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A calculus of sequents with probability

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§0. **Introduction.**

Sequents in **LKprob** are of the form $\Gamma \vdash_a^b \Delta$, meaning that 'the probability of provability of $\Gamma \vdash \Delta$ belongs to the interval $[a, b] \cap I$', where I is a finite subset of reals $[0, 1]$. The system **LKprob**, an extension of Gentzen's sequent calculus for classical propositional logic, is sound and complete with respect to a kind of Carnap–Popper–Leblanc–type probability logic semantics.

§1. **The probabilistic sequent calculus LKprob.**

The sequent $\Gamma \vdash \Delta$, as introduced by Gentzen, consists of two finite (possibly empty) sequences (or words) of formulae Γ — the antecedent, and Δ — the consequent, with the main interpretation as $\bigwedge \Gamma \rightarrow \bigvee \Delta$, where $\bigwedge \Gamma$ denotes the conjunction of all formulae appearing in Γ, and $\bigvee \Delta$ denotes the disjunction of all formulae appearing in Δ; particularly, if Γ or Δ is an empty sequence, then $\vdash \Delta$ is interpreted as $\bigvee \Delta$, $\Gamma \vdash$ as $\neg \bigwedge \Gamma$, and $\vdash$ can be understood as a pure contradiction. Propositional formulae are defined over propositional language consisting of a denumerable set of propositional letters: $\{p_1, p_2, ...\}$, logical connectives: $\neg$, $\wedge$, $\vee$ and $\rightarrow$, and two auxiliary symbols:) and (. The set of formulae is the smallest set containing propositional letters closed under the following formation rule: if A and B are formulae, then $(\neg A)$, $(A \wedge B)$, $(A \vee B)$ and $(A \rightarrow B)$ are formulae as well.

The axioms of **LKprob** are the following three sequents:

$$\Gamma \vdash_0^1 \Delta$$
$$\vdash^0$$
$$A \vdash_1 A$$

for any words Γ and Δ, and any formula A.

The structural rules of **LKprob** are as follows:

permutation: $$\frac{\Gamma A B \Pi \vdash_a^b \Delta}{\Gamma B A \Pi \vdash_a^b \Delta}(P \vdash_a^b) \qquad \frac{\Gamma \vdash_a^b \Delta A B \Lambda}{\Gamma \vdash_a^b \Delta B A \Lambda}(\vdash_a^b P)$$

contraction: $$\frac{\Gamma A A \vdash_a^b \Delta}{\Gamma A \vdash_a^b \Delta}(C \vdash_a^b) \qquad \frac{\Gamma \vdash_a^b A A \Delta}{\Gamma \vdash_a^b A \Delta}(\vdash_a^b C)$$

for any $a, b \in I$, *the cut rule:*

$$\frac{\Gamma \vdash_a^b A\Delta \quad \Pi A \vdash_c^d \Lambda}{\Gamma\Pi \vdash_{\max(0,a+c-1)}^{\min(b+d,1)} \Delta\Lambda}(\mathrm{cut}^{[a,b][c,d]})$$

for any $a, b, c, d \in I$

Specific structural rules:

weakening:

$$\frac{\Gamma \vdash_a^b \Delta \quad \vdash_c^d A}{\Gamma A \vdash_{\max(a,1-d)}^{\min(1,b+1-c)} \Delta}(W \vdash_a^b) \qquad \frac{\Gamma \vdash_a^b \Delta \quad \vdash_c^d A}{\Gamma \vdash_{\max(a,c)}^{\min(1,b+d)} A\Delta}(\vdash_a^b W)$$

for any $a, b, c, d \in I$,

monotonicity:

$$\frac{\Gamma \vdash_a^b \Delta}{\Gamma \vdash_c^d \Delta}(M \uparrow) \qquad \frac{\Gamma \vdash_a^b \Delta \quad \Gamma \vdash_c^d \Delta}{\Gamma \vdash_{\max(a,c)}^{\min(b,d)} \Delta}(M \downarrow)$$

for any $[a, b] \subseteq [c, d]$, and any $a \leq b$ and $c \leq d$, respectively, for $(M \uparrow)$ and $(M \downarrow)$, and the following specific rule regarding *additivity:*

$$\frac{AB \vdash_1 \quad \vdash_a^b A \quad \vdash_c^d B}{\vdash_{a+c}^{\min(1,b+d)} AB}(ADD)$$

