A Logic with Upper and Lower Probability Operators (LUPP)

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Outline of the talk

Example;
Syntax and Semantics;
Axioms and inference rules;
Construction of the canonical model and strong completeness theorem;
Decidability;

$LUPP_{Fr}(n)$. 
Outline of the talk

- Example;
- Syntax and Semantics;
- Axioms and inference rules;
- Construction of the canonical model and strong completeness theorem;
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- $LUPP^{Fr(n)}$. 
Example

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Example

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\[ P = \{ \mu \} \]

\[ P^\ast(X) = \sup \{ \mu(X) | \mu \in P \} \]

\[ P^\ast(X) = \inf \{ \mu(X) | \mu \in P \} \]
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Example

$P$ – a set of probability measures

$P^*(X) = \sup\{\mu(X) \mid \mu \in P\}, \quad P_*(X) = \inf\{\mu(X) \mid \mu \in P\}$
Syntax

Let $S$ be the set of rational numbers from $[0,1]$ and let $L = \{p, q, r, \ldots\}$ be a countable set of propositional letters. The language of logic LUPP consists of the elements of:

- set $L$,
- classical propositional connectives $\neg$ and $\land$,
- the lists of upper probability operators $U \geq s$ and $L \geq s$, for every $s \in S$. 
Let $S$ be the set of rational numbers from $[0, 1]$ and let $\mathcal{L} = \{p, q, r, \ldots\}$ be a countable set of propositional letters. The language of logic $LUPP$ consists of the elements of:

- set $\mathcal{L}$,
- classical propositional connectives $\neg$ and $\land$,
- the lists of upper probability operators $U_{\geq s}$ and $L_{\geq s}$, for every $s \in S$. 
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Example

\[ R \cup L = 0 \]

\[ B \cup U = 0.7 \]

\[ R \land L \geq 0.3 \land G \land U \leq 0.2 \Rightarrow L \geq 0.5 \]
Example

\[ L_{=0} R, L_{=0} B; \]
Example

\[ L = 0 R, L = 0 B; \quad U = 0.7 R, U = 0.7 B \]
Example

$L = 0R, L = 0B; \quad U = 0.7R, U = 0.7B$

$((U \leq 0.3G \land L \geq 0.3G) \land U \leq 0.2R) \Rightarrow L \geq 0.5B.$
Definition (LUPP-structure)

Any tuple $M = ⟨W, H, P, υ⟩$, where:

- $W$ is a nonempty set of worlds.
- $H$ is an algebra of subsets of $W$.
- $P$ is a set of finitely additive probability measures defined on $H$.
- $υ : W \times L \rightarrow \{true, false\}$ evaluations of the primitive propositions.

Definition (Satisfiability relation)

$M |= α$ iff $υ(w)(α) = true$, for all $w \in W$.

$M |= U ≥ s α$ iff $P⋆([α]) ≥ s$.

$M |= L ≥ s α$ iff $P⋆([α]) ≥ s$.

$M |= ¬φ$ iff it is not the case that $M |= φ$.

$M |= φ ∧ ψ$ iff $M |= φ$ and $M |= ψ$. 

Semantics
Semantics

Definition (LUPP-structure)
Any tuple \( M = \langle W, H, P, \nu \rangle \), where:
- \( W \) is a nonempty set of worlds.
- \( H \) is an algebra of subsets of \( W \).
- \( P \) is a set of finitely additive probability measures defined on \( H \).
- \( \nu : W \times \mathcal{L} \rightarrow \{true, false\} \) evaluations of the primitive propositions.
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Semantics

Definition (LUPP-structure)

Any tuple $M = \langle W, H, P, \nu \rangle$, where:
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- $P$ is a set of finitely additive probability measures defined on $H$.
- $\nu : W \times L \rightarrow \{true, false\}$ evaluations of the primitive propositions.

Definition (Satisfiability relation)

- $M \models \alpha$ iff $\nu(w)(\alpha) = true$, for all $w \in W$,
- $M \models U_s \alpha$ iff $P^*([\alpha]) \geq s$,
- $M \models L_s \alpha$ iff $P_*([\alpha]) \geq s$,
- $M \models \neg \phi$ iff it is not the case that $M \models \phi$,
- $M \models \phi \land \psi$ iff $M \models \phi$ and $M \models \psi$.  


Semantics

**Definition (LUPP-structure)**

Any tuple \(M = \langle W, H, P, \nu \rangle\), where:

- \(W\) is a nonempty set of *worlds*.
- \(H\) is an algebra of subsets of \(W\).
- \(P\) is a set of finitely additive probability measures defined on \(H\).
- \(\nu : W \times \mathcal{L} \longrightarrow \{true, false\}\) evaluations of the primitive propositions.

