Proposals for a Noncommutative Continuum

Revisiting Brouwer and Weyl after Connes

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- Brouwer and Weyl both used dyadic rational intervals $[\frac{m-1}{2^n}, \frac{m+1}{2^n}] \subset \mathbb{Q}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$. There is no special reason, here, to not allow other rational intervals.

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- Furthermore (vaguely): the continuum is not a "static" set of a priori given points; it should be viewed as a collection of processes — some sort of dynamical system.

Some modificatons, after non-commutative geometry

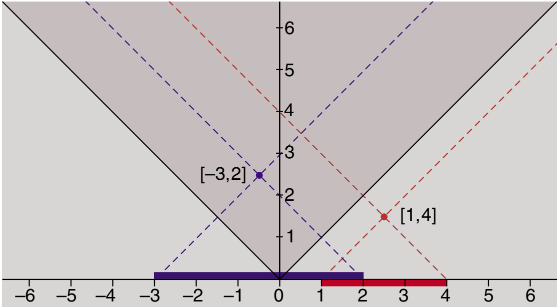
- The concept of space is algebraic: Gel'fand representation characterizes (certain) spaces as C^*-algebras of continuous functions.
- Connes, roughly: non commutative C^*-algebras correspond to "non-commutative" spaces.
- Question: Is there a C^*-algebra — a C^*-dynamical system — that captures some features of Brouwer's idea of the continuum?

The Game (not of thrones)

- Find a C^*-dynamical system that "models" descending chains of intervals.
- Do this without overhauling the foundational framework.
- Freely use any part of mathematics that may seem relevant.
- Suspend "foundational" judgments and see what happens.

Interval arithmetic in numerical analysis

Numerical analysts Warmus (in Poland) and Sunaga (in Japan) introduced the algebra of “approximate numbers”, namely intervals. Each interval is represented in the form $x + y\varepsilon = [x - y, x + y]$, where ε stands for $[-1, 1]$. Think of it as $x \pm y$.



ε-axis is vertical in the pic. $[1, 4] = 2.5 \pm 1.5 = 2.5 + 1.5\varepsilon$.

A bit of group theory

Let's look at the part of the interval algebra which represents actual intervals: $x + y\varepsilon$ with $y > 0$. This in turn can be identified with the (connected component of the) affine group,

$$\mathrm{Aff}(\mathbb{R}) = \mathbb{R} \rtimes \mathbb{R}^+ \cong \left\{ \begin{pmatrix} 1 & 0 \\ b & a \end{pmatrix} \;\middle|\; b \in \mathbb{R}, a \in \mathbb{R}^+ \right\}.$$

Also known as the "$ax + b$" group: the group of affine maps $f(x) = ax + b$ (with $a > 0$).

We now use $\mathrm{Aff}(\mathbb{R})$ to define a (noncommutative) multiplication of intervals.

Why look at the affine group

We are interested in finding an algebra that in some way represents sequences of intervals. In the Brouwer-Weyl model, these sequences should be nested.

Now consider the trivial computation:

$$\begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x + \frac{1}{2}y & \frac{1}{2}y \end{pmatrix}.$$

Right multiplication by $\alpha = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ has the effect of shrinking the interval $x + y\varepsilon$ to its right half:

$$x + y\varepsilon \to \left(x + \frac{1}{2}y\right) + \frac{1}{2}y\varepsilon$$

$$[x - y, x + y] \supset [x, x + y].$$

A noncommutative algebra of interval sequences

A sequence of multiplications by some (**not all!**) group elements is equivalent to a finite sequence of nested intervals:

$$\mathrm{int}(g_1) \supset \mathrm{int}(g_1 g_2) \supset \cdots \supset \mathrm{int}(g_1 g_2 \cdots g_n).$$

Informally: the group multiplication ecnodes the inclusion relation among intervals (considered as elements of the group). Choosing a generating set of $\mathrm{Aff}(\mathbb{R})$, for words u and v in the generators we (almost) have:

$$\mathrm{int}(u) \subseteq \mathrm{int}(v) \iff v \text{ is a prefix of } u.$$

But this is not correct. Not yet. (1) $\mathrm{Aff}(\mathbb{R})$ is uncountable, so it does not have a discrete set of generators; (2) we must choose generators in a special way: they should be contractions; then (3) there are issues with inverses of generators (inverse of a contraction is an expansion).

