# Some applications of Probabilistic first-order logic

M. Rašković, Z. Marković, Z. Ognjanović, N. Ikodinović

Dubrovnik, September, 2015.

# Outline

1. Introduction
2. Probabilistic first-order logic
3. Some applications and ideas for future work

# Outline

# ... Probabilistic first-order logic

- N. Ikodinović, M. Rašković, Z. Marković, Z. Ognjanović, *A first-order probabilistic logic with approximate conditional probabilities*, Logic Journal of the IGPL **22(4)**, 539-564, 2014.
- M. Rašković, Z. Marković, and Z. Ognjanović, *A logic with approximate conditional probabilities that can model default reasoning*, International Journal of Approximate Reasoning **49**, 52-66, 2008.
- Z. Ognjanović, M. Rašković, and Z. Marković, *Probability logics*, in Zbornik radova, subseries Logic in computer science **12(20)**, 35-111, Mathematical institute, 2009.
- M. Rasković, and R. Djordjević, *Probability Quantifiers and Operators*, VESTA, Beograd, 1996.

# Some applications ...

- CAUSAL REASONING J. Pearl, *Probabilistic Reasoning in Intelligent Systems*, Morgan Kaufmann, San Francisco, 1988.
- LEARNING FROM DATA R. E. Neapolitan, *Probabilistic Reasoning in Expert Systems*, Wiley, New York, 1990.
- MULTI-AGENT SYSTEMS R. Fagin, J. Y. Halpern, Y. Moses, M. Y. Vardi, *Reasoning About Knowledge*, MIT Press, Cambridge, 2003.
- ROBOTICS S. Thrun, W. Burgard, D. Fox, *Probabilistic Robotics*, MIT Press, Cambridge, 2005.
- LOGIC PROGRAMMING K. Kersting, L. D. Raedt, *Bayesian logic programming: Theory and tool*, in Getoor, L. and Taskar, B., editors, Introduction to Statistical Relational Learning, MIT Press, Cambridge, 2007
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# Probabilistic first-order logic $L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$

$L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$ is an extension of the ordinary first order classical logic $L_{\omega\omega}$, while $\mathrm{P}$ and $\mathbb{I}$ indicate that the values of the considered Probabilistic functions will belong to the unite interval $\mathbb{I}$ of some field.

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$\mathbb{Q}(\varepsilon) = \langle \mathbb{Q} \uplus \{\varepsilon\} \rangle_{\text{field}}$, $\varepsilon \in {}^*\mathbb{R}$

$\mathbb{Q}[\varepsilon] = \{c_0 + c_1\varepsilon + c_2\varepsilon^2 + \cdots + c_k\varepsilon^k \mid k \in \mathbb{N}, c_0, c_1, c_2, \ldots, c_k \in \mathbb{Q}\}$

$$c_0 + c_1\varepsilon + \cdots + c_k\varepsilon^k < d_0 + d_1\varepsilon + \cdots + d_\ell\varepsilon^\ell$$

$$\Updownarrow \text{ (def)}$$

$$c_i < d_i \text{ for some } i < \max\{k, \ell\} \text{ and } c_m = d_m \text{ for all } m < i$$

$$(c_i = 0, i > k \text{ and } d_j = 0, j > \ell)$$

$\mathbb{Q}(\varepsilon) = \{\frac{P(\varepsilon)}{Q(\varepsilon)} \mid P(\varepsilon), Q(\varepsilon) \in \mathbb{Q}[\varepsilon], Q(\varepsilon) > 0\}$

$$\frac{P(\varepsilon)}{Q(\varepsilon)} < \frac{R(\varepsilon)}{S(\varepsilon)} \Leftrightarrow P(\varepsilon) \cdot S(\varepsilon) < R(\varepsilon) \cdot Q(\varepsilon)$$

$$\boxed{\mathbb{I} = \{r \in \mathbb{Q}(\varepsilon) \mid 0 \leqslant r \leqslant 1\}}$$

# Why $\mathbb{Q}(\varepsilon)$?

- Adams, Pearl, Lehmann, Magidor ... have suggested a probabilistic interpretation of defaults: a default 'if $\alpha$, normally $\beta$' ($\alpha \rightarrowtail \beta$) means 'the probability of $\beta$ given $\alpha$ is *very close* to 1'.
- $\mathbb{Q}(\varepsilon)$ is (isomorphic to) a dense subfield of ${}^*\mathbb{R}$;
- the monad (halo) of a rational $q \in [0,1]$, defined by $\mathrm{monad}(x) = \{y \mid y \approx x\}$, can be characterized by:

$$\mathrm{monad}(q) = \bigcap_{n \in \mathbb{N}^+} \left[\max\left\{0, q - \frac{1}{n}\right\}, \min\left\{1, q + \frac{1}{n}\right\}\right];$$

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- $\mathbb{Q}(\varepsilon)$ is countable and recursive, i.e., its operations are computable and its ordering is decidable.

