Weak cyclic Cat-operads

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In order to ensure that all diagrams made of canonical arrows commute, it suffices to check a small number of commutations.

Mac Lane: MONOIDAL CATEGORY

\[ \mathbf{C}, \otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}, \quad \beta : (f \otimes g) \otimes h \to f \otimes (g \otimes h) \]

Coherence of monoidal categories: If the PENTAGON commutes, then all diagrams made of \( \beta \)-arrows commute.

\[
\begin{align*}
\beta \cdot 1 & \quad \beta \\
\beta & \quad (fg)(hk) \\
1 \cdot \beta & \quad \beta
\end{align*}
\]

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\]
In order to ensure that all diagrams made of canonical arrows commute, it suffices to check a small number of commutations.

**Mac Lane:** SYMMETRIC MONOIDAL CATEGORY

\[
\mathcal{C}, \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}, \quad \beta : (f \otimes g) \otimes h \to f \otimes (g \otimes h), \quad c : f \otimes g \to g \otimes f
\]

**Coherence of symmetric monoidal categories:** If the PENTAGON and the HEXAGON commute, then all diagrams made of \(\beta\)- and \(c\)-arrows commute.
Coherence theorems in operad theory: weakening the associativity of operadic composition

**Operad** (non-unital): a functor $\mathcal{O} : \text{Bij}^{\text{op}} \to \text{Set}$, together with *insertions* $\circ_{x} : \mathcal{O}(X) \times \mathcal{O}(Y) \to \mathcal{O}(X \setminus \{x\} \cup Y)$, such that

$$(f \circ_{x} g) \circ_{y} h = f \circ_{x} (g \circ_{y} h) \quad \text{and} \quad (f \circ_{x} g) \circ_{y} h = (f \circ_{y} h) \circ_{x} g.$$
Coherence theorems in operad theory: weakening the associativity of operadic composition

**Operad** (non-unital): a functor $\mathcal{O} : \text{Bij}^{op} \to \text{Set}$, together with *insertions* $o_x : \mathcal{O}(X) \times \mathcal{O}(Y) \to \mathcal{O}(X\setminus\{x\} \cup Y)$, such that

$$(f \circ_x g) \circ_y h = f \circ_x (g \circ_y h) \quad \text{and} \quad (f \circ_x g) \circ_y h = (f \circ_y h) \circ_x g.$$
Coherence theorems in operad theory: weakening the associativity of operadic composition

**Operad** (non-unital): a functor $\mathcal{O} : \text{Bij}^{op} \to \text{Set}$, together with *insertions* $\circ_x : \mathcal{O}(X) \times \mathcal{O}(Y) \to \mathcal{O}(X\setminus\{x\} \cup Y)$, such that

$$(f \circ_x g) \circ_y h = f \circ_x (g \circ_y h) \quad \text{and} \quad (f \circ_x g) \circ_y h = (f \circ_y h) \circ_x g.$$
Goal

Establish the notion of weak cyclic Cat-operad = cyclic operad enriched over Cat

Plan:
1. Recall the notion of cyclic operad
2. Introduce the notion of weak cyclic Cat-operad
3. Prove the coherence theorem
**Operad**: an operation has **inputs** and **an output**

**Cyclic operad**: an operation has **entries** and it can be composed with another operation along any of them
**Cyclic operads**

**Operad**: an operation has inputs and an output.

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**Cyclic operad** (non-unital): a functor $\mathcal{C} : \text{Bij}^{\text{op}} \to \text{Set}$, together with insertions $x \circ y : \mathcal{C}(X) \times \mathcal{C}(Y) \to \mathcal{C}(X \setminus \{x\} \cup Y \setminus \{y\})$, s.t. one of the following two associativity axioms hold:

$$(f \circ x \circ y \ g) \circ u \circ z \ h = f \circ x \circ y \ (g \circ u \circ z \ h) \quad \text{and} \quad (f \circ x \circ y \ g) \circ u \circ z \ h = (f \circ u \circ z \ h) \circ x \circ y \ g,$$

and the commutativity axiom $f \circ x \circ y \ g = g \circ x \circ y \ f$ holds.
**Operad**: an operation has inputs and an output

**Cyclic operad**: an operation has entries and it can be composed with another operation along any of them

**Cyclic operad (non-unital)**: a functor $\mathcal{C} : \text{Bij}^{op} \to \text{Set}$, together with insertions $x \circ y : \mathcal{C}(X) \times \mathcal{C}(Y) \to \mathcal{C}(X\setminus\{x\} \cup Y\setminus\{y\})$, s.t. one of the following two associativity axioms hold:

$$(f \circ_{x,y} g) \circ_{u,z} h = f \circ_{x,y} (g \circ_{u,z} h) \quad \text{and} \quad (f \circ_{x,y} g) \circ_{u,z} h = (f \circ_{u,z} h) \circ_{x,y} g,$$

and the commutativity axiom $f \circ_{x,y} g = g \circ_{x,y} f$ holds.
Cyclic operads

**Operad**: an operation has inputs and an output

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**Cyclic operad** (non-unital): a functor $\mathcal{C} : \text{Bij}^{op} \to \text{Set}$, together with insertions $x \circ y : \mathcal{C}(X) \times \mathcal{C}(Y) \to \mathcal{C}(X \setminus \{x\} \cup Y \setminus \{y\})$, s.t. one of the following two associativity axioms hold:

$$(f \circ x \circ y \circ g) \circ u \circ z \circ h = f \circ x \circ y \circ (g \circ u \circ z \circ h) \quad \text{and} \quad (f \circ x \circ y \circ g) \circ u \circ z \circ h = (f \circ u \circ z \circ h) \circ x \circ y \circ g,$$

and the commutativity axiom $f \circ x \circ y \circ g = g \circ x \circ y \circ f$ holds.
**Cyclic operads**

