A Logic with Upper and Lower Probability Operators(LUPP)

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LAP 2015, Dubrovnik, Croatia.

Outline of the talk

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- Example;
- Syntax and Semantics;
- Axioms and inference rules;
- Construction of the canonical model and strong completeness theorem;
- Decidability;
- $LUPP^{Fr(n)}$.

Example

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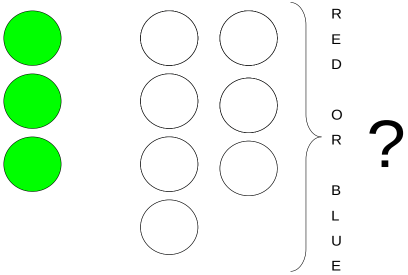

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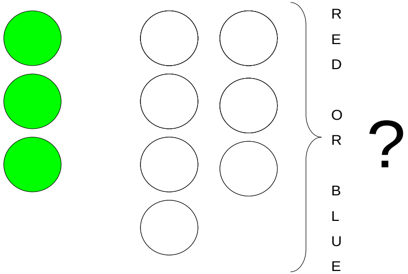

P – a set of probability measures

$P^{\star}(X) = \sup\{\mu(X) \mid \mu \in P\}, \quad P_{\star}(X) = \inf\{\mu(X) \mid \mu \in P\}$

Syntax

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Let S be the set of rational numbers from $[0, 1]$ and let $\mathcal{L} = \{p, q, r, \dots\}$ be a countable set of propositional letters. The language of logic *LUPP* consists of the elements of:

- set $\mathcal{L}$,
- classical propositional connectives $\neg$ and $\wedge$,
- the lists of upper probability operators $U_{\geq s}$ and $L_{\geq s}$, for every $s \in S$.

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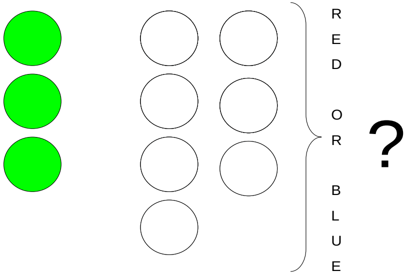

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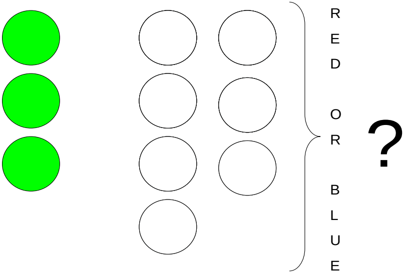

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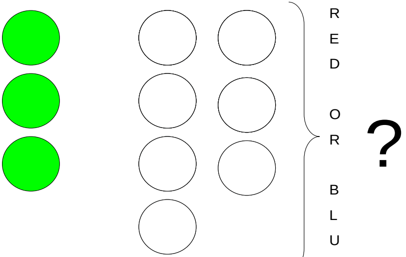

$$L_{=0}R, L_{=0}B; \qquad U_{=0.7}R, U_{=0.7}B$$

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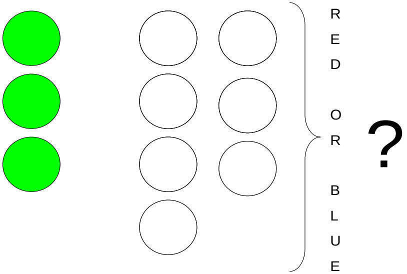

$$L_{=0}R,\ L_{=0}B; \qquad U_{=0.7}R,\ U_{=0.7}B$$

$$((U_{\leq 0.3}G \wedge L_{\geq 0.3}G) \wedge U_{\leq 0.2}R) \Rightarrow L_{\geq 0.5}B.$$

Semantics

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Definition (*LUPP*-structure)

Any tuple $M = \langle W, H, P, \upsilon \rangle$, where:

- W is a nonempty set of *worlds*.
- H is an algebra of subsets of W.
- P is a set of finitely additive probability measures defined on H.
- $\upsilon : W \times \mathcal{L} \longrightarrow \{\mathit{true}, \mathit{false}\}$ evaluations of the primitive propositions.

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Definition (Satisfiability relation)

- $M \models \alpha$ iff $\upsilon(w)(\alpha) = \mathit{true}$, for all $w \in W$,
- $M \models U_{\geq s}\alpha$ iff $P^{\star}([\alpha]) \geq s$,
- $M \models L_{\geq s}\alpha$ iff $P_{\star}([\alpha]) \geq s$,
- $M \models \neg\phi$ iff it is not the case that $M \models \phi$,
- $M \models \phi \wedge \psi$ iff $M \models \phi$ and $M \models \psi$.