**Definition (Satisfiability relation)**

- \(M \models \alpha\) iff \(\nu(w)(\alpha) = true\), for all \(w \in W\),
- \(M \models U_{\geq s} \alpha\) iff \(P^*([\alpha]) \geq s\),
- \(M \models L_{\geq s} \alpha\) iff \(P_*([\alpha]) \geq s\),
- \(M \models \neg \phi\) iff it is not the case that \(M \models \phi\),
- \(M \models \phi \land \psi\) iff \(M \models \phi\) and \(M \models \psi\).
Axiomatization issues
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1) Non-compactness of \textit{LUPP}-logic
   - consequence: there is no finitary axiomatization
Axiomatization issues

1) Non-compactness of $LUPP$-logic
   - consequence: there is no finitary axiomatization

2) Expressiveness of our propositional language
   - the representation theorem (Anger, Lembcke 1985)
Theorem (Anger and Lembcke, 1985)

Let $W$ be a set, $H$ an algebra of subsets of $W$, and $f$ a function $f : H \to [0, 1]$. There exists a set $P$ of probability measures such that $f = P$ if and only if $f$ satisfies the following three properties:

1. $f(\emptyset) = 0$,
2. $f(W) = 1$,
3. For all natural numbers $m$, $n$, $k$ and all subsets $A_1, \ldots, A_m$ in $H$, if $\{A_1, \ldots, A_m\}$ is an $(n, k)$-cover of $(A, W)$, then $k + nf(A) \leq \sum_{i=1}^{m} f(A_i)$.
Representation Theorem

Theorem (Anger and Lembcke, 1985)

Let $W$ be a set, $H$ an algebra of subsets of $W$, and $f$ a function $f : H \rightarrow [0, 1]$. There exists a set $P$ of probability measures such that $f = P^*$ iff $f$ satisfies the following three properties:

1. $f(\emptyset) = 0$,
2. $f(W) = 1$,
3. for all natural numbers $m, n, k$ and all subsets $A_1, \ldots, A_m$ in $H$, if $
\{\{A_1, \ldots, A_m\}\}$ is an $(n, k)$-cover of $(A, W)$, then $k + nf(A) \leq \sum_{i=1}^{m} f(A_i)$. 
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Axioms and Inference Rules

Axiom schemes

(1) all instances of the classical propositional tautologies

(2) \( U \leq 1 \alpha \)

(3) \( U \leq r \alpha \rightarrow U < s \alpha, s > r \)

(4) \( U < s \alpha \rightarrow U \leq s \alpha \)

(5) \( (U \leq r_1 \alpha_1 \land \cdots \land U \leq r_m \alpha_m) \rightarrow U \leq r \alpha, \text{if } \alpha \rightarrow \bigvee J \subseteq \{1, \ldots, m\}, |J| = k + n \land \bigwedge j \in J \alpha_j \) and \( \bigvee J \subseteq \{1, \ldots, m\}, |J| = k \land \bigwedge j \in J \alpha_j \) are propositional tautologies, where \( r = \sum_{i=1}^{m} r_i - k \), \( n \neq 0 \)

(6) \( \neg (U \leq r_1 \alpha_1 \land \cdots \land U \leq r_m \alpha_m), \text{if } \bigvee J \subseteq \{1, \ldots, m\}, |J| = k \land \bigwedge j \in J \alpha_j \) is a propositional tautology and \( \sum_{i=1}^{m} r_i < k \)

(7) \( L = 1(\alpha \rightarrow \beta) \rightarrow (U \geq s \alpha \rightarrow U \geq s \beta) \)
A Logic with Upper and Lower Probability Operators (LUPP)
Axioms and Inference Rules

**Axiom schemes**

1. all instances of the classical propositional tautologies
2. \( U_{\leq 1} \alpha \land L_{\leq 1} \alpha \)
3. \( U_{\leq r} \alpha \rightarrow U_{< s} \alpha, \ s > r \)
4. \( U_{< s} \alpha \rightarrow U_{\leq s} \alpha \)
5. \((U_{\leq r_1} \alpha_1 \land \cdots \land U_{\leq r_m} \alpha_m) \rightarrow U_{\leq r} \alpha, \) if \( \alpha \rightarrow \bigvee_{J \subseteq \{1, \ldots, m, |J|=k+n} \land_{j \in J} \alpha_j \) and \( \bigvee_{J \subseteq \{1, \ldots, m}, |J|=k \land_{j \in J} \alpha_j \) are propositional tautologies, where
   \[ r = \frac{\sum_{i=1}^{m} r_i - k}{n}, \ n \neq 0 \]
6. \(-((U_{\leq r_1} \alpha_1 \land \cdots \land U_{\leq r_m} \alpha_m), \) if \( \bigvee_{J \subseteq \{1, \ldots, m, |J|=k} \land_{j \in J} \alpha_j \) is a propositional tautology and \( \sum_{i=1}^{m} r_i < k \)
7. \( L_{=1}(\alpha \rightarrow \beta) \rightarrow (U_{\geq s} \alpha \rightarrow U_{\geq s} \beta) \)
Inference Rules