**Operad**: an operation has **inputs** and an **output**

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**Cyclic operad** (non-unital): a functor $\mathcal{C} : \text{Bij}^{\text{op}} \to \text{Set}$, together with *insertions* $x \circ_y : \mathcal{C}(X) \times \mathcal{C}(Y) \to \mathcal{C}(X \setminus \{x\} \cup Y \setminus \{y\})$, s.t. one of the following two associativity axioms hold:

$$(f \times_y g) \circ_z h = f \times_y (g \circ_z h) \quad \text{and} \quad (f \times_y g) \circ_z h = (f \circ_z h) \times_y g,$$

and the commutativity axiom $f \times_y g = g \times_y f$ holds.
Weakening the associativity and the commutativity

After replacing the equations from the previous slide with isomorphisms

$$\beta : (f \circ y \circ g) \circ z \circ h \to f \circ x \circ (g \circ u \circ z \circ h) \quad \text{and} \quad c : f \circ x \circ y \circ g \to g \circ x \circ y \circ f$$

we are in a setting resembling the one of a symmetric monoidal category.

*Are the coherences of Mac Lane the solution to our coherence problem?*
Weakening the associativity and the commutativity

After replacing the equations from the previous slide with isomorphisms

\[ \beta : (f \circ_{y} g) \circ_{z} h \rightarrow f \circ_{y} (g \circ_{z} h) \quad \text{and} \quad c : f \circ_{y} g \rightarrow g \circ_{y} f \]

we are in a setting resembling the one of a symmetric monoidal category.

Are the coherences of Mac Lane the solution to our coherence problem?

No!

Symmetric monoidal category: all possible \( \beta \)- and \( c \)-arrows exist

Cyclic operads: arrows induced by the shape of the underlying tree

\[ \beta : (f \circ_{y} g) \circ_{z} h \rightarrow f \circ_{y} (g \circ_{z} h) \] does not exist if

In particular, the hexagon of Mac Lane is not allowed!
A weak cyclic Cat-operad (non-unital):

- a functor $C : \text{Bij}^{op} \to \text{Cat}$, together with
- insertions $\circ_y : C(X) \times C(Y) \to C(X \backslash \{x\} \cup Y \backslash \{y\})$

and a family of natural isomorphisms

$$\beta : (f \circ_y g) \circ_z h \to f \circ_y (g \circ_z h) \quad \text{and} \quad c : f \circ_y g \to g \circ_y f$$

such that the diagrams commute.
A weak cyclic Cat-operad (non-unital):

- a functor $\mathcal{C} : \text{Bij}^{\text{op}} \to \text{Cat}$, together with
- insertions $x \circ y : \mathcal{C}(X) \times \mathcal{C}(Y) \to \mathcal{C}(X \setminus \{x\} \cup Y \setminus \{y\})$

and a family of natural isomorphisms

$$\beta : (f x \circ y g) u \circ z h \to f x \circ y (g u \circ z h) \quad \text{and} \quad c : f x \circ y g \to g x \circ y f$$

such that the diagrams commute.

One must be careful with the legitimacy of objects and arrows!
A weak cyclic Cat-operad (non-unital):

- a functor $\mathcal{C} : \text{Bij}^{op} \to \text{Cat}$, together with
- insertions $x \circ_{y} : \mathcal{C}(X) \times \mathcal{C}(Y) \to \mathcal{C}(X\backslash \{x\} \cup Y\backslash \{y\})$

and a family of natural isomorphisms

$$\beta : (f \circ_{x} g) \circ_{y} h \to f \circ_{y} (g \circ_{y} h) \quad \text{and} \quad c : f \circ_{x} g \to g \circ_{y} f$$

such that the diagrams commute.

One must be careful with the legitimacy of objects and arrows!
The category $\mathcal{WCO}$ is determined as follows:

- **objects** are some formal parenthesised words (these denote operations of a weak cyclic Cat-operad),
  - we (sometimes) use the notation $(T, w)$ for parenthesised words:
    - we denote $(fg)h$ with $((\cdot\cdot)\cdot, fgh)$

- **morphisms** are generated by
  
  $1 : f \to f$, $\beta : (fg)h \to f(gh)$ and $c : fg \to gf$,

quotiented by

\[
\begin{align*}
((fg)h)k & \xrightarrow{\beta \cdot 1} (f(gh))k \xrightarrow{\beta} (fg)(hk) \\
(f(gh))k & \xrightarrow{\beta} f((gh)k) \xrightarrow{1 \cdot \beta} f(g(hk)) \\
(fg)h & \xrightarrow{\beta} f(gh) \xrightarrow{c} (gh)f \\
(gf)h & \xrightarrow{c} h(gf) \xrightarrow{\beta} (hg)f \\
(gf)h & \xrightarrow{c \cdot 1} f(g(hk)) \\
(fg)h & \xrightarrow{c \cdot 1} f(gh) \xrightarrow{\beta} f(g(hk))
\end{align*}
\]