# $L^{P,\mathbb{I}}_{\omega\omega}$ – Syntax

## Logical symbols

- *Variables*:
  Var
- *Connectives*:
  $\neg, \wedge (\vee, \rightarrow, \leftrightarrow)$
- *Quantifiers*:
  $\forall$, $\exists$
- *Probabilistic quantififers*:
  $(\mathrm{CP} \cdots \begin{smallmatrix}\leq\\=\\\geq\end{smallmatrix} r)$, $r \in \mathbb{I}$
  $(\mathrm{CP} \cdots \approx q)$, $q \in \mathbb{I} \cap \mathbb{Q}$
- *Punctuation*:
  ) , (

## Non-logical symbols

- *Relation symbols*
  Rel
- *Function symbols*
  Fun

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– $\mathrm{Rel} \cap \mathrm{Fun} = \emptyset$, $L = \mathrm{Rel} \cup \mathrm{Fun}$

– **ar** : $\mathrm{Rel} \cup \mathrm{Fun} \rightarrow \mathbb{N}$

– $p \in \mathrm{Rel}$, $\mathrm{ar}(p) = 0$
propositional letter

– $c \in \mathrm{Fun}$, $\mathrm{ar}(f) = 0$
constant symbol

# $L^{\mathrm{P},\mathbb{I}}_{\omega\omega}$ – Semantics

- $(M, S^{\mathbf{M}})_{S \in L}$ is a classical first-order structure for $L$;
- $(M^n, \mathscr{F}_n, \mu_n)$ is a finitely-additive probability space.
  - $\mathscr{F}_n$ is a field of subsets of $M^n$;
  - $\mu_n : \mathscr{F}_n \to \mathbb{I}$ is a finitely-additive probability measure;

Moreover:

– for each $n$-ary relation symbol $R$ of $L$, $R^{\mathbf{M}} \in \mathscr{F}_n$;

– for all $i, j \leqslant n$, $\{(x_1, \ldots, x_n) \in A^n \mid x_i = x_j\} \in \mathscr{F}_n$;

$\vdots$

– if $X \in \mathscr{F}_n$, then $A \times X \in \mathscr{F}_{n+1}$;

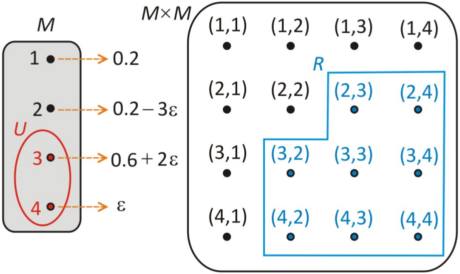
– if $X \in \mathscr{F}_n$, then $\mu_{n+1}(A \times X) = \mu_n(A)$.

# $L^{P,\mathbb{I}}_{\omega\omega}$ – Semantics

### Example

$L = \mathrm{Rel} = \{U, R\}$

$\mathbf{ar}(U) = 1,\ \mathbf{ar}(R) = 2$




- $\mathbf{M} = (M, U, R)$
- $\mu : M \to [0,1]$
  $\mu\colon\ 1 \mapsto 0.2,$
  $2 \mapsto 0.2 - 3\varepsilon,$
  $3 \mapsto 0.6 + 2\varepsilon,\ 4 \mapsto \varepsilon$
- $\mu : \mathscr{P}(M) \to [0,1]$
  $\mu(A) = \sum_{x \in A} \mu(x)$

# $L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$ – Semantics

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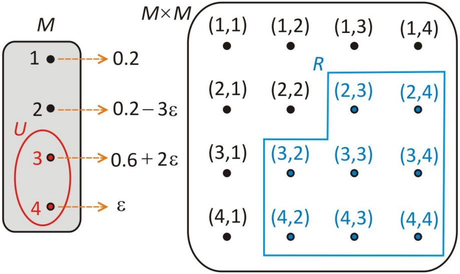


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- $\mu_2 : M \times M \to [0,1]$
  $\mu(x,y) = \mu(x) \cdot \mu(y)$
  $\mu_2 : \mathscr{P}(M \times M) \to [0,1]$

