \mid x \in X\} \in NF\)
- if \(c \in NF\), then \(\mu x.c \in NF\)
**μ-syntact as a rewriting system**

**Non-confluent** - critical pairs arise from the second equation viewed as a rewriting rule:

\[
c_2[\mu x . c_1 / y] \leftrightarrow \langle \mu x . c_1 \mid \mu y . c_2 \rangle \rightarrow c_1[\mu y . c_2 / x]
\]

**Terminating** (modulo the commutativity of \( \langle s \mid t \rangle \)) - the set \( NF \) of normal forms consists of terms produced only with the following rules:

- \( x \in NF \)
- if \( f \in \mathcal{C}(X) \) and \( t_x \in NF \) for all \( x \in X \), then \( f\{ t_x \mid x \in X \} \in NF \)
- if \( c \in NF \), then \( \mu_x . c \in NF \)
**μ-syntax as a rewriting system**

Non-confluent - critical pairs arise from the second equation viewed as a rewriting rule:

\[ c_2[\mu x.c_1/y] \leftarrow \langle \mu x.c_1 | \mu y.c_2 \rangle \rightarrow c_1[\mu y.c_2/x] \]

Terminating (modulo the commutativity of \( \langle s | t \rangle \)) - the set \( NF \) of normal forms consists of terms produced only with the following rules:

- \( x \in NF \)
- if \( f \in \mathbb{C}(X) \) and \( t_x \in NF \) for all \( x \in X \), then \( f\{t_x \mid x \in X\} \in NF \)
- if \( c \in NF \), then \( \mu x.c \in NF \)

The list shows different **parallel tree traversals** of the tree.
Cyclic operads revisited

The $\mu$-syntax

$\mu$-syntax does the job!

A cyclic operad can equivalently be characterised via the monad of
trees with half-edges (Markl) \( T(a) \) Vernon trees \( T(b) \) $\mu$-syntax
μ-syntax does the job!

A cyclic operad can equivalently be characterised via the monad of trees with half-edges (Markl) \( T(a) \overset{T(b)}{\leftrightarrow} \) Vernon trees \( \overset{\mu-syntax}{\leftrightarrow} \)

\[
T = (\{a, b, c, d, p, q, r, s, u, v\}, \lambda, \sigma), \\
\lambda = \{\{a, b, c, d\}, \{p, q, r, s\}, \{u, v\}\}, \\
\sigma = (a p)(b u) \\
+ \text{labeling } l : \text{Leaves}(T) \rightarrow X \\
+ f \in C(\{a, b, c, d\}), \ g \in C(\{p, q, r, s\}), \ h \in C(\{u, v\}) \\
\mathcal{T} = (f(a, b, x_1, x_2), g(p, x_4, x_5, x_6), h(u, x_3); \sigma), \\
\sigma = (a p)(b u) \\
f\{\mu u.h\{u, x_3\}, \mu p.g\{p, x_4, x_5, x_6\}, x_1, x_2\}
\]
A cyclic operad can equivalently be characterised via the monad of trees with half-edges (Markl) \(\mu\)-syntax  \(\equiv\) Vernon trees  \(\mu\)-syntax does the job!

\[
T = (\{a, b, c, d, p, q, r, s, u, v\}, \lambda, \sigma), \\
\lambda = \{\{a, b, c, d\}, \{p, q, r, s\}, \{u, v\}\}, \\
\sigma = (a p)(b u) \\
\text{+ labeling } l : \text{Leaves}(T) \rightarrow X \\
\text{+ } f \in C(\{a, b, c, d\}), \ g \in C(\{p, q, r, s\}), \ h \in C(\{u, v\})
\]

\[
\mathcal{T} = (f(a, b, x_1, x_2), g(p, x_4, x_5, x_6), h(u, x_3); \sigma), \\
\sigma = (a p)(b u) \\
f\{\mu u. h\{u, x_3\}, \mu p. g\{p, x_4, x_5, x_6\}, x_1, x_2\}
\]

**Theorem**

a) \(\text{Tree}_{\mathcal{C}}^{\text{he}}(X)/\sim \simeq VT_{\mathcal{C}}(X)/\alpha\)

b) \(\text{Comm}_{\mu}(X)/=_{\mu} \simeq VT_{\mathcal{C}}(X)/\alpha\)  \(\leftarrow\) discussed in the next slide
An insight in the proof of $\text{Comm}_\mu(X)\!/=_\mu \simeq \text{VT}_c(X)\!/\alpha$