The following rule, regarding *inconsistency:*

$$\frac{\Gamma \vdash^{\emptyset} \Delta}{\Pi \vdash^{\emptyset} \Lambda}(\bot)$$

The logical rules of **LKprob** are as follows:

$$\frac{\Gamma \vdash_a^b A\Delta}{\Gamma\neg A \vdash_a^b \Delta}(\neg \vdash_a^b) \qquad \frac{\Gamma A \vdash_a^b \Delta}{\Gamma \vdash_a^b \neg A\Delta}(\vdash_a^b \neg)$$

$$\frac{\Gamma AB \vdash_a^b \Delta}{\Gamma A \wedge B \vdash_a^b \Delta}(\wedge \vdash_a^b) \qquad \frac{\Gamma \vdash_a^b A\Delta \quad \Gamma \vdash_c^d B\Delta}{\Gamma \vdash_{\max(0,a+c-1)}^{\min(b,d)} A \wedge B\Delta}(\vdash_a^b \wedge)$$

$$\frac{\Gamma A \vdash_a^b \Delta \quad \Gamma B \vdash_c^d \Delta}{\Gamma A \vee B \vdash_{\max(0,a+c-1)}^{\min(b,d)} \Delta}(\vee \vdash_a^b) \qquad \frac{\Gamma \vdash_a^b AB\Delta}{\Gamma \vdash_a^b A \vee B\Delta}(\vdash_a^b \vee)$$

$$\frac{\Gamma \vdash_a^b A\Delta \quad \Gamma B \vdash_c^d \Delta}{\Gamma A \rightarrow B \vdash_{\max(0,a+c-1)}^{\min(b,d)} \Delta}(\rightarrow \vdash_a^b) \qquad \frac{\Gamma A \vdash_a^b B\Delta}{\Gamma \vdash_a^b A \rightarrow B\Delta}(\vdash_a^b \rightarrow)$$

Example 1. Let the formulas A, B, C, D and E have the following interpretation: A–the person is a female, B–has a Bachelor's degree, C–has Master's degree or doctorate, D–has a high salary, E–owns at least one property. The results of a questionnaire are: *(i)* the probability of having a high salary, if you are a female who has a Bachelor's degree is 0.873, *(ii)* the probability of owning at least one property, if you are a female who has a Bachelor's degree is 0.794 and *(iii)* the probability of not having master's or doctoral degree if you are a female is 0.951. That means that in our system **LKprob**, with $I = \{10^{-3}k | k = 0, 1, \ldots, 10^3\}$, the additional axioms are of the form *(i)* $AB \vdash_{0.873}^{0.873} D$, *(ii)* $AB \vdash_{0.794}^{0.794} E$ and *(iii)* $A \vdash_{0.951}^{0.951} \neg C$. Using the following proof

$$\dfrac{\dfrac{AB \vdash_{0.873}^{0.873} D \quad AB \vdash_{0.794}^{0.794} E}{AB \vdash_{0.667}^{0.794} D \wedge E}(\vdash \wedge) \qquad \dfrac{\dfrac{A \vdash_{0.951}^{0.951} \neg C}{AC \vdash_{0.951}^{0.951}}(\neg \vdash) \quad \vdash_0^1 D \wedge E}{AC \vdash_{0.951}^{1} D \wedge E}(\vdash W)}{A(B \vee C) \vdash_{0.618}^{0.794} D \wedge E}(\vee \vdash)$$

we can conclude that, if you are a female with Bachelor's, master's or doctoral degree, then the probability of having a high salary and owning at least one property belongs to the interval $[0.618, 0.794]$.

§2. Models for Probabilized Sequents.