**PENTAGON**

**HEXAGON**

**INVOLUTION**
The category $\mathcal{WCO}$ is determined as follows:

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  - we (sometimes) use the notation $(T, w)$ for parenthesized words:
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  \]

  quotiented by

  \[
  (((fg)h)k) \xrightarrow{\beta \cdot 1} (f(gh))k \xrightarrow{\beta} (fg)(hk) \xrightarrow{\beta} f((gh)k) \xrightarrow{1 \cdot \beta} f(g(hk))
  \]

  **PENTAGON**

  \[
  (fg)h \xrightarrow{\beta} f(gh) \xrightarrow{c} (gh)f
  \]

  **HEXAGON**

  \[
  (gf)h \xrightarrow{c} h(gf) \xrightarrow{\beta} (hg)f \xrightarrow{c \cdot 1} fg \xrightarrow{c} f(gf)
  \]

  **INVOLUTION**

  We shall prove the coherence of $\mathcal{WCO}$. 
\[ \theta : (fg)h \xrightarrow{c_{f,g} \cdot 1_h} (gf)h \xrightarrow{\beta_{g,f,h}} g(fh) \xrightarrow{c_{g, fh}} (fh)g \]

**Lemma 1**

The following diagrams commute in \( \mathbf{WCO} \):

- **\( (fg)h \) to \( (fg)k \) and \( (fh)g \) to \( (fh)k \)**
  
  \( ((fg)h)k \xrightarrow{\theta} ((fg)k)h \)
  
  \( ((fg)h)k \xrightarrow{\theta \cdot 1} ((fh)g)k \)

  - **\( (fgh)k \) to \( (f(gk))h \)**
    
    \( (f(gh))k \xrightarrow{\beta \cdot 1} (f(gk))h \)
    
    \( (f(gh))k \xrightarrow{\beta \cdot 1} ((fgh)k)h \)
    
    - **\( (fgh)k \) to \( (fgh)k \)**
      
      \( (f(gh)k) \xrightarrow{\beta \cdot \theta} f((gk)h) \)
      
      \( (f(gh)k) \xrightarrow{\beta \cdot \theta} ((fk)(gk))h \)
      
      \( (f(gh)k) \xrightarrow{\beta \cdot \theta} (f(gk))h \)

  - **\( (fhg)k \) to \( (fhg)k \)**
    
    \( ((fh)k)g \xrightarrow{\theta} ((fk)h)g \)
    
    \( ((fh)g)k \xrightarrow{\theta \cdot 1} ((fh)k)g \)
    
    \( ((fh)g)k \xrightarrow{\theta \cdot 1} ((fh)k)g \)
    
    - **\( (fhk)g \) to \( (fhk)g \)**
      
      \( ((fh)g)k \xrightarrow{\beta \cdot \theta} ((fhk)g)h \)
      
      \( ((fh)g)k \xrightarrow{\beta \cdot \theta} (f(gk))h \)
      
      \( ((fh)g)k \xrightarrow{\beta \cdot \theta} (f(gk))h \)
Lemma 1

The following diagrams commute in \( \mathcal{WCO} \):

\[
\begin{align*}
((fg)h)k & \xrightarrow{\theta} ((fg)k)h \\
(f(gh))k & \xrightarrow{\theta \cdot 1} ((fh)g)k \\
(f(kg)h) & \xrightarrow{\theta} ((fk)h)g \\
(fh)(gk) & \xrightarrow{\theta} (f(gk))h
\end{align*}
\]

Together with the PENTAGON, these make the four coherence conditions of a weak Cat-operad. However, the latter do not imply the HEXAGON.
Coherence for $\mathcal{WCO}$

Goal: $\mathcal{WCO}$ is a preorder ($\equiv$ is coherent)
Coherence for \( \mathcal{WCO} \)

Goal: \( \mathcal{WCO} \) is a preorder (\( \equiv \) is coherent)

\( \mathcal{WCO}_c = \mathcal{WCO} \) strictified in the monoidal structure (the structure of \( \beta \)'s)

\( \mathcal{WCO}_\beta = \mathcal{WCO} \) without the structure of \( c \)'s
Coherence for $\mathcal{WCO}$

Goal: $\mathcal{WCO}$ is a preorder ($\equiv$ is coherent)

$\mathcal{WCO}_c = \mathcal{WCO}$ strictified in the monoidal structure (the structure of $\beta$’s)

$\mathcal{WCO}_\beta = \mathcal{WCO}$ without the structure of $c$’s

The goal then follows from:

$\mathcal{WCO} \simeq \mathcal{WCO}_c$

$\mathcal{WCO}_c$ is a preorder
Coherence for $\mathcal{WCO}$

Goal: $\mathcal{WCO}$ is a preorder ($\equiv$ is coherent)

$\mathcal{WCO}_c = \mathcal{WCO}$ strictified in the monoidal structure (the structure of $\beta$’s)
$\mathcal{WCO}_\beta = \mathcal{WCO}$ without the structure of $c$’s

The goal then follows from:

$\mathcal{WCO} \simeq \mathcal{WCO}_c$  \quad $\mathcal{WCO}_c$ is a preorder

In order to prove these two claims, we must show that:

$\mathcal{WCO}_\beta$ is coherent  \quad $\mathcal{WCO}_c$ behaves like $S_n$
The category $\mathcal{WCO}_\beta$: monoidal coherence

$\mathcal{WCO}_\beta = \mathcal{WCO}$ without the structure of $c$’s

$\mathcal{WCO}_\beta$ are analogous to Mac Lane’s postulates for monoidal categories. We imitate his proof of monoidal coherence.
The category $\mathcal{WCO}_\beta$: monoidal coherence