After defining an *interpretation* $[[\_]] : \text{Exp}_\mu \to \text{Term}_c$ of the $\mu$-syntax into an arbitrary cyclic operad, and exhibiting the *cyclic operad structure of Vernon trees*, we associate with every command $c$ a Vernon tree $VT(c)$, as an instance of this interpretation.
An insight in the proof of $\text{Comm}_\mu(X)/\equiv_\mu \simeq VT_c(X)/\alpha$

After defining an interpretation $[[\_]] : \text{Exp}_\mu \to \text{Term}_c$ of the $\mu$-syntax into an arbitrary cyclic operad, and exhibiting the cyclic operad structure of Vernon trees, we associate with every command $c$ a Vernon tree $VT(c)$, as an instance of this interpretation.

Inj If for normal forms $c_1$ and $c_2$ we have $VT(c_1) = VT(c_2)$, then $c_1 =_\mu c_2$. 
An insight in the proof of \( \text{Comm}_\mu(X) / \equiv_{\mu} \simeq \text{VT}_c(X) / \alpha \)

After defining an interpretation \([[-]] : \text{Exp}_\mu \to \text{Term}_c\) of the \(\mu\)-syntax into an arbitrary cyclic operad, and exhibiting the cyclic operad structure of Vernon trees, we associate with every command \(c\) a Vernon tree \(\text{VT}(c)\), as an instance of this interpretation.

If for normal forms \(c_1\) and \(c_2\) we have \(\text{VT}(c_1) = \text{VT}(c_2)\), then \(c_1 =_{\mu} c_2\).

The equality relation \(=_{\mu}\) lives in the set of all commands. We introduce an equality \(=\)' that relates normal forms only:

\[
\text{if } \sigma(x) = \mu y.c, \text{ then } f\{\sigma\} =\' c[\mu x.f\{\sigma[x/x]\}]/y
\]

\[
\red{f}\{\mu y.g\{y, p, q, r, s\}, a, b, c\} =\' \red{g}\{\mu x.f\{x, a, b, c\}, p, q, r, s\}
\]
An insight in the proof of $\text{Comm}_\mu(X)/=\mu \simeq VT_c(X)/\alpha$

After defining an interpretation $[[\_]] : \text{Exp}_\mu \to \text{Term}_c$ of the $\mu$-syntax into an arbitrary cyclic operad, and exhibiting the cyclic operad structure of Vernon trees, we associate with every command $c$ a Vernon tree $VT(c)$, as an instance of this interpretation.

If for normal forms $c_1$ and $c_2$ we have $VT(c_1) = VT(c_2)$, then $c_1 =\mu c_2$.

The equality relation $=\mu$ lives in the set of all commands. We introduce an equality $='$ that relates normal forms only:

If $\sigma(x) = \mu y.c$, then $f\{\sigma\} =' c[\mu x.f\{\sigma[x/x]\}] / y$

$$f\{\mu y.g\{y, p, q, r, s\}, a, b, c\} =' g\{\mu x.f\{x, a, b, c\}, p, q, r, s\}$$

$c_1 =\mu c_2 \Rightarrow VT(c_1) = VT(c_2) \Rightarrow c_1 =' c_2 \Rightarrow c_1 =\mu c_2$
### Algebraic
(microcosm principle)

<table>
<thead>
<tr>
<th>Classical</th>
<th>Partial</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>May, Smirnov, Kelly</em></td>
<td><em>Fiore</em></td>
</tr>
<tr>
<td>Exchangable output</td>
<td>Entries only</td>
</tr>
<tr>
<td>??</td>
<td>??</td>
</tr>
</tbody>
</table>
Algebraic environment: **Species of structures (Joyal)**

$\mathcal{S} : \text{Bij}^{op} \to \text{Set}$ is a contravariant version of Joyal's species of structures!

The category of species $\text{Spec} := \text{Set}^{\text{Bij}^{op}}$

*The operadic composition in this context is recognized by examining the properties of basic operations on species:*
Algebraic environment: **Species of structures (Joyal)**

\( S : \text{Bij}^{op} \rightarrow \text{Set} \) is a contravariant version of Joyal's species of structures!

The category of species \( \text{Spec} := \text{Set}^{\text{Bij}^{op}} \)

*The operadic composition in this context is recognized by examining the properties of basic operations on species:*

The *sum of species \( S \) and \( T \):*

\[
(S + T)(X) := S(X) + T(X)
\]
Algebraic environment: Species of structures (Joyal)

$S : \text{Bij}^{op} \to \text{Set}$ is a contravariant version of Joyal’s species of structures!