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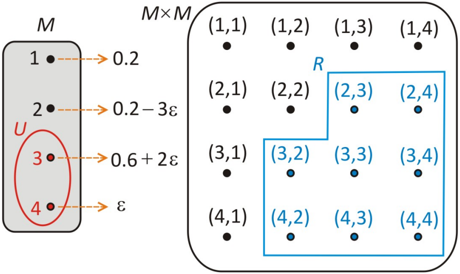


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- $\mu_n : \mathscr{P}(M^n) \to \mathbb{I}$, $n \in \mathbb{N}$

# $L^{\mathrm{P},\mathbb{I}}_{\omega\omega}$ – Semantics

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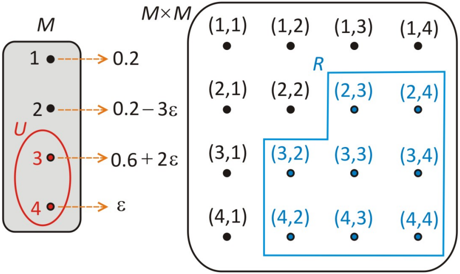


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$$(\underbrace{M, U, R}_{\mathbf{M}}, \mathscr{P}(M^n), \mu_n)$$

# $L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$ – Satisfiability relation

$(M, S^{\mathbf{M}}, \mathscr{F}_n, \mu_n)_{S \in L, n \in \mathbb{N}}$, $v : \mathrm{Var} \to M$

- $\mathbf{M}, v \models (\mathrm{CP}\vec{x} \leqslant r)(\alpha \mid \beta)$ iff

$$(\mu_n\{\vec{a} \in M^n \mid \mathbf{M}, v(\vec{x} := \vec{a}) \models \beta\} = 0 \,\&\, r = 1) \text{ OR } (\mu_n\{\vec{a} \in M^n \mid \mathbf{M}, v(\vec{x} := \vec{a}) \models \beta\} > 0 \,\& \, \frac{\mu_n\{\vec{a} \in M^n \mid \mathbf{M}, v(\vec{x} := \vec{a}) \models \alpha \wedge \beta\}}{\mu_n\{\vec{a} \in M^n \mid \mathbf{M}, v(\vec{x} := \vec{a}) \models \beta\}} \leqslant r)$$

- $\mathbf{M}, v \models (\mathrm{CP}\vec{x} \approx q)(\alpha \mid \beta)$ iff

$$(\mu_n\{\vec{a} \in M^n \mid \mathbf{M}, v(\vec{x} := \vec{a}) \models \beta\} = 0 \,\&\, q = 1) \text{ OR } (\mu_n\{\vec{a} \in M^n \mid \mathbf{M}, v(\vec{x} := \vec{a}) \models \beta\} > 0 \,\& \, \frac{\mu_n\{\vec{a} \in M^n \mid \mathbf{M}, v(\vec{x} := \vec{a}) \models \alpha \wedge \beta\}}{\mu_n\{\vec{a} \in M^n \mid \mathbf{M}, v(\vec{x} := \vec{a}) \models \beta\}} \in \mathrm{monad}(q))$$

# $L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$ – Satisfiability relation

Conventions

$$\mathrm{CP}x = r * \equiv \quad \mathrm{CP}x \leq r * \wedge \mathrm{CP}x \geq r *$$
$$\mathrm{CP}x > r * \equiv \neg \mathrm{CP}x \leq r *$$
$$\mathrm{CP}x < r * \equiv \neg \mathrm{CP}x \geq r *$$
$$\mathrm{P}x \gtreqless r * \equiv \quad \mathrm{CP}x \gtreqless r(* \mid \top)$$

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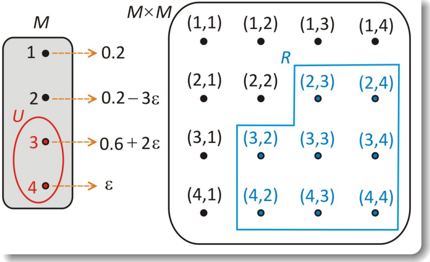


- $\mathbf{M}, (y \mapsto 1, \cdots) \not\models (\mathrm{CP}x < 0.25 - \varepsilon)(\neg U(x) \mid R(x,y))$
- $\mathbf{M}, (y \mapsto 2, \cdots) \models (\mathrm{CP}x < 0.25 - \varepsilon)(\neg U(x) \mid R(x,y))$
- $\mathbf{M} \models (\mathrm{CP}x < 0.25 - \varepsilon)(\neg U(x) \mid \exists y R(x,y))$
- $\mathbf{M} \not\models \forall y (\mathrm{CP}x < 0.25 - \varepsilon)(\neg U(x) \mid R(x,y))$