Let Seq be the set of all unlabelled sequents, i.e. of sequents of the form $\Gamma \vdash \Delta$, and I a finite subset of reals $[0, 1]$ closed under addition. Then a mapping $p : \mathrm{Seq} \to I$ will be a model, if it satisfies the following conditions:

(i) $p(A \vdash A) = 1$, for any formula A;

(ii) if $p(AB \vdash) = 1$, then $p(\vdash AB) = p(\vdash A) + p(\vdash B)$, for any formulas A and B;

(iii) if sequents $\Gamma \vdash \Delta$ and $\Pi \vdash \Lambda$ are equivalent in **LK**, in sense that there are proofs for both sequents $\bigwedge \Gamma \to \bigvee \Delta \vdash \bigwedge \Pi \to \bigvee \Lambda$ and $\bigwedge \Pi \to \bigvee \Lambda \vdash \bigwedge \Gamma \to \bigvee \Delta$ in **LK**, then $p(\Gamma \vdash \Delta) = p(\Pi \vdash \Lambda)$.

Satisfiability in a model for the probabilized sequents is defined by clause:

$$\models_p \Gamma \vdash_a^b \Delta \text{ iff } \ a \leq p(\Gamma \vdash \Delta) \leq b$$

and we say that the probabilized sequent $\Gamma \vdash_a^b \Delta$ is satisfied in a model p. A sequent $\Gamma \vdash_a^b \Delta$ is valid iff it is satisfied in each model, and this is denoted by $\models \Gamma \vdash_a^b \Delta$.

Lemma. *For any formulas A and B, the following equalities hold:*
(a) $p(\vdash \neg A) = 1 - p(\vdash A)$*;*
(b) $p(\vdash AB) = p(\vdash A) + p(\vdash B) - p(\vdash A \wedge B)$*;*
(c) $p(\vdash AB) \geq p(\vdash A)$*;*
(d) $p(A \vdash B) \leq p(A \vdash) + p(\vdash B)$*;*
(e) $p(A \vdash A) = p(A \vdash) + p(\vdash A)$*.*

Lemma. *For any formula A and each sequent $\Gamma \vdash \Delta$, we have:*
(a) $\models \Gamma \vdash_0^1 \Delta$*;*
(b) $\models \vdash^0$*;*
(c) $\models A \vdash_1 A$*.*

Lemma. *For any formulas A and B, and each words Γ, Δ, Π and Λ, we have:*
(a) if $a \leq p(\Gamma \vdash \Delta) \leq b$ and $c \leq p(\vdash A) \leq d$, then

$$\max(a, 1-d) \leq p(\Gamma A \vdash \Delta) \leq \min(1, b+1-c);$$

(b) if $a \leq p(\Gamma \vdash \Delta) \leq b$ and $c \leq p(\vdash A) \leq d$, then

$$\max(a, c) \leq p(\Gamma \vdash A\Delta) \leq \min(1, b+d);$$

(c) if $a \leq p(\Gamma \vdash A\Delta) \leq b$ and $c \leq p(\Gamma \vdash B\Delta) \leq d$, then

$$\max(0, a+c-1) \leq p(\Gamma \vdash A \wedge B\Delta) \leq \min(b, d);$$

(d) if $a \leq p(\Gamma A \vdash \Delta) \leq b$ and $c \leq p(\Gamma B \vdash \Delta) \leq d$, then

$$\max(0, a+c-1) \leq p(\Gamma A \vee B \vdash \Delta) \leq \min(b, d);$$

(e) if $a \leq p(\Gamma \vdash A\Delta) \leq b$ and $c \leq p(\Gamma B \vdash \Delta) \leq d$, then

$$\max(0, a+c-1) \leq p(\Gamma A \rightarrow B \vdash \Delta) \leq \min(b, d).$$

Proof. *(d) Suppose that* $p(\Gamma A \vdash \Delta) \in [a,b]$ *and* $p(\Gamma B \vdash \Delta) \in [c,d]$. *We have that*

$$
\begin{aligned}
p(\Gamma(A \vee B) \vdash \Delta) &= p(\vdash (\neg A \wedge \neg B)\Delta\neg(\bigwedge \Gamma)) \\
&= p(\vdash (\neg A \vee \Delta \vee \neg(\bigwedge \Gamma)) \wedge (\neg B \vee \Delta \vee \neg(\bigwedge \Gamma))) \\
&= p(\vdash \neg A\Delta\neg(\bigwedge \Gamma)) + p(\vdash \neg B\Delta\neg(\bigwedge \Gamma)) - p(\vdash \neg A\neg B\Delta\neg(\bigwedge \Gamma)) \\
&= p(\Gamma A \vdash \Delta) + p(\Gamma B \vdash \Delta) - p(\vdash \neg A\neg B\Delta\neg(\bigwedge \Gamma))
\end{aligned}
$$

Therefore, $p(\Gamma(A \vee B) \vdash \Delta) \in [max(0, a+c-1), min(b,d)]$.