The category of species $\text{Spec} := \text{Set}^{\text{Bij}^{op}}$

The operadic composition in this context is recognized by examining the properties of basic operations on species:

The sum of species $S$ and $T$:

$$(S + T)(X) := S(X) + T(X)$$

The product of species $S$ and $T$:

$$(S \cdot T)(X) := \sum_{(X_1, X_2)} S(X_1) \times T(X_2)$$
Algebraic environment: Species of structures (Joyal)

$S : \textbf{Bij}^{op} \to \textbf{Set}$ is a contravariant version of Joyal's species of structures!

The category of species $\textbf{Spec} := \textbf{Set}^{\textbf{Bij}^{op}}$

The operadic composition in this context is recognized by examining the properties of basic operations on species:

The sum of species $S$ and $T$:

$$(S + T)(X) := S(X) + T(X)$$

The product of species $S$ and $T$:

$$(S \cdot T)(X) := \sum_{(X_1, X_2)} S(X_1) \times T(X_2)$$

The substitution product of species $S$ and $T$ (with $S(\emptyset) = \emptyset$):

$$(S \circ T)(X) = \sum_{\pi \in \mathcal{P}(X)} S(\pi) \times \prod_{p \in \pi} T(p)$$
Algebraic environment: Species of structures (Joyal)

$S : \text{Bij}^{\text{op}} \rightarrow \textbf{Set}$ is a contravariant version of Joyal’s species of structures!

The category of species $\textbf{Spec} := \text{Set}^{\text{Bij}^{\text{op}}}$

The operadic composition in this context is recognized by examining the properties of basic operations on species:

- **The sum of species $S$ and $T$:**
  
  $(S + T)(X) := S(X) + T(X)$

- **The product of species $S$ and $T$:**
  
  $(S \cdot T)(X) := \sum_{(X_1, X_2)} S(X_1) \times T(X_2)$

- **The substitution product of species $S$ and $T$ (with $S(\emptyset) = \emptyset$):**
  
  $(S \circ T)(X) = \sum_{\pi \in P(X)} S(\pi) \times \prod_{p \in \pi} T(p)$

- **The derivative of a species $S$:**
  
  $\partial S(X) = S(X + \{\star\})$
Symmetric operads: classical, algebraic

How does an element of \((T \circ S)(X)\) look like?

\[
g\{f_y\{x \mid x \in X_y\} \mid y \in Y\}
\]
Symmetric operads: classical, algebraic

How does an element of \((T \circ S)(X)\) look like?

\[ g\{f_y\{x \mid x \in X_y\} \mid y \in Y\} \]

And what are the properties of the substitution product?

- It is associative (up to isomorphism of species)
- It has the species of singletons \(I\) as neutral element.

\[ \rightarrow (\text{Spec}, \circ) \text{ is a monoidal category (with unit } I\). \]
Symmetric operads: classical, algebraic

How does an element of \((T \circ S)(X)\) look like?

\[
g\{f_y \{x \mid x \in X_y\} \mid y \in Y\}
\]

And what are the properties of the substitution product?

- It is associative (up to isomorphism of species)
- It has the species of singletons \(I\) as neutral element.

\[\rightarrow (\text{Spec}, \circ)\] is a monoidal category (with unit \(I\)).

Definition 1

An operad is a monoid \((S, \mu : S \circ S \rightarrow S)\) in the category of species.
Symmetric operads: classical, algebraic

How does an element of \((T \circ S)(X)\) look like?

\[
g\{ f_y \{ x \mid x \in X_y \} \mid y \in Y \}
\]

And what are the properties of the substitution product?

- It is associative (up to isomorphism of species)
- It has the species of singletons \(I\) as neutral element.

\[\rightarrow (\text{Spec}, \circ)\] is a monoidal category (with unit \(I\)).

**Definition 1**

An operad is a monoid \((S, \mu : S \circ S \rightarrow S)\) in the category of species.
Symmetric operads: partial, algebraic (Fiore)

The product on species needed to formulate partial composition algebraically is

\[ T \ast S := (\partial T) \cdot S. \]

**Microcosm principle:** *What are the properties of this product?*
Symmetric operads: partial, algebraic (Fiore)

The product on species needed to formulate partial composition algebraically is

\[ T \ast S := (\partial T) \cdot S. \]

**Microcosm principle:** *What are the properties of this product?*

Comparing the species \( U \ast (T \ast S) \) and \( (U \ast T) \ast S \), we conclude that

