LAP21016
Dubrovnik, Sept. 2016

Interpreting Sequent Calculi as Client–Server Games

Chris Fermüller

Theory and Logic Group
Vienna University of Technology

Background

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- to breathe life into the resource metaphor, we need dynamics

  $\Longrightarrow$ game semantics for substructural sequent calculi

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(since 1990s: Blass, Abramsky, Jagadeesan, Hyland, Ong, . . .)

– abstract semantic models of (fragments and variants) of linear logic
– leads to a fully abstract semantic model of PCF

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(3) Client/Server games (**C**/**S**-games)

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  Notation: $G_1, \ldots, G_n \triangleright H$

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- extraction proceeds stepwise, in rounds, initiated by **C**
- **C** succeeds (wins) if H is atomic and $\in \{G_1, \ldots, G_n\}$ the final state.
  We are interested in winning strategies for **C**.

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in each state $\Gamma \triangleright H$ the client **C** may request one of two actions from **S**:

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$(U^+_{\perp})$ $G = \perp$: game ends, **C** wins

CHECK-rules: depend on **C**'s current IP H.

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(C_{given}) $H = (F_1 \text{ given } F_2)$: **S** adds F_2 to Γ, F_1 replaces H

(C^+_{atom}) H is atomic: game ends, **C** wins if $H \in \Gamma$

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Note: (winning) strategies for **C** are trees of states
that branch for all choices of **S**

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Sequent calculus proofs in disguise

C's winning strategy for $[(a, b), (b, c)] \triangleright$ some_of(b, d) corresponds to

$$\dfrac{\dfrac{\dfrac{b,\ a \wedge b,\ (a \wedge b) \vee (b \wedge c) \vdash b}{a \wedge b,\ (a \wedge b) \vee (b \wedge c) \vdash b}\ (\wedge, l) \qquad \dfrac{b,\ a \wedge b,\ (a \wedge b) \vee (b \wedge c) \vdash b}{a \wedge b,\ (a \wedge b) \vee (b \wedge c) \vdash b}\ (\wedge, l)}{(a \wedge b) \vee (b \wedge c) \vdash b}\ (\vee, l)}{(a \wedge b) \vee (b \wedge c) \vdash b \vee d}\ (\vee, r)$$

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Note:

- intuitionistic rules
- no structural rules

Gentzen's original LI/LK

Initial sequents: $A \vdash A$

Cut rule: $$\frac{\Gamma \vdash \Delta, A \qquad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\ (cut)$$

Structural rules:

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\ (w, r) \quad \frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ (w, l) \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ (c, r) \quad \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\ (c, l)$$

Logical rules:

$$\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\ (\neg, r) \qquad \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\ (\neg, r)$$

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$$\frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}\ (\to, r) \qquad \frac{\Gamma \vdash \Delta, A \qquad B, \Gamma \vdash \Delta}{A \to B, \Gamma \vdash \Delta}\ (\to, l)$$

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LIp – a proof search friendly version of LI:

- Initial sequents: $A, \Gamma \vdash \Delta, A \ / \ \bot, \Gamma \vdash \Delta \quad \Rightarrow$ no weakening
- contraction built into logical rules, cut-free

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Corollary to the (cut-free!) soundness and completeness of **LIp**:

Theorem

C has a winning strategy for $G_1, \ldots, G_n \triangleright F$ iff $G_1, \ldots, G_n \models F$ holds in intuitionistic logic.

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intuitionistic logic is hardly 'substructural'
$\Rightarrow$ find versions of the game that model resource consciousness

Eliminating implicit contraction

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- $\Rightarrow$ contraction free intuitionistic logic

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- corresponds to weakening (w, l) of **LI**

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- new constructor: arbitrary_many(F)

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- game rules for arbitrary_many(F):
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- arbitrary_many(F) corresponds to $!F$ of linear logic
- dismissing, copying, and replacing correspond to

$$\frac{\Gamma \vdash \Delta}{!A, \Gamma \vdash \Delta}\ (w!) \qquad \frac{!A, !A, \Gamma \vdash \Delta}{!A, \Gamma \vdash \Delta}\ (c!) \qquad \frac{A, \Gamma \vdash \Delta}{!A, \Gamma \vdash \Delta}\ L!$$

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- to obtain a **C**/**S**-game for full intuitionistic linear logic (ILL):
  - replace (U_{given}) by a 'splitting version' of it
  - **C** can always add $\emptyset$ (empty IP – corresponding to Girard's **1**) to **S**'s Γ
  - modify the winning conditions:
    C wins in the following states: $A \triangleright A$ $\quad \bot, \Gamma \triangleright A$ $\quad \triangleright \emptyset$

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- the 'bunch of information' provided by **S** might be a list (sequence)
- if **S** CHECKs an conditional IP of **C**, the 'conditioning IP' is added either first or last:
  $\Rightarrow$ F_1 given F_2 splits into F_1 given$\searrow$ F_2, F_1 given$\nearrow$ F_2 corresponding to

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A \backslash B}\ (\backslash, r) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash B / A}\ (/, r)$$

- UNPACKing conditional information provided by **S** follows

$$\frac{\Gamma \vdash A \qquad \Pi, B, \Sigma \vdash \Delta}{\Pi, \Gamma, A \backslash B, \Sigma \vdash \Delta}\ (\backslash, l) \qquad \frac{\Gamma \vdash A \qquad \Pi, B, \Sigma \vdash \Delta}{\Pi, A / B, \Gamma, \Sigma \vdash \Delta}\ (/, l)$$

- combined with a 'sequence version of conjunction' (fusion) this leads to an **C**/**S**-game for full Lambek calculus **FL**

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Topics for further investigation

- interpreting multi-conclusion calculi, in particular full **LL**
- systematic connections to other game semantics
- hypersequent systems modeled by parallel games
- ...