Weak cyclic Cat-operads

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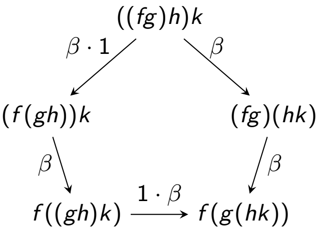
Coherence theorems in category theory

In order to ensure that all diagrams made of canonical arrows commute, it suffices to check a small number of commutations.

Mac Lane: MONOIDAL CATEGORY

$$\mathbf{C}, \ \otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}, \ \beta : (f \otimes g) \otimes h \to f \otimes (g \otimes h)$$

Coherence of monoidal categories: If the PENTAGON commutes, then all diagrams made of β-arrows commute.



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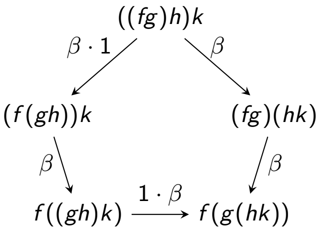
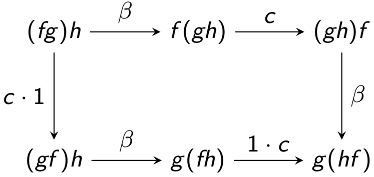
Coherence of symmetric monoidal categories: If the PENTAGON and the HEXAGON commute, then all diagrams made of β- and c-arrows commute.

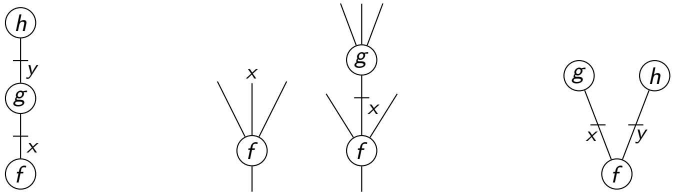




Coherence theorems in operad theory: weakening the associativity of operadic composition

Operad (non-unital): a functor $\mathcal{O} : \mathbf{Bij}^{op} \to \mathbf{Set}$, together with *insertions* $\circ_x : \mathcal{O}(X) \times \mathcal{O}(Y) \to \mathcal{O}(X \backslash \{x\} \cup Y)$, such that

$$(f \circ_x g) \circ_y h = f \circ_x (g \circ_y h) \qquad \text{and} \qquad (f \circ_x g) \circ_y h = (f \circ_y h) \circ_x g.$$



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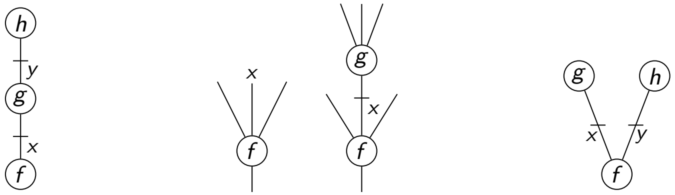

Cat-operad: **Set** replaced by **Cat**

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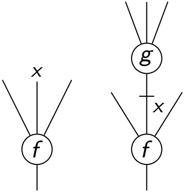
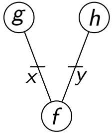




Cat-operad: **Set** replaced by **Cat**

Došen and Petrić (2015):

Weak Cat-operad: associativity equations replaced by isomorphisms $\beta : (f \circ_x g) \circ_y h \to f \circ_x (g \circ_y h)$ and $\theta : (f \circ_x g) \circ_y h \to (f \circ_y h) \circ_x g$

Goal

Establish the notion of

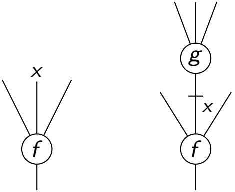
weak cyclic Cat-operad = cyclic operad enriched over Cat

Plan:

1. Recall the notion of cyclic operad
2. Introduce the notion of weak cyclic Cat-operad
3. Prove the coherence theorem

Cyclic operads

Operad: an operation has inputs and an output



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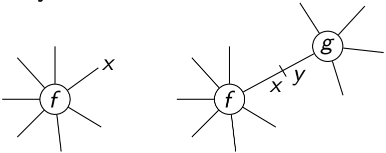

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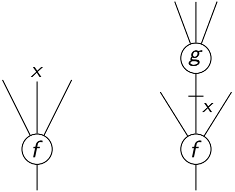