Group and semigroup algebras

Obviously there will some corrections and decisions to make. Still, this is the basic idea: we have a group multiplication that we like, so turn it into an algebra in the usual way. I.e., make it a group algebra. Use group elements as the basis of a vector space and extend the group multiplication by linearity.

But contractions are a semigroup, not a group: no inverses. So if we want *nested* sequences, we should have a semigroup algebra. For example, the matrices

$$\alpha = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

generate the free semigroup. The free semigroup algebra on two generators is simply the algebra of noncommuting polynomials in two variables.

Several options at this stage

In this thought experiment, the Brouwer-Weyl continuum appears as (more or less) the semigroup algebra $\mathbb{Q}B$, where B is the semigroup generated by the affine matrices corresponding to dyadic rational intervals.

Since we are not interested in ditching $\mathbb{R}$, or $\mathbb{C}$, we have more options. We can work with real or complex group algebras, or C^* algebras.

A definition

Let M be the submonoid of the rational affine group, consisting of subintervals of $[-1,1] \cap \mathbb{Q}$.

Lemma

For intervals $a + b\varepsilon$ and $c + d\varepsilon$: $a + \varepsilon b \subseteq c + \varepsilon d$ if and only if $a + b\varepsilon = (c + d\varepsilon)(x + y\varepsilon)$ in $\mathrm{Aff}(\mathbb{Q})$ for some $x + \varepsilon y \in M$ (i.e. $|x| + y \leq 1$).

Less formally…

By discussions above, the prefix order on $\mathrm{Aff}(\mathbb{Q})$ induced by the monoid M, namely:

$$x \leq_M y \iff y \in xM$$

is precisely the reverse inclusion; because right multiplication by elements of M acts as an affine contraction.

Main definition

Let $\mathcal{C}(M)$ be the C^*-algebra of operators on $\ell^2(M)$ obtained by restricting the left-regular representation of the rational affine group.

This algebra has the structure of a C^*-dynamical system.

The structure

For each interval $x \in M$, let P_x be the projection onto the subspace $\ell^2(xM)$ of $\ell^2(M)$. This is the projection onto the subspace of intervals of x.

These projections generate an abelian algebra A_M since $P_x P_y = P_{x \cap y}$ (or zero if the intersection is empty or a point).

The monoid M acts on this abelian algebra by endomorphisms $\varphi_x(P_y) = P_{xy}$. Formally the endomorphisms are implemented by isometries v_x such that $v_x P_y v_x^* = P_{xy}$.

$\mathcal{C}(M)$ is the C^*-algebra generated by isometries v_x: it is the semidirect product $A_M \rtimes M$. This is the structure of a C^*-dynamical system.

Heyting algebra semantics

In topos-theoretic foundations of physics the lattice of right ideals of a monoid—which has the structure a Heyting algebra—has been used to define a semantics of propositions, for example by Isham. In this thought experiment, we have:

Theorem

The maximal abelian subalgebra of the continuum $\mathcal{C}(M)$ provides a natural Heyting algebra semantics, with propositions represented by projections associated to right ideals of the monoid M.

But there is a kind of dynamics as well, arising from the action of the monoid on projections: $\varphi_x(P_J) = P_{xJ}$, for right ideals J of M.

Dual structure

Let Ω_M be the set of filters on M as a poset. With the topology induced from 2^M, Ω_M is a compact space now known as the Nica spectrum of M.

The algebra of functions $C(\Omega_M)$ is the Gel'fand dual of the abelian algebra of projections P_x.

The monoid M acts on $C(\Omega_M)$ by affine substitutions (basically). By NIca's results, $\mathcal{C}(M)$ is isomorphic to the crossed product $C(\Omega_M) \rtimes M$.

More generally...