- The product is **not associative**
- The product satisfies the **pre-Lie equality**, given by the isomorphism

\[ \beta : ((U \ast T) \ast S) + (U \ast (S \ast T)) \rightarrow (U \ast (T \ast S)) + ((U \ast S) \ast T), \]
Symmetric operads: partial, algebraic (Fiore)

The product on species needed to formulate partial composition algebraically is

\[ T \ast S := (\partial T) \cdot S. \]

**Microcosm principle:** What are the properties of this product?

Comparing the species \( U \ast (T \ast S) \) and \( (U \ast T) \ast S \), we conclude that

- The product is not associative
- The product satisfies the pre-Lie equality, given by the isomorphism

\[
\beta : ((U \ast T) \ast S) + (U \ast (S \ast T)) \to (U \ast (T \ast S)) + ((U \ast S) \ast T),
\]

**Definition 2**

An operad is a pair \((S, \nu : S \ast S \to S)\), such that \(\nu_2 \circ \beta = \nu_1\), where \(\nu_1\) and \(\nu_2\) are induced by \(\nu\).
From ordinary to cyclic operads

Our cornerstone is an ordinary operad \((S, \nu : S \ast S \to S)\).

*How to enrich this structure, so that it encompasses the actions* 

\[ D_{xz} : S(X) \to S((X \setminus \{x\}) \cup \{z\})? \]
From ordinary to cyclic operads

Our cornerstone is an ordinary operad \((S, \nu : S \ast S \to S)\). 

*How to enrich this structure, so that it encompasses the actions* 

\[ D_{xz} : S(X) \to S((X\{x\}) \cup \{z\}) \] 

In particular, we should translate the compatibility of \(D_{xz}\) with the two possible partial compositions subject to it:

\[ \nu_3 : \partial S \cdot \partial S \to \partial S \]

and

\[ \nu_4 : \partial(\partial S) \cdot S \to \partial S \]
From ordinary to cyclic operads

Our cornerstone is an ordinary operad \((S, \nu : S \ast S \to S)\).

*How to enrich this structure, so that it encompasses the actions*

\[ D_{xz} : S(X) \to S((X\setminus\{x\}) \cup \{z\})? \]

In particular, we should translate the compatibility of \(D_{xz}\) with the two possible partial compositions subject to it:

\[ \nu_3 : \partial S \cdot \partial S \to \partial S \quad \text{and} \quad \nu_4 : \partial(\partial S) \cdot S \to \partial S \]

_This suggests to mimick the actions of \(D_{xz}\) by a natural transformation \(D : \partial S \to \partial S\), which will commute with \(\nu_3\) and \(\nu_4\) in the appropriate way._
Cyclic operads: exchangeable output

Definition 3

A cyclic operad is a triple \((S, \nu : S \star S \to S, D : \partial S \to \partial S)\), such that:

- \((S, \nu : S \star S \to S)\) is an operad, and
- the morphism \(D\) is satisfies the laws
  \[ D^2 = id \text{ and } (\partial D \circ ex)^3 = id, \]
  where \(ex : \partial(\partial S) \to \partial(\partial S)\) exchanges the two distinguished elements in \((X + \{\ast\}) + \{\diamond\},\)
- as well as the laws given by the following commuting diagrams:

\[
\begin{array}{ccc}
\partial S \cdot \partial S & \xrightarrow{D \cdot D} & \partial S \cdot \partial S \\
\downarrow \nu_3 & & \downarrow \text{Swap} \\
\partial S & \xrightarrow{\partial S \cdot \partial S} & \partial S \cdot \partial S \\
\downarrow D & & \downarrow \nu_3 \\
\partial S & \xrightarrow{\partial S} & \partial S \\
\end{array}
\]

\[
\begin{array}{ccc}
\partial(\partial S) \cdot S & \xrightarrow{(ex \circ \partial D \circ ex) \cdot id} & \partial(\partial S) \cdot S \\
\downarrow \nu_4 & & \downarrow \nu_4 \\
\partial S & \xrightarrow{\partial(\partial S) \cdot S} & \partial S \\
\downarrow D & & \downarrow D \\
\partial S & \xrightarrow{\partial S} & \partial S \\
\end{array}
\]
Cyclic operads: entries only

The relevant product on species is now

\[ S \triangle T = (\partial S) \cdot (\partial T). \]
Cyclic operads revisited

The relevant product on species is now

\[ S △ T = (\partial S) \cdot (\partial T). \]

Following the microcosm principle, we examine its properties:

- It is commutative
- It satisfies the identity given by the isomorphism

\[ \gamma : (S △ T) △ U + T △ (S △ U) + (T △ U) △ S \rightarrow S △ (T △ U) + (S △ U) △ T + U △ (S △ T) \]
Cyclic operads revisited

The \( \mu \)-syntax

Microcosm principle for cyclic operads

Cyclic operads: entries only

The relevant product on species is now

\[
S \triangleright T = (\partial S) \cdot (\partial T).
\]

Following the **microcosm principle**, we examine its properties:

- It is commutative
- It satisfies the identity given by the isomorphism

\[
\gamma : (S \triangleright T) \triangleright U + T \triangleright (S \triangleright U) + (T \triangleright U) \triangleright S \to S \triangleright (T \triangleright U) + (S \triangleright U) \triangleright T + U \triangleright (S \triangleright T)
\]

**Definition 4**

A **cyclic operad** is a pair \((S, \rho : S \triangleright S \to S)\), such that \(\rho_2 \circ \gamma = \rho_1\), where \(\rho_1\) and \(\rho_2\) are induced by \(\rho\).
Question:

Can we reconstruct a species $T$, given $\partial T$?
Descent theory for species (Lamarche)

**Question:**

*Can we reconstruct a species $T$, given $\partial T$?*

The answer is **no**, but it becomes **yes** if additional data is provided. Lamarche (private communication) defines a **descent data** as a pair

$$(S, D : \partial S \to \partial S),$$

such that $D^2 = id$ and $(\partial D \circ \text{ex})^3 = id$.

Lamarche has also shown that the functor

$$T \mapsto (\partial T, \text{ex})$$

is an equivalence of categories when restricted to species which are empty on the empty set.
“exchangeable output” ⇔ “entries only”

Theorem 2

This equivalence carries over to an equivalence between the previous two definitions of cyclic operads.
"exchangeable output" \iff "entries only"

**Theorem 2**

This equivalence carries over to an equivalence between the previous two definitions of cyclic operads.

The steps of the proof are as follows:

- **Given** \( \rho : T \blacktriangle T \to T \), we define \( \nu : \partial T \ast \partial T \to \partial T \) as follows

\[
\nu : \partial \partial T \cdot \partial T \xrightarrow{\text{ex} \cdot \text{id}} \partial \partial T \cdot \partial T \xrightarrow{\text{inj}} \partial(\partial T \cdot \partial T) \xrightarrow{\partial \rho} \partial T
\]

- **Given** \( \nu : \partial T \ast \partial T \to \partial T \), to define \( \rho : T \blacktriangle T \to T \) amounts to define \( \rho' : \partial(\partial T \cdot \partial T) \to \partial T \) (\( \rho \) is then uniquely defined via \( \partial \rho = \rho' \)):

\[
\begin{align*}
\rho'_1 & : \partial \partial T \cdot \partial T \xrightarrow{\text{ex} \cdot \text{id}} \partial \partial T \cdot \partial T \xrightarrow{\nu} \partial T \\
\rho'_2 & : \partial T \cdot \partial \partial T \xrightarrow{\text{Swap}} \partial \partial T \cdot \partial T \xrightarrow{\rho'_1} \partial T
\end{align*}
\]

\( \rho' = [\rho'_1, \rho'_2] \)

- And then we have to check that everything commutes in the appropriate way...
...for example:

\[
\partial \partial \partial T \cdot \partial T \cdot \partial T \\
\overset{\text{\(\partial \text{ex} \cdot \text{Swap}\)}}{\rightarrow} \\
\partial \partial \partial T \cdot \partial T \cdot \partial T
\]

\[
\partial (\partial \partial T \cdot \partial T) \cdot \partial T \\
\overset{\text{\(\text{id} \cdot \text{Swap}\)}}{\leftarrow} \\
\partial \partial \partial T \cdot \partial T \cdot \partial T
\]

\[
\partial (\partial \partial T \cdot \partial T) \cdot \partial T \\
\overset{\text{\(\text{inj} \cdot \text{id}\)}}{\rightarrow} \\
\partial \partial \partial T \cdot \partial T \cdot \partial T
\]

\[
\partial (\partial \partial T \cdot \partial T) \cdot \partial T \\
\overset{\text{\(\text{ex} \cdot \text{Swap}\)}}{\leftarrow} \\
\partial \partial \partial T \cdot \partial T \cdot \partial T
\]
Some directions of investigations:

- Figure out the microcosm principle for modular and wheeled operads
- Study the “up to homotopy” versions of all these (infinity operads, properads, wheeled properads,...)

Hope that you enjoyed the promenade!