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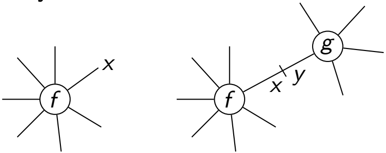

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$$(f\ {}_x\circ_y\ g)\ {}_u\circ_z\ h = f\ {}_x\circ_y\ (g\ {}_u\circ_z\ h) \quad \text{and} \quad (f\ {}_x\circ_y\ g)\ {}_u\circ_z\ h = (f\ {}_u\circ_z\ h)\ {}_x\circ_y\ g,$$

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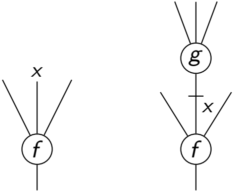

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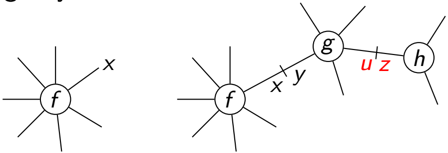

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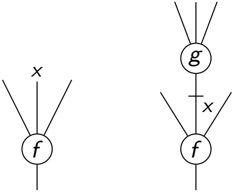

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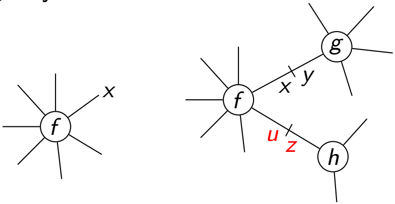

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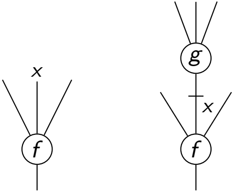

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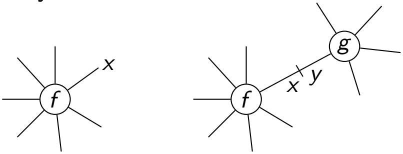

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Weakening the associativity and the commutativity

After replacing the equations from the previous slide with isomorphisms

$$\beta : (f\,{}_x{\circ}_y\, g)\,{}_u{\circ}_z\, h \rightarrow f\,{}_x{\circ}_y\,(g\,{}_u{\circ}_z\, h) \quad \text{and} \quad c : f\,{}_x{\circ}_y\, g \rightarrow g\,{}_x{\circ}_y\, f$$

we are in a setting resembling the one of a symmetric monoidal category.

Are the coherences of Mac Lane the solution to our coherence problem?

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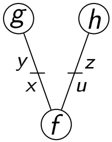
Are the coherences of Mac Lane the solution to our coherence problem?

No!

Symmetric monoidal category: *all* possible β- and c-arrows exist
Cyclic operads: *arrows induced by the shape of the underlying tree*

$$\beta : (f\ {}_x{\circ}_y\ g)\ {}_u{\circ}_z\ h \to f\ {}_x{\circ}_y\ (g\ {}_u{\circ}_z\ h) \ \textit{does not exist if}$$



In particular, the HEXAGON *of Mac Lane is not allowed!*

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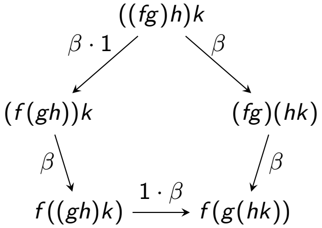

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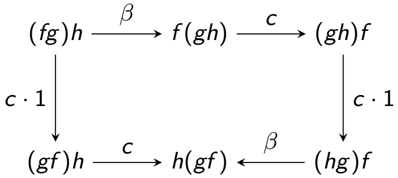

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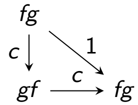

INVOLUTION

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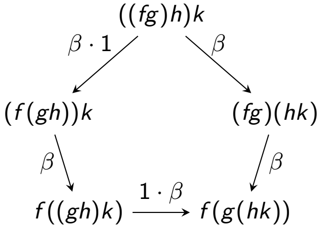

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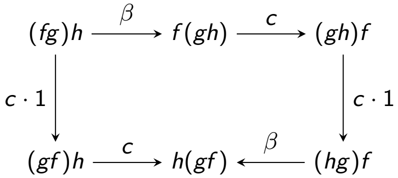

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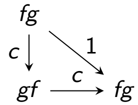