$\mathcal{WCO}_\beta = \mathcal{WCO}$ without the structure of $c$'s

$\rightarrow$ The equations of $\mathcal{WCO}_\beta$ are analogous to Mac Lane’s postulates for monoidal categories. We imitate his proof of monoidal coherence.

\[
\begin{align*}
(T, v) &\xrightarrow{\beta} (T_1, v) \xleftarrow{\beta} (T_2, v) \xrightarrow{\beta} (T_3, v) \xleftarrow{\beta} (T', v) \\
(N, v) &\xrightarrow{\beta^{-1}} (N, v) = (N, v) = (N, v) = (N, v) = (N, v)
\end{align*}
\]

Not all of parenthesized words and $\beta$-arrows are in $\mathcal{WCO}_\beta$!

We show that there is still enough arrows to carry out the proof!

**Theorem 2**

$\mathcal{WCO}_\beta$ is a preorder.
The category $\mathcal{WCO}_c$: strictifying the monoidal structure

$\mathcal{WCO}_c = \mathcal{WCO}$ strictified in the monoidal structure (the structure of $\beta$’s)

- The objects of $\mathcal{WCO}_c$ are $\beta^{-1}$-normal forms of $\mathcal{WCO}$, identified with nonparenthesized words.
- The morphisms of $\mathcal{WCO}_c$ are generated by the family $c_{w,w'} : ww' \rightarrow w'w$, indexed by all pairs of words $w, w'$ for which there exists $T, T'$ such that $((T, T'), ww')$ is an object of $\mathcal{WCO}$, quotiented by the strictified version of the HEXAGON:
The category \( \mathcal{WCO}_c \): strictifying the monoidal structure

\[ \mathcal{WCO}_c = \mathcal{WCO} \text{ strictified in the monoidal structure (the structure of } \beta \text{'s)} \]

- The objects of \( \mathcal{WCO}_c \) are \( \beta^{-1} \)-normal forms of \( \mathcal{WCO} \), identified with nonparenthesized words.
- The morphisms of \( \mathcal{WCO}_c \) are generated by the family \( c_{w,w'} : ww' \to w'w \), indexed by all pairs of words \( w, w' \) for which there exists \( T, T' \) such that \(( (T, T'), ww' )\) is an object of \( \mathcal{WCO} \), quotiented by the strictified version of the HEXAGON:

\[
\begin{align*}
(fg)h &\xrightarrow{\beta_{f,g,h}} f(gh) \xrightarrow{c_{f,gh}} (gh)f \\
(gf)h &\xrightarrow{c_{gf,h}} h(gf) \xrightarrow{\beta_{h,g,f}} (hg)f \\
\end{align*}
\]
The category $\mathcal{WCO}_c$: strictifying the monoidal structure

$\mathcal{WCO}_c = \mathcal{WCO}$ strictified in the monoidal structure (the structure of $\beta$’s)

- The objects of $\mathcal{WCO}_c$ are $\beta^{-1}$-normal forms of $\mathcal{WCO}$, identified with *nonparenthesized* words.
- The morphisms of $\mathcal{WCO}_c$ are generated by the family $c_{w,w'} : ww' \rightarrow w'w$, indexed by all pairs of words $w$, $w'$ for which there exists $T$, $T'$ such that $((T, T'), ww')$ is an object of $\mathcal{WCO}$, quotiented by the strictified version of the **HEXAGON**:

\[
\begin{array}{ccc}
fgf & \xrightarrow{c_{f,gh}} & ghf \\
\downarrow c_{f,g} \cdot 1 & & \downarrow c_{g,h} \cdot 1 \\
gfh & \xrightarrow{c_{gf,h}} & hgf
\end{array}
\]
From $\mathcal{WCO}$ to $\mathcal{WCO}_c$: easy observations

Let $w = \cdots ab \cdots$ and $w' = \cdots ba \cdots$ be objects of $\mathcal{WCO}_c$ such that there exists an arrow $\varphi : (T, w) \rightarrow (T', w')$ in $\mathcal{WCO}$. Then

1. there exists an arrow from $(N, w)$ to $(N', w')$ in $\mathcal{WCO}$,
2. there exists an arrow from $w$ to $w'$ in $\mathcal{WCO}_c$. 
Let \( w = \cdots ab \cdots \) and \( w' = \cdots ba \cdots \) be objects of \( \mathcal{WCO}_c \) such that there exists an arrow \( \varphi : (T, w) \to (T', w') \) in \( \mathcal{WCO} \). Then

1. there exists an arrow from \((N, w)\) to \((N', w')\) in \( \mathcal{WCO} \),
2. there exists an arrow from \(w\) to \(w'\) in \( \mathcal{WCO}_c \).
Let $w = \cdots ab\cdots$ and $w' = \cdots ba\cdots$ be objects of $\mathcal{WCO}_c$ such that there exists an arrow $\varphi : (T, w) \to (T', w')$ in $\mathcal{WCO}$. Then

1. there exists an arrow from $(N, w)$ to $(N', w')$ in $\mathcal{WCO}$,
2. there exists an arrow from $w$ to $w'$ in $\mathcal{WCO}_c$. 
Let $w = \cdots ab \cdots$ and $w' = \cdots ba \cdots$ be objects of $\mathcal{WCO}_c$ such that there exists an arrow $\varphi : (T, w) \to (T', w')$ in $\mathcal{WCO}$. Then

1. there exists an arrow from $(N, w)$ to $(N', w')$ in $\mathcal{WCO}$,
2. there exists an arrow from $w$ to $w'$ in $\mathcal{WCO}_c$.