(1) From $\rho$ and $\rho \to \sigma$ infer $\sigma$.

(2) From $\alpha$ infer $L \geq 1 - \alpha$.

(3) From the set of premises $\{\phi \to U \geq s - 1 \alpha | k \geq 1\}$ infer $\phi \to U \geq s \alpha$.

(4) From the set of premises $\{\phi \to L \geq s - 1 \alpha | k \geq 1\}$ infer $\phi \to L \geq s \alpha$. 
Inference Rules

(1) From $\rho$ and $\rho \rightarrow \sigma$ infer $\sigma$

(2) From $\alpha$ infer $L \geq 1 \alpha$

(3) From the set of premises

$$\{ \phi \rightarrow U_{\geq s - \frac{1}{k}} \alpha \mid k \geq \frac{1}{s} \}$$

infer $\phi \rightarrow U_{\geq s} \alpha$

(4) From the set of premises

$$\{ \phi \rightarrow L_{\geq s - \frac{1}{k}} \alpha \mid k \geq \frac{1}{s} \}$$

infer $\phi \rightarrow L_{\geq s} \alpha$. 
Theorem

Every consistent set can be extended to a maximal consistent set.

Sketch of the proof:

Let \( T \) be a consistent set of formulas. We define a sequence of sets \( T_i \), as follows:

1. \( T_0 = T \cup \text{Con}(T) \cup \{ \alpha \mid \alpha \in \text{Con}(T) \} \)
2. For every \( i \geq 0 \),
   (a) if \( T_i \cup \{ \phi_i \} \) is consistent, then \( T_{i+1} = T_i \cup \{ \phi_i \} \), otherwise
   (b) if \( \phi_i \) is of the form \( \psi \rightarrow U \geq s \beta \), then \( T_{i+1} = T_i \cup \{ \neg \phi_i, \psi \rightarrow \neg U \geq s - 1 \} \), for some positive integer \( n \), so that \( T_{i+1} \) is consistent, otherwise
   (c) if \( \phi_i \) is of the form \( \psi \rightarrow L \geq s \beta \), then \( T_{i+1} = T_i \cup \{ \neg \phi_i, \psi \rightarrow \neg L \geq s - 1 \} \), for some positive integer \( n \), so that \( T_{i+1} \) is consistent, otherwise
   (d) \( T_{i+1} = T_i \cup \{ \neg \phi_i \} \).
3. \( T^* = \bigcup_{i=0}^{\infty} T_i \).
Theorem

*Every consistent set can be extended to a maximal consistent set.*

Sketch of the proof:
Theorem

Every consistent set can be extended to a maximal consistent set.

Sketch of the proof: Let $T$ be a consistent set of formulas. We define a sequence of sets $T_i$, as follows:

1. $T_0 = T \cup Cn_C(T) \cup \{L_{\geq 1}\alpha \mid \alpha \in Cn_C(T)\}$
2. For every $i \geq 0$,
   - if $T_i \cup \{\phi_i\}$ is consistent, then $T_{i+1} = T_i \cup \{\phi_i\}$, otherwise
   - if $\phi_i$ is of the form $\psi \rightarrow U_{\geq s}\beta$, then $T_{i+1} = T_i \cup \{\neg \phi_i, \psi \rightarrow \neg U_{\geq s-\frac{1}{n}}\beta\}$, for some positive integer $n$, so that $T_{i+1}$ is consistent, otherwise
   - if $\phi_i$ is of the form $\psi \rightarrow L_{\geq s}\beta$, then $T_{i+1} = T_i \cup \{\neg \phi_i, \psi \rightarrow \neg L_{\geq s-\frac{1}{n}}\beta\}$, for some positive integer $n$, so that $T_{i+1}$ is consistent, otherwise
   - $T_{i+1} = T_i \cup \{\neg \phi_i\}$.
3. $T^* = \bigcup_{i=0}^{\infty} T_i$. 
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Construction of the canonical model

Theorem (Strong completeness)

A set of formulas $T$ is consistent iff it is LUPP $\mathcal{Meas}$-satisfiable.