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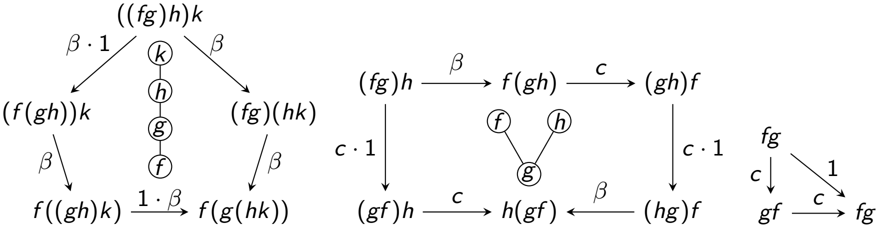

PENTAGON HEXAGON INVOLUTION

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- **objects** are *some* formal parenthesised words (these denote operations of a weak cyclic Cat-operad),
  - we (sometimes) use the notation (T, w) for parenthesized words:

    we denote $(fg)h$ with $((\cdot\cdot)\cdot, fgh)$
- **morphisms** are generated by

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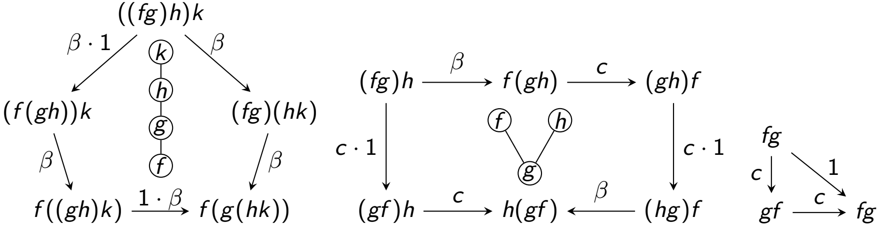

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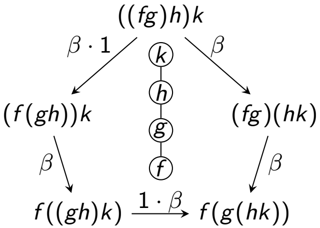

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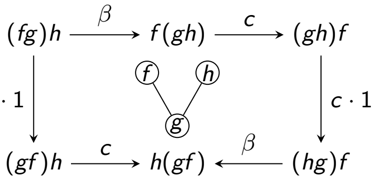

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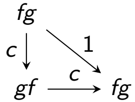

INVOLUTION

We shall prove the coherence of $\mathcal{WCO}$.

$\mathcal{W}CO$: important properties

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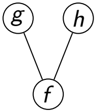

Lemma 1

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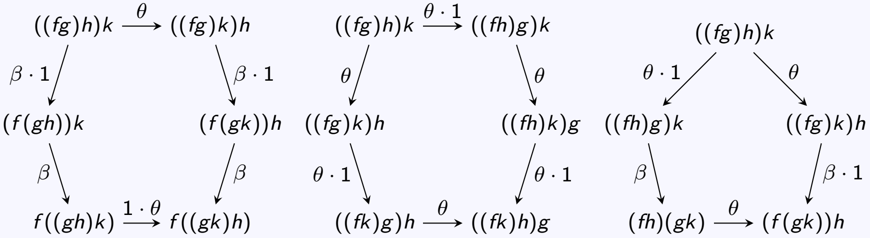

$\beta\theta$-HEXAGON θ-HEXAGON $\beta\theta$-PENTAGON

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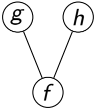

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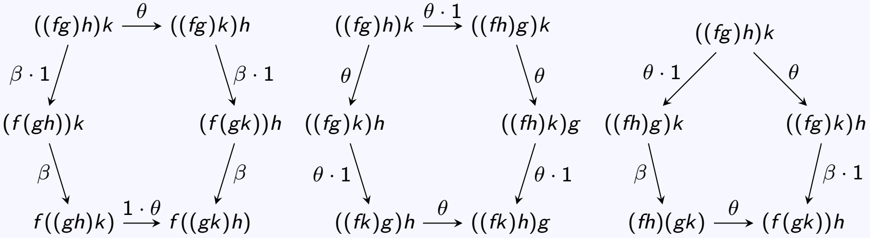

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Together with the PENTAGON, these make the *four coherence conditions* of a *weak Cat-operad*. However, the latter *do not* imply the HEXAGON.