# $L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$ – Axiomatization

(Axioms)

- All $L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$-instances of the axioms for $L_{\omega\omega}$
- $(\mathrm{CP}\vec{x} \leqslant r_1)(\alpha \mid \beta) \to (\mathrm{CP}\vec{x} < r_2)(\alpha \mid \beta)$, $r_1 < r_2$
- $(\mathrm{P}\vec{x} \geqslant 1)\alpha \leftrightarrow \beta \to ((\mathrm{P}\vec{x} = r)\alpha \to (\mathrm{P}\vec{x} = r)\beta)$
- $((\mathrm{P}\vec{x} = r_1)\alpha \wedge (\mathrm{P}\vec{x} = r_2)\beta \wedge (\mathrm{P}\vec{x} = 0)(\alpha \wedge \beta)) \to (\mathrm{P}\vec{x} = \min\{1, r_1 + r_2\})(\alpha \vee \beta)$

⋮

(Rules)

⋮

**(Approx)** $\dfrac{\{\gamma\to(\mathrm{CP}\vec{x}\geqslant q-\frac{1}{n})(\alpha|\beta):n\geqslant\frac{1}{q}\}\cup\{\gamma\to(\mathrm{CP}\vec{x}\leqslant q+\frac{1}{n})(\alpha|\beta):n\geqslant\frac{1}{1-q}\}}{\gamma\to(\mathrm{CP}\vec{x}\approx q)(\alpha|\beta)}$,

**(Range)** $\dfrac{\alpha \to (\mathrm{P}\vec{x} \neq r)\beta(\vec{x}), r \in \mathbb{I}}{\alpha \to \perp}$;

# $L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$ – Formal proofs

### Definition

A proof of a formula $\alpha$ from a set $\Gamma$ of formulas is a countable sequence of formulas $\alpha_\kappa$ indexed by countable ordinal numbers such that the last formula is $\alpha$, and each formula in the sequence is an axiom, or a formula in $\Gamma$ or it is derived from the preceding formulas by a rule of inference with no application of **(Gen)** to a formula when the variable is free in formulas of $\Gamma$.

$\Gamma \vdash \alpha$ if there is a proof of $\alpha$ from $\Gamma$.

A set $\Gamma$ of formulas is consistent if there is at least one $L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$ formula that is not deducible from $\Gamma$, otherwise $\Gamma$ is inconsistent.

A set $\Gamma$ is maximal consistent iff $\Gamma$ is consistent and for every formula $\alpha$, $\alpha \in \Gamma$ or $\neg\alpha \in \Gamma$. Note that if $\Gamma$ is maximal consistent and $\Gamma \vdash \alpha$ then $\alpha \in \Gamma$.

# $L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$ – Formal proofs

**Lemma**

1. $\vdash (\mathrm{CP}\vec{x} \geq r_1)(\alpha \mid \beta) \to (\mathrm{CP}\vec{x} \geq r_2)(\alpha \mid \beta),\ r_1 > r_2$
2. $\vdash (\mathrm{CP}\vec{x} \leq r_1)(\alpha \mid \beta) \to (\mathrm{CP}\vec{x} \leq r_2)(\alpha \mid \beta),\ r_1 < r_2$
3. $\vdash (\mathrm{CP}\vec{x} = q)(\alpha \mid \beta) \to (\mathrm{CP}\vec{x} \approx q)(\alpha \mid \beta),\ q \in \mathbb{I}_{\mathbb{Q}}$

*Proof for (3)*

1. $(\mathrm{CP}\vec{x} = q)(\alpha \mid \beta) \leftrightarrow (\mathrm{CP}\vec{x} \geq q)(\alpha \mid \beta) \wedge (\mathrm{CP}\vec{x} \leq q)(\alpha \mid \beta)$

$2_n$. $(\mathrm{CP}\vec{x} \geq q)(\alpha \mid \beta) \to (\mathrm{CP}\vec{x} \geq q - \frac{1}{n})(\alpha \mid \beta),\ n \geq \frac{1}{q}$ [(1) of this lemma]

$3_n$. $(\mathrm{CP}\vec{x} \leq q)(\alpha \mid \beta) \to (\mathrm{CP}\vec{x} \leq q + \frac{1}{n})(\alpha \mid \beta),\ n \geq \frac{1}{1-q}$ [(2) of this lemma]