We shall examine $(N, w) \to (N', w')$ in order to exhibit $w \to w'$. We want to define a class of generators of $\mathcal{WCO}_c$ which “act as transpositions”.
Let \( w = \cdots ab \cdots \) and \( w' = \cdots ba \cdots \) be objects of \( \mathcal{WCO}_c \) such that there exists an arrow \( \varphi : (T, w) \to (T', w') \) in \( \mathcal{WCO} \). Then

1. there exists an arrow from \( (N, w) \) to \( (N', w') \) in \( \mathcal{WCO} \),
2. there exists an arrow from \( w \) to \( w' \) in \( \mathcal{WCO}_c \).

We shall examine \( (N, w) \to (N', w') \) in order to exhibit \( w \to w' \).

We want to define a class of generators of \( \mathcal{WCO}_c \) which “act as transpositions”.

**Lemma 3**

\((N, w)\) can be transformed, by a sequence of \( \beta \)-arrows, into a parenthesised word \((S, w)\) whose form is either

\[ \cdots (ab) \cdots, \text{ or } \cdots ((ta)b) \cdots, \text{ or } \cdots (a(bt)) \cdots, \]

where, in the last two cases, \( a_T(T(t), a) = a_T(T(t), b) = 1 \).
"Minimal" arrow \((N, w) \rightarrow (N', w')\)

Example.

\[
\varphi : (kh)(gf) \xrightarrow{\beta_{k,h,gf}} k(h(gf)) \xrightarrow{1 \cdot \beta^{-1}_{h,g,f}} k((hg)f) \xrightarrow{1 \cdot c_{h,g} \cdot 1} k((gh)f)
\]

\[
((kh)g)f = (N, w) \quad \text{and} \quad (k(gh))f = (N', w')
\]
“Minimal” arrow \((N, w) \rightarrow (N', w')\)

Example.

\[
\mathcal{T} = \begin{array}{c}
\circ \quad \circ \quad \circ \quad \circ \\
\end{array}
\]

\[
\begin{array}{c}
\varphi : (kh)(gf) \xrightarrow{\beta_{k,h,gf}} k(h(gf)) \xrightarrow{1 \cdot \beta^{-1}_{h,g,f}} k((hg)f) \xrightarrow{1 \cdot c_{h,g} \cdot 1} k((gh)f) \\
((kh)g)f \xrightarrow{\beta_{k,h,g} \cdot 1} (k(hg))f \xrightarrow{\beta_{k,h,g}^{-1} \cdot 1} (N, w) \xrightarrow{\beta_{k,h,g} \cdot 1} (S, w) \xrightarrow{\beta_{k,h,g}^{-1} \cdot 1} (N', w')
\end{array}
\]
Example.

\[
\begin{align*}
\mathcal{T} = \quad & \bullet \overset{f}{\rightarrow} \overset{g}{\rightarrow} \overset{h}{\rightarrow} \overset{k}{\rightarrow} \\
\varphi : (kh)(gf) \xrightarrow{\beta_{k,h,gf}} & k(h(gf)) \xrightarrow{1 \cdot \beta^{-1}_{h,g,f}} k((hg)f) \xrightarrow{1 \cdot c_{h,g} \cdot 1} k((gh)f) \\
((kh)g)f \xrightarrow{\beta^{-1}_{kh,g,f}} & (k(hg))f \xrightarrow{1 \cdot c_{h,g} \cdot 1} (k(gh))f \\
(N, w) = & \quad (S, w) = \quad (N', w') =
\end{align*}
\]
“Minimal” arrow \((N, w) \rightarrow (N', w')\)

\[
\begin{align*}
\theta : (ta)b & \xrightarrow{c_{t,a} \cdot 1_{b}} (at)b \xrightarrow{b_{a,t,b}} a(tb) \xrightarrow{c_{a,tb}} (tb)a \\
\theta^r : a(bt) & \xrightarrow{c_{a,bt}} (bt)a \xrightarrow{b_{b,t,a}} b(ta) \xrightarrow{1_{b} \cdot c_{t,a}} b(at)
\end{align*}
\]
Lemma 4

Depending on the shape of \((S, w)\), the diagram

\[
\begin{array}{ccc}
\phi : (T, w) & \xrightarrow{\beta \text{'s}} & (T', w') \\
\downarrow & & \downarrow \\
(N, w) & \xrightarrow{\beta_u, b, a} & (S, w) \\
\end{array}
\]

can be completed with one of the following four morphisms:

\[c_{a, b}, \quad \beta_{u, b, a}^{-1} \circ c_{a, b}, \quad \theta_{t, a, b}, \quad \theta_{r, a, b, t}.\]
"Minimal" arrow \((N, w) \to (N', w')\)

\[
\theta : (ta)b \xrightarrow{c_{t, a} \cdot 1_b} (at)b \xrightarrow{\beta_{a, t, b}} a(tb) \xrightarrow{c_{a, tb}} (tb)a \\
\theta^r : a(bt) \xrightarrow{c_{a, bt}} (bt)a \xrightarrow{\beta_{b, t, a}} b(ta) \xrightarrow{1_b \cdot c_{t, a}} b(at)
\]