Lemma. *Let* $p(\vdash A) = a$, $p(\vdash B) = b$, $p(\vdash C) = c$, $p(A \vdash B) = r$ *and* $p(B \vdash C) = s$, *with* $a + r \geq 1$. *Then:*

(a) (T. Hailperin (1984)) *(modus ponens probabilized)*

$$a + r - 1 \leq p(\vdash B) \leq r$$

(b) (C. G. Wagner (2004)) *(modus tollens probabilized)*

$$r - b \leq p(A \vdash) \leq r$$

(c) (hypothetical syllogism rule probabilized)

$$\max(1 - a, b) + \max(1 - b, c) - 1 \leq p(A \vdash C) \leq 2 - a - b + c$$

(d) (hypothetical syllogism rule probabilized)

$$\max(r - a, r + s - 1) \leq p(A \vdash C) \leq \min(s + 1 - a, r + c)$$

The bounds in (a), (b), (c) and (d) are the best possible.

Corollary. *If* $a \leq p(\Gamma \vdash A\Delta) \leq b$ *and* $c \leq p(\Pi A \vdash \Lambda) \leq d$, *then*

$$\max(0, a + c - 1) \leq p(\Gamma\Pi \vdash \Delta\Lambda) \leq \min(b + d, 1).$$

§3. **Consistent LKprob—theories.**

Let σ_i $(1 \le i \le n)$ be a finite list of sequents of the form $\Gamma_i \vdash_{a_i}^{b_i} \Delta_i$, for $a_i, b_i \in I$ $(1 \le i \le n)$. Then, by **LKprob**$(\sigma_1, \dots, \sigma_n)$, we denote an extension of **LKprob** by sequents $\sigma_1, \dots, \sigma_n$ as additional axioms and call it *an* **LKprob**—*theory over* $\sigma_1, \dots, \sigma_n$. We say that a theory **LKprob**$(\sigma_1, \dots, \sigma_n)$ is *inconsistent* if there are two sequents $\Gamma \vdash_a^b \Delta$ and $\Gamma \vdash_c^d \Delta$, both provable in **LKprob**$(\sigma_1, \dots, \sigma_n)$ such that $[a, b] \cap [c, d] = \emptyset$; otherwise, **LKprob**$(\sigma_1, \dots, \sigma_n)$ is *consistent.* A sequent $\Gamma \vdash_a^b \Delta$ is said to be *consistent with respect to* **LKprob**$(\sigma_1, \dots, \sigma_n)$ if there is no sequent $\Gamma \vdash_c^d \Delta$ provable in **LKprob**$(\sigma_1, \dots, \sigma_n)$, for $[a, b] \cap [c, d] = \emptyset$. A finite set of sequents $\{\tau_1, \dots, \tau_k\}$ is consistent with respect to **LKprob**$(\sigma_1, \dots, \sigma_n)$ if, for each i $(1 \le i \le k)$, τ_i is consistent with respect to **LKprob**$(\sigma_1, \dots, \sigma_n, \tau_1, \dots, \tau_{i-1}, \tau_{i+1}, \dots, \tau_k)$. A denumerable set of sequents is consistent with respect to **LKprob**$(\sigma_1, \dots, \sigma_n)$ if each of its finite subsets is consistent with respect to **LKprob**$(\sigma_1, \dots, \sigma_n)$. A consistent theory is called a *maximal consistent theory* if each of its proper extensions is inconsistent.