Coherence for $\mathcal{W}CO$

Goal: $\mathcal{W}CO$ is a preorder (= is coherent)

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$\mathcal{W}CO_{\boldsymbol{c}} = \mathcal{W}CO$ strictified in the monoidal structure (the structure of β's)
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In order to prove these two claims, we must show that:

$\mathcal{W}CO_\beta$ is coherent $\mathcal{W}CO_c$ behaves like S_n

The category $\mathcal{W}CO_\beta$: monoidal coherence

$\mathcal{W}CO_\beta = \mathcal{W}CO$ without the structure of c's

$\longrightarrow$ The equations of $\mathcal{W}CO_\beta$ are analogous to Mac Lane's postulates for monoidal categories. We imitate his proof of monoidal coherence.

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$$\begin{array}{ccccccccc}
(T,v) & \xrightarrow{\beta} & (T_1,v) & \xleftarrow{\beta} & (T_2,v) & \xrightarrow{\beta} & (T_3,v) & \xleftarrow{\beta} & (T',v) \\
\scriptstyle{\beta^{-1}\text{'s normalisation arrow}}\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\scriptstyle{\beta^{-1}\text{'s normalisation arrow}} \\
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\end{array}$$

Not all of parenthesized words and β-arrows are in $\mathcal{W}CO_\beta$!

We show that there is still *enough* arrows to carry out the proof!

Theorem 2

$\mathcal{W}CO_\beta$ *is a preorder.*

The category $\mathcal{W}CO_c$: strictifying the monoidal structure

$\mathcal{W}CO_c = \mathcal{W}CO$ strictified in the monoidal structure (the structure of β's)

- The objects of $\mathcal{W}CO_c$ are β^{-1}-normal forms of $\mathcal{W}CO$, identified with *nonparenthesized* words.
- The morphisms of $\mathcal{W}CO_c$ are generated by the family $c_{w,w'} : ww' \to w'w$, indexed by all pairs of words w, w' for which there exists T, T' such that $((T, T'), ww')$ is an object of $\mathcal{W}CO$, quotiented by the strictified version of the HEXAGON:

$$\begin{array}{ccc} fgh & \xrightarrow{c_{f,gh}} & ghf \\ {\scriptstyle c_{f,g}\cdot 1}\downarrow & & \downarrow{\scriptstyle c_{g,h}\cdot 1} \\ gfh & \xrightarrow[c_{gf,h}]{} & hgf \end{array}$$

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$$\begin{array}{ccccc}
(fg)h & \xrightarrow{\beta_{f,g,h}} & f(gh) & \xrightarrow{c_{f,gh}} & (gh)f \\
\downarrow{\scriptstyle c_{f,g}\cdot 1_h} & & & & \downarrow{\scriptstyle c_{g,h}\cdot 1_f} \\
(gf)h & \xrightarrow{c_{gf,h}} & h(gf) & \xleftarrow{\beta_{h,g,f}} & (hg)f
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From $\mathcal{W}CO$ to $\mathcal{W}CO_c$: easy observations

Let $w = \cdots ab \cdots$ and $w' = \cdots ba \cdots$ be objects of $\mathcal{W}CO_c$ such that there exists an arrow $\varphi : (T, w) \to (T', w')$ in $\mathcal{W}CO$. Then

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We shall examine $(N, w) \to (N', w')$ in order to exhibit $w \to w'$.

We want to define a class of generators of $\mathcal{W}CO_c$ which "act as transpositions".

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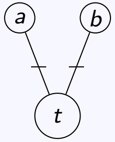
We want to define a class of generators of $\mathcal{W}CO_c$ which "act as transpositions".

Lemma 3

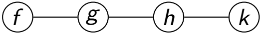
(N, w) can be transformed, by a sequence of β-arrows, into a parenthesised word (S, w) whose form is either

$$\cdots(ab)\cdots, \text{ or } \cdots((ta)b)\cdots, \text{ or } \cdots(a(bt))\cdots,$$

where, in the last two cases, $a_{\mathcal{T}}(T(t), a) = a_{\mathcal{T}}(T(t), b) = 1$.