$4_n$. $(\mathrm{CP}\vec{x} = q)(\alpha \mid \beta) \to (\mathrm{CP}\vec{x} \geq q - \frac{1}{n})(\alpha \mid \beta),\ n \geq \frac{1}{q}$

$5_n$. $(\mathrm{CP}\vec{x} = q)(\alpha \mid \beta) \to (\mathrm{CP}\vec{x} \leq q + \frac{1}{n})(\alpha \mid \beta),\ n \geq \frac{1}{1-q}$

6. $(\mathrm{CP}\vec{x} = q)(\alpha \mid \beta) \to (\mathrm{CP}\vec{x} \approx q)(\alpha \mid \beta)$

[from $4_n$, $n \geq \frac{1}{q}$ and $5_n$, $n \geq \frac{1}{1-q}$ by (**Approx**)]

# $L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$ – Soundness and Completeness

**Theorem (Deduction theorem)**

*If $\Gamma$ is a set of formulas and $\Gamma, \varphi \vdash \psi$, then $\Gamma \vdash \varphi \to \psi$.*

**Theorem (Soundness theorem)**

*The axiomatic system for $L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$ is sound with respect to the class of $L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$-models.*

**Theorem (Completeness theorem)**

*If $T$ is a consistent set of formulas, then $T$ has a model.*

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# Decidability

$L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$ is clearly undecidable, but the following fragments are decidable:

1. Boolean combinations of sentences of the form $(\mathrm{CP}\vec{x} \diamond r)(\alpha(\vec{x}) \mid \beta(\vec{x}))$, where $\diamond \in \{\leqslant, \geqslant, \approx\}$ and $\alpha(\vec{x})$ and $\beta(\vec{x})$ are classical formulas without function symbols and equality sign with at most four classical quantifiers, but only one alteration (i.e., the quantifier prefix is at most $\exists\exists\forall\forall$ or $\forall\forall\exists\exists$).
2. Similar as previous fragment except that formulas $\alpha$ and $\beta$ may contain equality sign and one unary function symbol, but the quantifier prefix is restricted to $\forall\exists$ or $\exists\forall$.

RESEARCH PROBLEM: Find more decidable fragments of $L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$.

# Decidability – First steps

**Theorem (Gödel 1932, Kalmár 1933, Schütte 1934)**

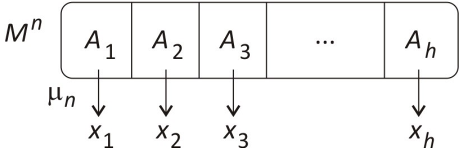
*The satisfiability and the finite satisfiability problems are decidable for the class* $[\exists^*\forall^2\exists^*, \mathrm{all}]$.

**Theorem (Shelah 1977)**

*The satisfiability and the finite satisfiability problems are decidable for the classes* $[\exists^*\forall\exists^*, \mathrm{all}, (1)]_=$, *i.e., for the class of formulas whose prenex form has the quantifier prefix of the form* $\exists^*\forall\exists^*$ *and whose language contains equality sign, arbitrary relation and constant symbols, and at most one unary function symbol.*

# Decidability – Final steps

$\Phi$, that should be satisfied, induces a finite partition on each 'admissible' classical model. Fortunately, all partitions are of the same size $h$, but unfortunately, $h$ is of exponential size related to the complexity of the formula.




$\Phi$ is satisfiable iff a system with unknowns $x_i$ has a solution in $\mathbb{Q}(\varepsilon)$:

$$\left|\begin{array}{l} \sum_{i=1}^{h} x_i = 1, x_i \geq 0 \\ \dfrac{x_{i_1} + \cdots + x_{i_k}}{x_{i_1} + \cdots + x_{i_k} + x_{i_{k+1}} + \cdots + x_{i_m}} \begin{array}{c} \geq \\ \leq \\ \approx \\ \not\approx \end{array} r, \quad x_{i_1} + \cdots + x_{i_m} > 0 \end{array}\right.$$