Lemma. *Each consistent theory can be extended to a maximal consistent theory.*

Proof. Let $\mathcal{T}$ be a consistent theory, and let $\alpha_1, \alpha_2, \dots, \alpha_n, \dots$ be the sequence of all unlabelled sequents i.e. α_n is $\Gamma_n \vdash \Delta_n$, and for each $c \in I$, let $\alpha_1^c, \alpha_2^c, \dots, \alpha_n^c, \dots$ be the sequence of the corresponding labelled sequents, i.e. α_n^c is $\Gamma_n \vdash_c^c \Delta_n$. Let the sequence $(\mathcal{T}_n)$ of theories be defined inductively as follows: $\mathcal{T}_0 = \mathcal{T}$, and $\mathcal{T}_{n+1} = \mathcal{T}_n \cup \{\alpha_n^{c_1}\}$, if $\alpha_n^{c_1}$ is consistent with respect to $\mathcal{T}_n$, but if it is not consistent, then: $\mathcal{T}_{n+1} = \mathcal{T}_n \cup \{\alpha_n^{c_2}\}$, if $\alpha_n^{c_2}$ is consistent with respect to $\mathcal{T}_n$, but if it is not, then ... $\mathcal{T}_{n+1} = \mathcal{T}_n \cup \{\alpha_n^{c_{m-1}}\}$, if $\alpha_n^{c_{m-1}}$ is consistent with respect to $\mathcal{T}_n$, and finally, $\mathcal{T}_{n+1} = \mathcal{T}_n \cup \{\alpha_n^{c_m}\}$, otherwise; where $\{c_1, c_2, \dots, c_m\} = I$. Let us note that the final result of this construction depends on the order of points $c_1, c_2, \dots, c_m$ of the set I. Let

$$\mathcal{T}' = \cup_{n \in \omega} \mathcal{T}_n$$

Then, by induction on n we will prove that $\mathcal{T}'$ is a maximal consistent extension of $\mathcal{T}$. First, we prove that if $\mathcal{T}_n$ is consistent, then $\mathcal{T}_{n+1}$ is consistent. The only interesting case is when $\mathcal{T}_{n+1} = \mathcal{T}_n \cup \{\alpha_n^{c_m}\}$. Suppose that $\mathcal{T}_{n+1}$ is inconsistent, i.e. that the sequent $\alpha_n^{c_m}$ is not consistent with respect to $\mathcal{T}_n$. Then there exists an interval $[a, b] \subset [0, 1]$ such that $c_m \notin [a, b]$ and $\Gamma_n \vdash_a^b \Delta_n$ is provable in $\mathcal{T}_n$, which is impossible because the theory $\mathcal{T}_n \cup \{\alpha_n^{c_j}\}$ is inconsistent for each j $(1 \leq j \leq m-1)$. In order to prove that $\mathcal{T}'$ is a *maximal* consistent extension of $\mathcal{T}$ we extend $\mathcal{T}'$ by the sequent $\Gamma_k \vdash_a^b \Delta_k$. In case that this is a proper extension, we already have that the theory $\mathcal{T}_{k+1} \subset \mathcal{T}'$ contains $\Gamma_k \vdash_c^c \Delta_k$ for some $c \notin [a, b]$, and, consequently, this extension will be inconsistent.□

§4. Soundness and Completeness.

Soundness Theorem. *If an* **LKprob***–theory has a model, then it is consistent.*

In order to prove the completeness part, we define the notion of *canonical model.* Let $\mathrm{Cn}(\mathbf{LKprob}(\sigma_1, \ldots, \sigma_n))$ be the set of all $\mathbf{LKprob}(\sigma_1, \ldots, \sigma_n)$–provable sequents and $\mathrm{ConExt}(\mathrm{Cn}(\mathbf{LKprob}(\sigma_1, \ldots, \sigma_n)))$ the class of all its maximal consistent extensions. Then, for any $X \in \mathrm{ConExt}(\mathrm{Cn}(\mathbf{LKprob}(\sigma_1, \ldots, \sigma_n)))$ we define $\models_{p^X} \Gamma \vdash_a^b \Delta$ iff $a \leq \max\{c | \Gamma \vdash_c^1 \Delta \in X\}$ and $b \geq \min\{c | \Gamma \vdash_0^c \Delta \in X\}$. Obviously, such a definition provides that the mapping p^X, depending on X, has the adequate values. In that case we have that:

Lemma. $\models_{p^X} \Gamma \vdash_a^b \Delta$ *iff* $\Gamma \vdash_a^b \Delta \in X$.

Completeness Theorem. *Each consistent* **LKprob***–theory has a model.*