Essentially, reverse inclusion of intervals is an order relation defined by the monoid M on the rational affine group.

The construction works for any "quasi-lattice ordered group" (a term introduced by Nica): their order can be defined as the prefix order induced from a submonoid (under some conditions).

Therefore we could look at various "discretizations" of this idea of the continuum, of which dyadic intervals of Brouwer and Weyl are just one example.

A simple example

Let F be the monoid generated by the subintervals $[-1, 0]$ and $[0, 1]$ of $[-1, 1]$: the free monoid on two generators. Ω_F is the usual Cantor space associated with a binary tree (left-right partitions), and its nodes correspond to the Cayley graph F. The action of F on this space corresponds to pushing along the tree by left multiplication.

The algebra $\mathcal{C}(F)$ is known as the Cuntz-Toeplitz algebra $\mathcal{TO}_2$, i.e. the C^*-algebra defined by the generators v_1, v_2 and relations: $v_i^* v_j = \delta_{ij}$.

Discretizations make a difference

Theorem

A partition of the unit interval into n subintervals generates the free monoid F_n on n generators. The corresponding algebra $\mathcal{C}(F_n)$ is isomorphic to the Cuntz-Toplitz algebra $\mathcal{TO}_n$. In particular, different partitions can yield different "continua".

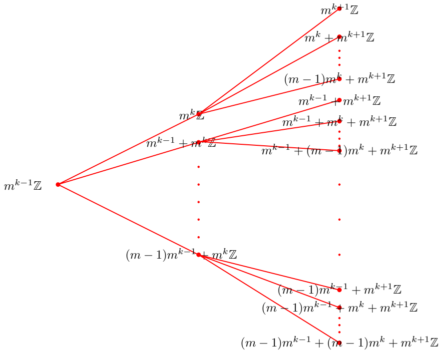
The semigroup B generated by dyadic intervals of Brouwer and Weyl is not free. In the affine group, it generates the subgroup known as the Baumslag-Solitar group $B(1, 2)$. Also known as the "wavelet group", generated by affine maps $x \to x + 1$ and $x \to 2x$.

Groups acting on trees

The Baumslag-Solitar group $B(1,m)$ is an action of $\mathbb{Z}$ on

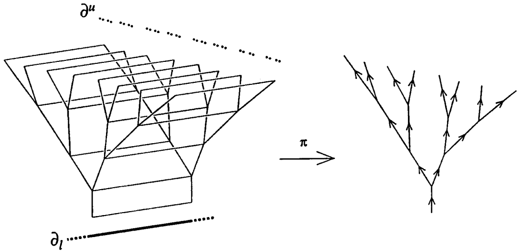
$$\mathbb{Z}[1/m] = \bigcup_{n\in\mathbb{Z}} m^n\mathbb{Z}, \quad m^n\mathbb{Z} \to m^{n+1}\mathbb{Z}$$

$\mathbb{Z}[1/m]$ is essentially a tree:



Picture from geometric group theory

The graph of B is coarsely isometric ("quasi-isometric") to a foliation with each leaf a copy of the hyperbolic plane:



The geometry has two limit cases, real and p-adic: in the pic, $\partial_l = \mathbb{R}$ and $\partial^u = \mathbb{Q}_2$.

Continua as noncommutative spaces

- Some natural homomorphic images of the contuina defined here have been studied from the viewpoint of non-commutative geometry.
- The work is ongoing, although no one seems to have noticed the interval algebra interpretation.
- For some reason, algebras that appear in this context also appear in contemporary mathematical physics.

Higher Dimensions

- Similar ideas can be made to work in higher dimensions.
- For example, elements of the affine group $\mathbb{Q}^2 \rtimes \mathbb{Q}^+$ can be thought of as disks (rather than intervals), with the positive component representing radius.
- This group is not quasi-lattice ordered since intersection of disks is not a disk; the difficulty can be resolved but the algebra has a complicated presentation.
- In case of 3D balls, the analog of the Nica spectrum would be the compactification of the space of future cones in the sense of special relativity.

Thank you

- Questions?
- Comments?
- Suggestions?

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