"Minimal" arrow $(N, w) \to (N', w')$

Example. $\mathcal{T} =$ 

$$
\begin{array}{ccccccc}
\varphi : (kh)(gf) & \xrightarrow{\beta_{k,h,gf}} & k(h(gf)) & \xrightarrow{1 \cdot \beta^{-1}_{h,g,f}} & k((hg)f) & \xrightarrow{1 \cdot c_{h,g} \cdot 1} & k((gh)f) \\
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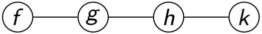
Example. $\mathcal{T} =$ 

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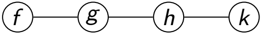
Example. $\mathcal{T} =$ 

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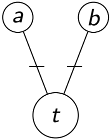

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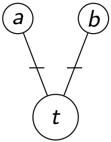

Lemma 4

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can be completed with one of the following four morphisms:

$$c_{a,b}, \quad \beta^{-1}_{u,b,a} \circ c_{a,b}, \quad \theta_{t,a,b}, \quad \theta^r_{a,b,t}.$$

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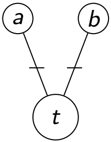

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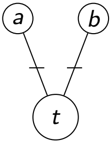

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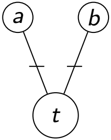

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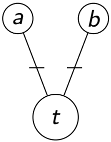

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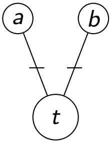

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What “remains” of $(N, w) \to (N', w')$ when we “pass it” to $\mathcal{WCO}_C$?

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Lemma 4 tells that the following morphisms exist in $\mathcal{W}CO_c$:

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Permissible simple transpositions (PST) of $\mathcal{W}CO_c$

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We call them permissible simple transpositions (PST) of $\mathcal{W}CO_c$.
We denote with $s_i : w \to w'$ the PST that permutes a_i and a_{i+1}.

Coherence of $S_n \Longrightarrow$ Coherence of $\mathcal{W}CO_c$

Lemma 5

PST's generate the morphisms of $\mathcal{W}CO_c$.

Proof. We use the (strictified versions of) $\beta\theta$-HEXAGON, θ-HEXAGON and $\beta\theta$-PENTAGON.

Coherence of $S_n \Longrightarrow$ Coherence of $\mathcal{W}CO_c$

Lemma 5

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Lemma 6

PST's satisfy the equalities matching the relations satisfied by the generators $\tau_i = (i, i+1)$ of the symmetric group S_n.

Proof. We prove the equalities

1. $s_i^2 = 1$, $i = 1, \ldots, n-1$,
2. $s_i s_j = s_j s_i$, $1 \leq i < j-1 \leq n-2$, and
3. $(s_i s_{i+1})^3 = 1$

by using using the TRIANGLE and the HEXAGON.

Lemma 5

PST's generate the morphisms of $\mathcal{W}CO_c$.

Proof. We use the (strictified versions of) $\beta\theta$-HEXAGON, θ-HEXAGON and $\beta\theta$-PENTAGON.

Lemma 6

PST's satisfy the equalities matching the relations satisfied by the generators $\tau_i = (i, i+1)$ of the symmetric group S_n.

Proof. We prove the equalities

1. $s_i^2 = 1$, $i = 1, \ldots, n-1$,
2. $s_i s_j = s_j s_i$, $1 \leq i < j-1 \leq n-2$, and
3. $(s_i s_{i+1})^3 = 1$

by using using the TRIANGLE and the HEXAGON.

Theorem 7

$\mathcal{W}CO_c$ is a preorder.

Lemma 8

The categories $\mathcal{W}CO$ and $\mathcal{W}CO_c$ are equivalent.

Proof. $F : \mathcal{W}CO \to \mathcal{W}CO_c$ is the forgetful functor.

$G : \mathcal{W}CO_c \to \mathcal{W}CO$ is defined as follows:

- $G(w) = (N, w)$,
- $G(c_{w,w'} : ww' \to w'w) = \varphi_2 \circ c \circ \varphi_1$, where φ_1 and φ_2 are arbitrary β-arrows which "rearrange" parenthesis in the appropriate way, and
- $G(s \cdot t) = \varphi^{-1} \circ (G(s) \cdot G(t)) \circ \varphi$, where φ is an arbitrary β-arrow which"rearranges" parenthesis in the appropriate way.

The coherence of $\mathcal{W}CO_\beta$ ensures that G is well-defined. It is also used for the definition of a natural isomorphism $\eta : 1 \to GF$:

$\eta_{(T,w)} : (T, w) \to (N, w)$ is *any* normalisation arrow for (T, w).

Coherence of $\mathcal{W}CO$

Lemma 8

The categories $\mathcal{W}CO$ and $\mathcal{W}CO_c$ are equivalent.

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Theorem 9

$\mathcal{W}CO$ is a preorder.