Finite model property of interpretability logics via filtrations

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- for each $w \in W$, $S_w \subseteq R[w] \times \mathcal{P}(R[w])$
  - if wRu then $uS_w\{u\}$
  - if uS_wV and vS_wZ_v for all $v \in V$ then $uS_w(\cup Z_v)$
  - if $wRuRv$ then $uS_w\{v\}$

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Existence: $\widetilde{R}$ is a filtration iff $R^{min} \subseteq \widetilde{R} \subseteq R^{max}$, where:

- $R^{min} = \{([w], [u]) : wRu\}$
- $[w]R^{max}[u]$ iff for all $\Box A \in \Gamma$ we have: if $w \Vdash \Box A$ then $u \Vdash A$

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Example: each formula of the basic modal language which has a transitive model, also has a finite transitive model.
The proof is the same, but using the particular filtration which preserves transitivity: $[w]R^t[u]$ iff for all $\Box A \in \Gamma$ we have: if $w \Vdash \Box A$ then $u \Vdash A \wedge \Box A$.

Refining filtration

It is not always possible to define a suitable filtration which preserves a desired property. Shehtman (1993) proposes a refinement of filtration using an appropriate equivalence relation $\sim \subseteq \equiv_\Gamma$. Filtration is defined in the same way, but over $W/_\sim$.

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In the definition of generalized Veltman models there are plenty of properties we need to preserve under filtration. Refining filtration using bisimulations shows to be a good tool to accomplish this.

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Let W be a generalized Veltman model, Γ an adequate set of formulas and $\sim \subseteq \equiv_\Gamma$ an equivalence relation on W.
For $V \subseteq W$, put $V_\sim = \{[w] : w \in V\}$.

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Existence of a filtration over $W/_\sim$: we prove that the filtration over this particular $\sim$ is in fact a generalized Veltman model.

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Using this main idea, we obtained an alternative proof of the finite model property of interpretability logic **IL** w.r.t. Veltman models, and we proved the finite model property of the systems **ILM** and $\mathbf{ILM_0}$ w.r.t. generalized Veltman models.