# Decidability – Final steps

$\Phi$ is satisfiable iff a system with unknowns $x_i$ has a solution in $\mathbb{Q}(\varepsilon)$:

$$\sum_{i=1}^{h} x_i = 1, x_i \geq 0$$

$$\frac{x_{i_1} + \cdots + x_{i_k}}{x_{i_1} + \cdots + x_{i_k} + x_{i_{k+1}} + \cdots + x_{i_m}} \; \substack{\geq \\ = \\ \leq \\ \approx \\ \not\approx} \; r, \quad x_{i_1} + \cdots + x_{i_m} > 0$$

iff a system with unknowns $x_i$ has a solution in $\mathbb{Q}(\varepsilon, K)$:

$$\sum_{i=1}^{h} x_i = 1, x_i \geq 0$$

$$\frac{x_{i_1} + \cdots + x_{i_k}}{x_{i_1} + \cdots + x_{i_k} + x_{i_{k+1}} + \cdots + x_{i_m}} \; \begin{array}{l} \substack{\geq \\ = \\ \leq} \; r \\ \geq r - K\varepsilon \\ \leq r + K\varepsilon \\ > r + \frac{1}{K} \\ < r - \frac{1}{K} \end{array} \quad x_{i_1} + \cdots + x_{i_m} > 0$$

# Decidability – Final steps

$\mathbb{Q}[\varepsilon, K]$: $Q_0(K)\varepsilon^0 + Q_1(K)\varepsilon^1 + \ldots + Q_n(K)\varepsilon^n$, where $Q_i$'s are polynomials in $K$ with rational coefficients.

$$\left(\frac{1}{2}K^2 + K\right) + \left(\frac{1}{3}K^3 + 7K\right)\varepsilon + \varepsilon^2 < \left(\frac{1}{2}K^2 + K\right) + \left(\frac{1}{3}K^3 + \frac{2}{3}K^2 + 1\right)\varepsilon$$

$\frac{1}{2}K^2 + K = \frac{1}{2}K^2 + K$ $(\frac{1}{2} = \frac{1}{2},\ 1 = 1)$
$\frac{1}{3}K^3 + 0K^2 + 7K + 0 < \frac{1}{3}K^3 + \frac{2}{3}K^2 + 0K + 1$ $(\frac{1}{3} = \frac{1}{3},\ 0 < \frac{2}{3})$

$$\mathbb{Q}(\varepsilon, K) = \left\{ \frac{P(\varepsilon,K)}{Q(\varepsilon,K)} \mid P(\varepsilon, K), Q(\varepsilon, K) \in \mathbb{Q}[\varepsilon, K], Q(\varepsilon, K) > 0 \right\}$$

$$\frac{P(\varepsilon,K)}{Q(\varepsilon,K)} < \frac{R(\varepsilon,K)}{S(\varepsilon,K)} \Leftrightarrow P(\varepsilon, K) \cdot S(\varepsilon, K) < R(\varepsilon, K) \cdot Q(\varepsilon, K)$$

RESEARCH PROBLEM: Analyse algebraic aspects of the procedure for solving systems with approx-signs.

# Non-Archimedean probability spaces in Applications

RESEARCH PROBLEM: Adapt the procedure for solving systems with approx-signs to fit other areas where one deals with infinitesimal probabilities:

- G. B. Asheim, *The Consistent Preference Approach to Deductive Reasoning in Games*, Springer, 2006.
- Robert Anderson, *Infinitesimal Methods in Mathematical Economics*, PhD thesis, Department of Economics and Department of Mathematics University of California at Berkeley, 2008.
- Joseph Halpern, *A Nonstandard (non-Archimedean) Characterization of Sequential Equilibrium, Perfect Equilibrium, and Proper Equilibrium*, International Journal of Game Theory 3838:1, 2009.
- A. Brandenburger, A. Friedenberg, H. J. Keisler, *Admissibility in Games*, Econometrica, Vol. 76, No. 2, 2008.

# Complexity

Complexity of the procedure is huge.

- T. Stojanović, T. Davidović, Z. Ognjanović, *Bee Colony Optimization for the satisfiability problem in probabilistic logic*, Applied Soft Computing **31**, 339–347, 2015
- T. Stojanović, T. Davidović, Z. Ognjanović, N. Ikodinović *A Heuristic approach to the Satisfiability problem in Default logic* (in preparation)

BCO is a stochastic, random search technique that belongs to the class of population based algorithms.

RESEARCH PROBLEM: Apply BCO or some other heuristics to the Satisfiability problem in $L^{\mathrm{P},\mathbb{I}}_{\omega\omega}$.

