# Finite model property of interpretability logics via filtrations

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Satisfaction:  $w \Vdash A \rhd B$  if for all u s.t. wRu and  $u \Vdash A$  there is v s.t.  $uS_wv$  and  $v \Vdash B$ 

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  - if wRu then  $uS_w\{u\}$
  - if  $uS_wV$  and  $vS_wZ_v$  for all  $v \in V$  then  $uS_w(\cup Z_v)$
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Existence:  $\widetilde{R}$  is a filtration iff  $R^{min}\subseteq\widetilde{R}\subseteq R^{max}$ , where:

- $ightharpoonup R^{min} = \{([w], [u]) : wRu\}$
- ▶  $[w]R^{max}[u]$  iff for all  $\Box A \in \Gamma$  we have: if  $w \Vdash \Box A$  then  $u \Vdash A$

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 $w \Vdash \Box A \text{ then } u \Vdash A \land \Box A.$ 

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In the definition of generalized Veltman models there are plenty of properties we need to preserve under filtration. Refining filtration using bisimulations shows to be a good tool to accomplish this.

Let W be a generalized Veltman model,  $\Gamma$  an adequate set of formulas and  $\sim \subseteq \equiv_{\Gamma}$  an equivalence relation on W. For  $V \subseteq W$ , put  $V_{\sim} = \{[w] : w \in V\}$ .

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Existence: using a particular  $\sim$  which is defined using bisimulations



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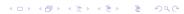
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Existence of a filtration over  $W/_{\sim}$ : we prove that the filtration over this particular  $\sim$  is in fact a generalized Veltman model.



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Using this main idea, we obtained an alternative proof of the finite model property of interpretability logic **IL** w.r.t. Veltman models, and we proved the finite model property of the systems **ILM** and **ILM**<sub>0</sub> w.r.t. generalized Veltman models.