Lemma 4

Depending on the shape of \((S, w)\), the diagram

\[
\varphi : (T, w) \xrightarrow{(N, w)} (S, w) \xrightarrow{c_{a, b}} (N', w')
\]

can be completed with one of the following four morphisms:

\[
c_{a, b}, \quad \beta_{u, b, a}^{-1} \circ c_{a, b}, \quad \theta_{t, a, b}, \quad \theta_{a, b, t}^r.
\]
Lemma 4

Depending on the shape of \((S, w)\), the diagram

\[
\varphi : (T, w) \xrightarrow{(\beta \text{'s})} (S, w) \xrightarrow{\beta_{u,b,a}^{-1} \circ c_{a,b}} (N', w')
\]

can be completed with one of the following four morphisms:

\[
c_{a,b}, \quad \beta_{u,b,a}^{-1} \circ c_{a,b}, \quad \theta_{t,a,b}, \quad \theta_{a,b,t}^r.
\]
“Minimal” arrow \((N, w) \rightarrow (N', w')\)

\[\theta : (ta)b \xrightarrow{c_{t,a} \cdot 1_b} (at)b \xrightarrow{\beta_{a,t,b}} a(tb) \xrightarrow{c_{a,tb}} (tb)a\]

\[\theta^r : a(bt) \xrightarrow{c_{a,bt}} (bt)a \xrightarrow{\beta_{b,t,a}} b(ta) \xrightarrow{1_b \cdot c_{t,a}} b(at)\]

**Lemma 4**

*Depending on the shape of \((S, w)\), the diagram*

\[
\varphi : (T, w) \xrightarrow{\beta's} (S, w) \xrightarrow{\theta_{t,a,b}} (N', w')
\]

*can be completed with one of the following four morphisms:*

\[c_{a,b}, \quad \beta_{u,b,a}^{-1} \circ c_{a,b}, \quad \theta_{t,a,b}, \quad \theta_{r_{a,b,t}}\]
“Minimal” arrow \((N, w) \rightarrow (N', w')\)

\[
\begin{align*}
\theta : (ta)b & \xrightarrow{c_{t,a} \cdot 1_b} (at)b \xrightarrow{\beta_{a,t,b}} a(tb) \xrightarrow{c_{a,tb}} (tb)a \\
\theta^r : a(bt) & \xrightarrow{c_{a,bt}} (bt)a \xrightarrow{\beta_{b,t,a}} b(ta) \xrightarrow{1_b \cdot c_{t,a}} b(at)
\end{align*}
\]

**Lemma 4**

*Depending on the shape of\((S, w)\), the diagram*

\[
\phi : (T, w) \xrightarrow{\beta's} (S, w) \xrightarrow{\theta^r_{a,b,t}} (N', w')
\]

*can be completed with one of the following four morphisms:*

\[
\begin{align*}
c_{a,b}, & \ & \beta_{u,b,a}^{-1} \circ c_{a,b}, & \ & \theta_{t,a,b}, & \ & \theta^r_{a,b,t}.
\end{align*}
\]
“Minimal” arrow \((N, w) \rightarrow (N', w')\)

\[
\begin{align*}
\theta : (ta)b & \xrightarrow{c_{t,a} \cdot 1_b} (at)b & \beta_{a,t,b} : a(tb) \xrightarrow{c_{a,tb}} (tb)a \\
\theta^r : a(bt) & \xrightarrow{c_{a,bt}} (bt)a & \beta_{b,t,a} : b(ta) \xrightarrow{1_b \cdot c_{t,a}} b(at)
\end{align*}
\]

Lemma 4

Depending on the shape of \((S, w)\), the diagram

\[
\varphi : (T, w) \xrightarrow{\beta 's} (S, w) \xrightarrow{\theta^r_{a,b,t}} (N', w')
\]

can be completed with one of the following four morphisms:

\[c_{a,b}, \quad \beta^{-1}_{u,b,a} \circ c_{a,b}, \quad \theta_{t,a,b}, \quad \theta^r_{a,b,t} \cdot \]

What “remains” of \((N, w) \rightarrow (N', w')\) when we “pass it” to \(\mathcal{WCO}_C\)?
Permissible simple transpositions (PST) of $\mathcal{WCO}_c$

\[ \theta : (ta)b \xrightarrow{c_{t,a} \cdot 1_b} (at)b \xrightarrow{\beta_{a,t,b}} a(tb) \xrightarrow{c_{a,tb}} (tb)a \]
\[ \theta^r : a(bt) \xrightarrow{c_{a, bt}} (bt)a \xrightarrow{\beta_{b,t,a}} b(ta) \xrightarrow{1_b \cdot c_{t,a}} b(at) \]

\[ \varphi : (T, w) \xrightarrow{\beta' \text{'s}} (T', w') \]
\[ (N, w) \xrightarrow{\beta' \text{'s}} (S, w) \]
\[ (N', w') \]