Sketch of the proof:

1. Every consistent set $T$ can be extended to a maximal consistent set $T^{\star}$.

2. We use $T^{\star}$ to construct a canonical model.

Definition

If $T^{\star}$ is the maximally consistent set of formulas, then a tuple $M_{T^{\star}} = \langle W, H, P, \nu \rangle$ is defined:

- $W = \{ w \mid w\rhd C_{n}C(T) \}$,
- $H = \{ [\alpha] \mid \alpha \in \text{For } C \}$, where $[\alpha] = \{ w \in W \mid w\rhd \alpha \}$,
- $P$ is any set of probability measures such that $P^{\star}([\alpha]) = \sup \{ s \mid U \geq s \}$, $\alpha \in T^{\star}$,

for every world $w$ and every propositional letter $p$,

$\nu(w, p) = \text{true}$ iff $w\rhd p$.
Theorem (Strong completeness)

A set of formulas $T$ is consistent iff it is $\text{LUPP}_{\text{Meas}}$ — satisfiable.

Sketch of the proof:
Theorem (Strong completeness)

A set of formulas $T$ is consistent iff it is $\text{LUPP}_{\text{Meas}}$ - satisfiable.

Sketch of the proof:

1. Every consistent set $T$ can be extended to a maximal consistent set $T^*$.
2. We use $T^*$ to construct a canonical model.

Definition

If $T^*$ is the maximally consistent set of formulas, then a tuple $M_{T^*} = \langle W, H, P, \nu \rangle$ is defined:

- $W = \{ w \mid w \models Cn_C(T) \}$,
- $H = \{ [\alpha] \mid \alpha \in \text{For}_C \}$, where $[\alpha] = \{ w \in W \mid w \models \alpha \}$,
- $P$ is any set of probability measures such that $P^*([\alpha]) = \sup \{ s \mid U_{\geq s} \alpha \in T^* \}$,
- for every world $w$ and every propositional letter $p$, $\nu(w, p) = \text{true}$ iff $w \models p$. 
Theorem (Strong completeness)

A set of formulas $T$ is consistent iff it is $LUPP_{\text{Meas}}$-satisfiable.

Sketch of the proof:

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Definition

If $T^*$ is the maximally consistent set of formulas, then a tuple $M_{T^*} = \langle W, H, P, \nu \rangle$ is defined:

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- $H = \{ [\alpha] \mid \alpha \in For_C \}$, where $[\alpha] = \{ w \in W \mid w \models \alpha \}$,
- $P$ is any set of probability measures such that $P^*([\alpha]) = \sup \{ s \mid U_{\geq s} \alpha \in T^* \}$,
- for every world $w$ and every propositional letter $p$, $\nu(w, p) = \text{true}$ iff $w \models p$. 
Decidability
Decidability

**Theorem (Decidability)**

An *satisfiability problem* for LUPP-formulas is *NP-complete*. 
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$LUPP^{Fr}(n)$

**Definition** ($LUPP^{Fr}(n)$-structure)

Any tuple $M = \langle W, H, P, \nu \rangle$, where:

- $W$ is a nonempty set of worlds.
- $H$ is an algebra of subsets of $W$.
- $P$ is a set of finitely additive probability measures such that for all $\mu \in P$,
  
  $\mu : H \to \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$

- $\nu : W \times L \to \{true, false\}$ evaluations of the primitive propositions.

**Consequence:**

- The axiomatization is finite.
Definition \((LUPP^{Fr(n)}\text{-structure})\)

Any tuple \(M = \langle W, H, P, \nu \rangle\), where:

- \(W\) is a nonempty set of \textit{worlds}.
- \(H\) is an algebra of subsets of \(W\).
- \(P\) is a set of finitely additive probability measures such that for all \(\mu \in P\),
  \(\mu : H \rightarrow \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}\).
- \(\nu : W \times \mathcal{L} \rightarrow \{\text{true}, \text{false}\}\) evaluations of the primitive propositions.

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Definition \((LUPP^{Fr(n)}\)-structure)\)

Any tuple \(M = \langle W, H, P, \nu \rangle\), where:

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- \(P\) is a set of finitely additive probability measures such that for all \(\mu \in P\), \(\mu : H \rightarrow \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}\).
- \(\nu : W \times \mathcal{L} \rightarrow \{true, false\}\) evaluations of the primitive propositions.

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Further Work
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- Iterations of lower and upper probability operators
- First order lower and upper probability logic
References