# Expressive power of $L^{\mathrm{P},\mathbb{I}}_{\omega\omega}$

Example

- ‘Most birds fly’: $(\mathrm{CP}x > 0.5)(\mathtt{fly}(x) \mid \mathtt{bird}(x))$, where ‘$> 0.5$’ corresponds to ‘Most’,
- ‘90% of birds can fly’: $(\mathrm{CP}x = 0.9)(\mathtt{fly}(x) \mid \mathtt{bird}(x))$,
- ‘Approximately 90% of birds fly’: $(\mathrm{CP}x \approx 0.9)(\mathtt{fly}(x) \mid \mathtt{bird}(x))$,
- ‘More than 90% of birds can fly’: $(\mathrm{CP}x > 0.9)(\mathtt{fly}(x) \mid \mathtt{bird}(x))$,
- ‘Almost all birds fly’: $(\mathrm{CP}x \approx 1)(\mathtt{fly}(x) \mid \mathtt{bird}(x))$, etc.

# Expressive power of $L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$

Example

In medicine, sensitivity is defined as the percentage of true positive cases relative to the sum of true positives and false negatives (i.e., the total number of tested patients having the disease), and specificity is defined as the percentage of true negative cases relative to the sum of true negatives and false positives (i.e., the total number of tasted patients who do not have the disease).

'The sensitivity of the test is at least 95%'
$(\mathrm{CP}x \geqslant 0.95)(\mathtt{positive}(x) \mid \mathtt{tested}(x) \wedge \mathtt{disease}(x))$

'The specificity of the test is higher than 90%'
$(\mathrm{CP}x > 0.9)(\neg\mathtt{positive}(x) \mid \mathtt{tested}(x) \wedge \neg\mathtt{disease}(x))$

# Deduction in $L^{\mathrm{P},\mathbb{I}}_{\omega\omega}$

**Example**

STATISTICAL REASONING

Most birds fly, Penguins do not fly $\vdash$ Most birds are not penguins

$L^{\mathrm{P},\mathbb{I}}_{\omega\omega}$-REASONING

$$(\mathrm{CP}x > 0.5)(\mathtt{F}(x) \mid \mathtt{B}(x)), \forall x(\mathtt{P}(x) \to \neg\mathtt{F}(x)) \vdash_{L^{\mathrm{P},\mathbb{I}}_{\omega\omega}} (\mathrm{CP}x > 0.5)(\neg\mathtt{P}(x) \mid \mathtt{B}(x))$$

# Deduction in $L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$

## Example

STATISTICAL REASONING

Most birds fly, Penguins do not fly $\vdash$ Most birds are not penguins

$L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$-REASONING

$(\mathrm{CP}x > 0.5)(\mathrm{F}(x) \mid \mathrm{B}(x)), \forall x(\mathrm{P}(x) \to \neg\mathrm{F}(x)) \vdash_{L_{\omega\omega}^{\mathrm{P},\mathbb{I}}} (\mathrm{CP}x > 0.5)(\neg\mathrm{P}(x) \mid \mathrm{B}(x))$

## Example

COMMONSENSE REASONING

Almost all birds fly, Penguins do not fly $\vdash$ Almost all birds are not penguins

DEFAULT REASONING

$\mathrm{B}(x) \rightarrowtail \mathrm{F}(x), \mathrm{P}(x) \to \neg\mathrm{F}(x) \vdash_{KLM} \mathrm{B}(x) \rightarrowtail \neg\mathrm{P}(x)$

$L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$-REASONING

$(\mathrm{CP}x \approx 1)(\mathrm{F}(x) \mid \mathrm{B}(x)), \forall x(\mathrm{P}(x) \to \neg\mathrm{F}(x)) \vdash_{L_{\omega\omega}^{\mathrm{P},\mathbb{I}}} (\mathrm{CP}x \approx 1)(\neg\mathrm{P}(x) \mid \mathrm{B}(x))$

# KLM (Kraus-Lehman-Magidor)

## KLM

- $\dfrac{}{\alpha \mid\sim \alpha}$
- $\dfrac{\alpha \mid\sim \gamma}{\beta \mid\sim \gamma}$ if $\vdash \alpha \leftrightarrow \beta$
- $\dfrac{\alpha \mid\sim \beta}{\alpha \mid\sim \gamma}$ if $\vdash \beta \rightarrow \gamma$
- $\dfrac{\alpha \mid\sim \beta \;\; \alpha \mid\sim \gamma}{\alpha \wedge \beta \mid\sim \gamma}$
- $\dfrac{\alpha \mid\sim \beta \;\; \alpha \mid\sim \gamma}{\alpha \mid\sim \beta \wedge \gamma}$
- $\dfrac{\alpha \mid\sim \gamma \; \beta \mid\sim \gamma}{\alpha \vee \beta \mid\sim \gamma}$
- $\dfrac{\alpha \mid\not\sim \neg\beta \;\; \alpha \mid\sim \gamma}{\alpha \wedge \beta \mid\sim \gamma}$

