Finite model property of interpretability logics via filtrations

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Interpretability logic

Syntax: the basic modal language enriched with a binary modal operator ▷
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Semantics: Veltman models

- $W \neq \emptyset$
- $R \subseteq W \times W$ transitive and reverse well-founded

Satisfaction: $w \models A \triangleright B$ if for all $u$ s.t. $wRu$ and $u \models A$ there is $v$ s.t. $u \triangleleft w v$ and $v \models B$
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  - if $wR u$ then $uS_w u$
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Interpretability logic

Syntax: the basic modal language enriched with a binary modal operator ⊢

Semantics: generalized Veltman models

- $W \neq \emptyset$
- $R \subseteq W \times W$ transitive and reverse well-founded
- for each $w \in W$, $S_w \subseteq R[w] \times \mathcal{P}(R[w])$
  - if $wRu$ then $uS_w\{u\}$
  - if $uS_wV$ and $vS_wZ_v$ for all $v \in V$ then $uS_w(\bigcup Z_v)$
  - if $wRuRv$ then $uS_w\{v\}$

Satisfaction: $w \models A \triangleright B$ if for all $u$ s.t. $wRu$ and $u \models A$ there is $V$ s.t. $uS_wV$ and $v \models B$ for all $v \in V$
Filtrations of Kripke models

Let $\Gamma$ be a set of formulas closed under taking subformulas.
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Let $\Gamma$ be a set of formulas closed under taking subformulas. For $w, u \in W$, put $w \equiv_{\Gamma} u$ if for all $A \in \Gamma$ we have $w \vDash A$ iff $u \vDash A$. Then $\equiv_{\Gamma}$ is an equivalence relation. Filtration is a model over $W/\equiv_{\Gamma}$ s.t.

- if $wRu$ then $[w] \tilde{R}[u]$
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Filtration is a model over $W/\equiv_{\Gamma}$ s.t.

- if $wRu$ then $[w][u]$
- if $w \models \Box A \in \Gamma$ and $[w][u]$ then $u \models A$

Filtration theorem: for all $w \in W$ and $A \in \Gamma$, $[w] \models A$ iff $w \models A$.

Proof: by induction

Existence: $\tilde{R}$ is a filtration iff $R_{\text{min}} \subseteq \tilde{R} \subseteq R_{\text{max}}$, where:

- $R_{\text{min}} = \{(w, u) : wRu\}$
- $[w]R_{\text{max}}[u]$ iff for all $\Box A \in \Gamma$ we have: if $w \models \Box A$ then $u \models A$
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- $[w] \models p$ iff $w \models p$, for each propositional variable $p \in \Gamma$
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Filtration is a model over \( W/\equiv_\Gamma \) s.t.

\begin{itemize}
  \item if \( wRu \) then \([w]\tilde{R}[u]\)
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Finite model property via filtration

Each satisfiable formula of the basic modal language has a finite model.

Proof: Let $A$ be a satisfiable formula, $W$ a model and $w \in W$ s.t. $w \models A$. Let $\Gamma$ be the set of all subformulas of $A$. Since $\Gamma$ is finite, $W/\equiv\Gamma$ is also finite, and by the filtration theorem we have $w \models A$.

In this proof we can use any filtration of $W$. Particular filtrations are used to prove fmp w.r.t. characteristic classes of models.

Example: each formula of the basic modal language which has a transitive model, also has a finite transitive model. The proof is the same, but using the particular filtration which preserves transitivity: $w R_t u$ iff for all $\Box A \in \Gamma$ we have: if $w \models \Box A$ then $u \models A \land \Box A$. 
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In this proof we can use any filtration of $W$. Particular filtrations are used to prove \textsc{fmp} w.r.t. characteristic classes of models. Example: each formula of the basic modal language which has a transitive model, also has a finite transitive model. The proof is the same, but using the particular filtration which preserves transitivity: 

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Refining filtration

It is not always possible to define a suitable filtration which preserves a desired property. Shehtman (1993) proposes a refinement of filtration using an appropriate equivalence relation $\sim \subseteq \equiv \Gamma$. Filtration is defined in the same way, but over $W/\sim$. Shehtman (2005) uses a particular $\sim$ defined using bisimulations to prove fmp for some product modal logics. In the definition of generalized Veltman models there are plenty of properties we need to preserve under filtration. Refining filtration using bisimulations shows to be a good tool to accomplish this.
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In the definition of generalized Veltman models there are plenty of properties we need to preserve under filtration. Refining filtration using bisimulations shows to be a good tool to accomplish this.
Filtrations of generalized Veltman models

Let $W$ be a generalized Veltman model, $\Gamma$ an adequate set of formulas and $\sim \subseteq \equiv_{\Gamma}$ an equivalence relation on $W$. For $V \subseteq W$, put $V_{\sim} = \{[w]: w \in V\}$.

Filtration theorem: for all $w \in W$ and $A \in \Gamma$, $[w] \vDash A$ iff $w \vDash A$. Proof: by induction. Existence: using a particular $\sim$ which is defined using bisimulations.
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- if $wRu$ and there is $\Box A \in \Gamma$ s.t. $w \not\models \Box A$ and $u \models \Box A$, then $[w]\tilde{R}[u]$
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- $[u] S_{[w]} V_\sim$ iff for all $w' \in [w]$ and $u' \in [u]$ s.t. $w' Ru'$ we have $u' S_{w'} V'$ for some $V'$ s.t. $V_\sim \subseteq V_\sim$

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Filtration via bisimulations

Vrgoč and Vuković (2010) define an appropriate notion of bisimulation between generalized Veltman models, with good properties:

- If there is a bisimulation $Z$ between $W$ and $W'$ s.t. $wZw'$, then $w$ and $w'$ are modally equivalent.
- The identity is a bisimulation, the inverse of a bisimulation is a bisimulation, the composition of bisimulations is a bisimulation, the union of bisimulations is a bisimulation.

Consequences:
- For $w, u \in W$, put $w \sim u$ iff there is a bisimulation $Z \subseteq W \times W$ s.t. $wZu$. Then $\sim$ is an equivalence relation.
- For any set of formulas $\Gamma$, we have $\sim \subseteq \equiv \Gamma$.

Existence of a filtration over $W/\sim$: We prove that the filtration over this particular $\sim$ is in fact a generalized Veltman model.
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To prove the finite model property using filtration, we need \( \sim \) to generate only finitely many equivalence classes.
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- the relation $\sim$ defined using bisimulations preserves properties of structures, but it refines $\equiv_\Gamma$, so we may lose finiteness

Fortunately, in the case of generalized Veltman semantics, we can obtain the finite model property by using filtration twice:

- first we obtain a generalized Veltman model with bounded length of $\tilde{R}$-chains,
- and then we use this to show that repeated filtration gives a finite model.

Using this main idea, we obtained an alternative proof of the finite model property of interpretability logic $\text{IL}$. And we proved the finite model property of the systems $\text{ILM}$ and $\text{ILM}_0$ w.r.t. generalized Veltman models.
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Fortunately, in the case of generalized Veltman semantics, we can obtain the finite model property by using filtration twice:

- first we obtain a generalized Veltman model with bounded length of \( \tilde{R} \)-chains,
- and then we use this to show that repeated filtration gives a finite model.

Using this main idea, we obtained an alternative proof of the finite model property of interpretability logic \( \text{IL} \) w.r.t. Veltman models, and we proved the finite model property of the systems \( \text{ILM} \) and \( \text{ILM}_0 \) w.r.t. generalized Veltman models.