Lemma 4 tells that the following morphisms exist in $\mathcal{WCO}_c$:

\[ c_{a,b} : w \rightarrow w', \quad \text{if} \ (S, w) = \cdots (ab) \cdots , \]
\[ c_{a,tb} \circ (c_{t,a} \cdot 1_b) : w \rightarrow w', \quad \text{if} \ (S, w) = \cdots ((ta)b) \cdots , \]
\[ (1_b \cdot c_{t,a}) \circ c_{a, bt} : w \rightarrow w', \quad \text{if} \ (S, w) = \cdots (a(bt)) \cdots . \]
Permissible simple transpositions (PST) of $\mathcal{WCO}_c$

$$
\begin{align*}
\theta : (ta)b & \xrightarrow{c_{t,a} \cdot 1_b} (at)b \xrightarrow{\beta_{a,t,b}} a(tb) \xrightarrow{c_{a,tb}} (tb)a \\
\theta^r : a(bt) & \xrightarrow{c_{a,bt}} (bt)a \xrightarrow{\beta_{b,t,a}} b(ta) \xrightarrow{1_b \cdot c_{t,a}} b(at)
\end{align*}
$$

\[\varphi : (T, w) \xrightarrow{\beta's} (S, w) \xrightarrow{c_{a,b}} (N', w')\]

Lemma 4 tells that the following morphisms exist in $\mathcal{WCO}_c$:

\[
\begin{align*}
    c_{a,b} : w & \rightarrow w', \quad \text{if } (S, w) = \cdots (ab) \cdots , \\
    c_{a,tb} \circ (c_{t,a} \cdot 1_b) : w & \rightarrow w', \quad \text{if } (S, w) = \cdots ((ta)b) \cdots , \\
    (1_b \cdot c_{t,a}) \circ c_{a, bt} : w & \rightarrow w', \quad \text{if } (S, w) = \cdots (a(bt)) \cdots .
\end{align*}
\]
Permissible simple transpositions (PST) of $\mathcal{W} CO_c$

$$\theta : (ta)b \xrightarrow{c_{t,a} \cdot 1_b} (at)b \xrightarrow{\beta_{a,t,b}} a(tb) \xrightarrow{c_{a,tb}} (tb)a$$

$$\theta^r : a(bt) \xrightarrow{c_{a, bt}} (bt)a \xrightarrow{\beta_{b,t,a}} b(ta) \xrightarrow{1_b \cdot c_{t,a}} b(at)$$

$$\varphi : (T, w) \xrightarrow{(T', w')} (N, w) \xrightarrow{\beta_s} (S, w) \xrightarrow{\beta_{u, b, a} \circ c_{a, b}} (N', w')$$

Lemma 4 tells that the following morphisms exist in $\mathcal{W} CO_c$:

- $c_{a, b} : w \to w'$, if $(S, w) = \cdots (ab) \cdots$,
- $c_{a, tb} \circ (c_{t, a} \cdot 1_b) : w \to w'$, if $(S, w) = \cdots ((ta)b) \cdots$,
- $(1_b \cdot c_{t, a}) \circ c_{a, bt} : w \to w'$, if $(S, w) = \cdots (a(bt)) \cdots$. 

Permissible simple transpositions (PST) of $\mathcal{W}CO_c$

$$\theta : (ta)b \xrightarrow{c_{t,a} \cdot 1_b} (at)b \xrightarrow{\beta_{a,t,b}} a(tb) \xrightarrow{c_{a,tb}} (tb)a$$

$$\theta^r : a(bt) \xrightarrow{c_{a, bt}} (bt)a \xrightarrow{\beta_{b, t, a}} b(ta) \xrightarrow{1_b \cdot c_{t, a}} b(at)$$

$$\varphi : (T, w) \xrightarrow{\beta^\prime s} (S, w) \xrightarrow{\theta_{t, a, b}} (N', w')$$

Lemma 4 tells that the following morphisms exist in $\mathcal{W}CO_c$:

$$c_{a, b} : w \to w', \quad \text{if } (S, w) = \cdots (ab) \cdots ,$$

$$c_{a, tb} \circ (c_{t, a} \cdot 1_b) : w \to w', \quad \text{if } (S, w) = \cdots ((ta)b) \cdots ,$$

$$(1_b \cdot c_{t, a}) \circ c_{a, bt} : w \to w', \quad \text{if } (S, w) = \cdots (a(bt)) \cdots .$$
Permissible simple transpositions (PST) of $\mathcal{WCO}_c$

$$\theta : (ta)b \xrightarrow{c_{t,a} \cdot 1_b} (at)b \xrightarrow{\beta_{a,t,b}} a(tb) \xrightarrow{c_{a,tb}} (tb)a$$

$$\theta^r : a(bt) \xrightarrow{c_{a, bt}} (bt)a \xrightarrow{\beta_{b,t,a}} b(ta) \xrightarrow{1_b \cdot c_{t,a}} b(at)$$

$$\varphi : (T, w) \xrightarrow{} (T', w')$$

$$\varphi : (N, w) \xrightarrow{\beta's} (S, w) \xrightarrow{\theta^r_{a,b,t}} (N', w')$$

Lemma 4 tells that the following morphisms exist in $\mathcal{WCO}_c$:

$$c_{a,b} : w \rightarrow w', \text{ if } (S, w) = \cdots (ab) \cdots ,$$

$$c_{a, tb} \circ (c_{t, a} \cdot 1_b) : w \rightarrow w', \text{ if } (S, w) = \cdots ((ta)b) \cdots ,$$

$$(1_b \cdot c_{t, a}) \circ c_{a, bt} : w \rightarrow w', \text{ if } (S, w) = \cdots (a(bt)) \cdots .$$
Permissible simple transpositions (PST) of $\mathcal{WCO}_c$

$$\theta : (ta)b \xrightarrow{c_{t,a} \cdot 1_b} (at)b \xrightarrow{\beta_{a,t,b}} a(tb) \xrightarrow{c_{a,tb}} (tb)a$$

$$\theta^r : a(bt) \xrightarrow{c_{a, bt}} (bt)a \xrightarrow{\beta_{b,t,a}} b(ta) \xrightarrow{1_b \cdot c_{t,a}} b(at)$$

$$\varphi : (T, w) \xrightarrow{\beta' s} (S, w) \xrightarrow{\theta^r_{a,b,t}} (N', w')$$