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2) Expressiveness of our propositional language
   - the representation theorem (Anger, Lembcke 1985)

Representation Theorem

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Theorem (Anger and Lembcke, 1985)

Let W be a set, H an algebra of subsets of W, and f a function $f : H \longrightarrow [0,1]$. There exists a set P of probability measures such that $f = P^\star$ iff f satisfies the following three properties:

(1) $f(\emptyset) = 0$,

(2) $f(W) = 1$,

(3) *for all natural numbers m, n, k and all subsets $A_1, \ldots, A_m$ in H, if $\{\{A_1, \ldots, A_m\}\}$ is an (n,k)-cover of (A, W), then $k + nf(A) \leq \sum_{i=1}^{m} f(A_i)$.*

Axiom schemes

Axiom schemes

(1) all instances of the classical propositional tautologies

(2) $U_{\leq 1}\alpha \wedge L_{\leq 1}\alpha$

(3) $U_{\leq r}\alpha \rightarrow U_{<s}\alpha$, $s > r$

(4) $U_{<s}\alpha \rightarrow U_{\leq s}\alpha$

(5) $(U_{\leq r_1}\alpha_1 \wedge \cdots \wedge U_{\leq r_m}\alpha_m) \rightarrow U_{\leq r}\alpha$, if $\alpha \rightarrow \bigvee_{J \subseteq \{1,\ldots,m\}, |J|=k+n} \bigwedge_{j \in J} \alpha_j$ and $\bigvee_{J \subseteq \{1,\ldots,m\}, |J|=k} \bigwedge_{j \in J} \alpha_j$ are propositional tautologies, where $r = \frac{\sum_{i=1}^{m} r_i - k}{n}$, $n \neq 0$

(6) $\neg(U_{\leq r_1}\alpha_1 \wedge \cdots \wedge U_{\leq r_m}\alpha_m)$, if $\bigvee_{J \subseteq \{1,\ldots,m\}, |J|=k} \bigwedge_{j \in J} \alpha_j$ is a propositional tautology and $\sum_{i=1}^{m} r_i < k$

(7) $L_{=1}(\alpha \rightarrow \beta) \rightarrow (U_{\geq s}\alpha \rightarrow U_{\geq s}\beta)$

Inference Rules

Inference Rules

(1) From ρ and $\rho \rightarrow \sigma$ infer σ

(2) From α infer $L_{\geq 1}\alpha$

(3) From the set of premises

$$\{\phi \rightarrow U_{\geq s-\frac{1}{k}}\alpha \mid k \geq \frac{1}{s}\}$$

infer $\phi \rightarrow U_{\geq s}\alpha$

(4) From the set of premises

$$\{\phi \rightarrow L_{\geq s-\frac{1}{k}}\alpha \mid k \geq \frac{1}{s}\}$$

infer $\phi \rightarrow L_{\geq s}\alpha$.

Construction of the canonical model

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Theorem

Every consistent set can be extended to a maximal consistent set.

Sketch of the proof:

Construction of the canonical model

Theorem

Every consistent set can be extended to a maximal consistent set.

Sketch of the proof:Let T be a consistent set of formulas. We define a sequence of sets T_i, as follows:

(1) $T_0 = T \cup Cn_C(T) \cup \{L_{\geq 1}\alpha \mid \alpha \in Cn_C(T)\}$

(2) for every $i \geq 0$,

  (a) if $T_i \cup \{\phi_i\}$ is consistent, then $T_{i+1} = T_i \cup \{\phi_i\}$, otherwise
  (b) if ϕ_i is of the form $\psi \rightarrow U_{\geq s}\beta$, then $T_{i+1} = T_i \cup \{\neg\phi_i, \psi \rightarrow \neg U_{\geq s-\frac{1}{n}}\beta\}$, for some positive integer n, so that T_{i+1} is consistent, otherwise
  (c) if ϕ_i is of the form $\psi \rightarrow L_{\geq s}\beta$, then $T_{i+1} = T_i \cup \{\neg\phi_i, \psi \rightarrow \neg L_{\geq s-\frac{1}{n}}\beta\}$, for some positive integer n, so that T_{i+1} is consistent, otherwise
  (d) $T_{i+1} = T_i \cup \{\neg\phi_i\}$.

(3) $T^\star = \bigcup_{i=0}^{\infty} T_i$.

Theorem (Strong completeness)

A set of formulas T is consistent iff it is $LUPP_{Meas}$ − satisfiable.

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A set of formulas T is consistent iff it is $LUPP_{Meas}-$satisfiable.

Sketch of the proof:

1. Every consistent set T can be extended to a maximal consistent set $T^\star$.
2. We use $T^\star$ to construct a canonical model.

Definition

If $T^\star$ is the maximally consistent set of formulas, then a tuple $M_{T^\star} = \langle W, H, P, v\rangle$ is defined:

- $W = \{w \mid w \models Cn_C(T)\}$,
- $H = \{[\alpha] \mid \alpha \in For_C\}$, where $[\alpha] = \{w \in W \mid w \models \alpha\}$,
- P is any set of probability measures such that $P^\star([\alpha]) = \sup\{s \mid U_{\geq s}\alpha \in T^\star\}$,
- for every world w and every propositional letter p, $v(w, p) = \mathit{true}$ iff $w \models p$.

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Decidability

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Theorem (Decidability)

A satisfiability problem for LUPP-formulas is NP-complete.

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Consequence:

- The axiomatization is finite.

Further Work

Further Work

- Iterations of lower and upper probability operators
- First order lower and upper probability logic

References

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