## Probabilistic first-order logic

$$\boxed{(\mathrm{CP}\vec{x} \approx 1)(\alpha \mid \beta) \equiv \beta \rightsquigarrow_{\vec{x}} \alpha}$$

$\mathtt{Married}(x,y) \rightsquigarrow_{x,y} \neg\mathtt{Relatives}(x,y)$

$\vdots$

- $\alpha \rightsquigarrow_{\vec{x}} \beta, \beta \rightarrow \gamma \vdash \alpha \rightsquigarrow_{\vec{x}} \gamma$
- $\alpha \rightsquigarrow_{\vec{x}} \beta, \alpha \rightsquigarrow_{\vec{x}} \gamma \vdash \alpha \wedge \beta \rightsquigarrow_{\vec{x}} \gamma$

$\vdots$

# Default reasoning in general

RESEARCH PROBLEM: From $L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$-point of view, consider all 'problematic' properties for a non-monotonic consequence relation. These properties (Rationality, Specificity, Property inheritance, Ambiguity preservation, Syntax independence) are listed in:

- S. Benferhat, A. Saffiotti, P. Smets, *Belief functions and default reasoning*, Artificial Intelligence **122**, 1–69, 2000.

# $L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$ as a framework for other logics

Many logical systems can be regarded as fragments of first-order classical logic (modal logics, some description logics, ...)

RESEARCH PROBLEM: More or less analogously, $L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$ might be considered as a framework for many interesting logics (probabilistic modal logic, probabilistic description logic ...)

# Extensions of $L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$

## Research problem:

### Logical symbols

- *Variables*
- *Connectives*
- *Probability operators*: $(\mathrm{CP} \stackrel{\leq}{\stackrel{=}{\geq}} r)$, $r \in \mathbb{I}$ $(\mathrm{CP} \approx q)$, $q \in \mathbb{I} \cap \mathbb{Q}$
- *Quantifiers*
- *Probability quantifiers*: $(\mathrm{CP} \cdots \stackrel{\leq}{\stackrel{=}{\geq}} r)$, $r \in \mathbb{I}$ $(\mathrm{CP} \cdots \approx q)$, $q \in \mathbb{I} \cap \mathbb{I}$
- *Punctuation*

### Non-logical symbols

- *Relation symbols* $\mathrm{Rel}$
- *Function symbols* $\mathrm{Fun}$

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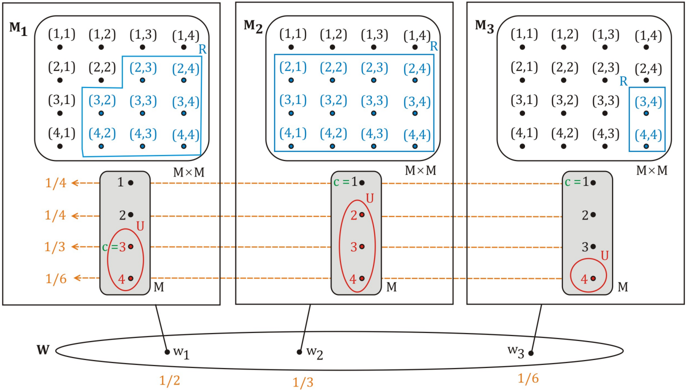
– $\mathrm{Rel} \cap \mathrm{Fun} = \emptyset$, $L = \mathrm{Rel} \cup \mathrm{Fun}$

– $\mathbf{ar} : \mathrm{Rel} \cup \mathrm{Fun} \to \mathbb{N}$

– $p \in \mathrm{Rel}$, $\mathrm{ar}(p) = 0$ propositional letter

– $c \in \mathrm{Fun}$, $\mathrm{ar}(f) = 0$ constant symbol

# Extensions of $L_{\omega\omega}^{\mathrm{P},\mathbb{I}}$




$\mathbf{W} \models \mathrm{P}_{=5/6}(\mathrm{P}xy > 0,09)R(x,y)$
$\mathbf{W} \models \mathrm{P}_{=1}(U(c) \Rightarrow R(c,c))$
$\mathbf{W} \models \mathrm{CP}_{=3/5}(U(c) \mid (\mathrm{P}x \geqslant 0,5)U(x))$

# Thank you for your attention