Lemma 4 tells that the following morphisms exist in $\mathcal{WCO}_c$:

$$c_{a,b} : w \rightarrow w', \quad \text{if } (S, w) = \cdots (ab) \cdots ,$$

$$c_{a,tb} \circ (c_{t,a} \cdot 1_b) : w \rightarrow w', \quad \text{if } (S, w) = \cdots ((ta)b) \cdots ,$$

$$(1_b \cdot c_{t,a}) \circ c_{a, bt} : w \rightarrow w', \quad \text{if } (S, w) = \cdots (a(bt)) \cdots .$$

We call them permissible simple transpositions (PST) of $\mathcal{WCO}_c$. We denote with $s_i : w \rightarrow w'$ the PST that permutes $a_i$ and $a_{i+1}$. 
Lemma 5

PST’s generate the morphisms of $\mathcal{W}_{CO_c}$.

Proof. We use the (strictified versions of) $\beta\theta$-HEXAGON, $\theta$-HEXAGON and $\beta\theta$-PENTAGON.
Lemma 5

**PST’s generate the morphisms of** $\mathcal{WCO}_c$.

Proof. We use the (strictified versions of) $\beta\theta$-HEXAGON, $\theta$-HEXAGON and $\beta\theta$-PENTAGON.

Lemma 6

**PST’s satisfy the equalities matching the relations satisfied by the generators** $\tau_i = (i, i + 1)$ of the symmetric group $S_n$.

Proof. We prove the equalities

1. $s_i^2 = 1, \ i = 1, \ldots, n - 1,$
2. $s_is_j = s_js_i, \ 1 \leq i < j - 1 \leq n - 2,$ and
3. $(s_is_{i+1})^3 = 1$

by using using the TRIANGLE and the HEXAGON.
Coherence of $S_n$ $\implies$ Coherence of $\mathcal{WCOC}_c$

**Lemma 5**

*PST’s generate the morphisms of $\mathcal{WCOC}_c$.*

Proof. We use the (strictified versions of) $\beta\theta$-HEXAGON, $\theta$-HEXAGON and $\beta\theta$-PENTAGON.

**Lemma 6**

*PST’s satisfy the equalities matching the relations satisfied by the generators $\tau_i = (i, i + 1)$ of the symmetric group $S_n$.*

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1. $s_i^2 = 1$, $i = 1, \ldots, n - 1$,
2. $s_is_j = s_js_i$, $1 \leq i < j - 1 \leq n - 2$, and
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by using using the TRIANGLE and the HEXAGON.

**Theorem 7**

$\mathcal{WCOC}_c$ is a preorder.
Lemma 8

The categories $\mathcal{WCO}$ and $\mathcal{WCO}_c$ are equivalent.

Proof. $F : \mathcal{WCO} \to \mathcal{WCO}_c$ is the forgetful functor.

$G : \mathcal{WCO}_c \to \mathcal{WCO}$ is defined as follows:

- $G(w) = (N, w)$,
- $G(c_{w, w'} : ww' \to w'w) = \varphi_2 \circ c \circ \varphi_1$, where $\varphi_1$ and $\varphi_2$ are arbitrary $\beta$-arrows which “rearrange” parenthesis in the appropriate way, and
- $G(s \cdot t) = \varphi^{-1} \circ (G(s) \cdot G(t)) \circ \varphi$, where $\varphi$ is an arbitrary $\beta$-arrow which “rearranges” parenthesis in the appropriate way.

The coherence of $\mathcal{WCO}_\beta$ ensures that $G$ is well-defined. It is also used for the definition of a natural isomorphism $\eta : 1 \to GF$:

$$\eta_{(T, w)} : (T, w) \to (N, w)$$

is any normalisation arrow for $(T, w)$. 

Theorem 9

$\mathcal{WCO}$ is a preorder.
Lemma 8

The categories $\mathcal{WCO}$ and $\mathcal{WCO}_c$ are equivalent.

Proof. $F : \mathcal{WCO} \rightarrow \mathcal{WCO}_c$ is the forgetful functor. $G : \mathcal{WCO}_c \rightarrow \mathcal{WCO}$ is defined as follows:

- $G(w) = (N, w)$,
- $G(c_{w, w'}, Ww' \rightarrow w'w) = \varphi_2 \circ c \circ \varphi_1$, where $\varphi_1$ and $\varphi_2$ are arbitrary $\beta$-arrows which “rearrange” parenthesis in the appropriate way, and
- $G(s \cdot t) = \varphi^{-1} \circ (G(s) \cdot G(t)) \circ \varphi$, where $\varphi$ is an arbitrary $\beta$-arrow which “rearranges” parenthesis in the appropriate way.

The coherence of $\mathcal{WCO}_\beta$ ensures that $G$ is well-defined. It is also used for the definition of a natural isomorphism $\eta : 1 \rightarrow GF$:

$\eta_{(T, w)} : (T, w) \rightarrow (N, w)$ is any normalisation arrow for $(T, w)$.

Theorem 9

$\mathcal{WCO}$ is a preorder.