Probabilistic reasoning in type systems

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Outline of the talk

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- The Idea

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- Strong completeness

The Idea

Probabilistic Logic (Extension of the classical propositional logic)

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Completeness theorem for the Lambda Calculus with Intersection Types

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With the notion of filter lambda model, completeness of the type assignment was proved:

Theorem (Completeness)

$\Gamma \vdash M : \sigma \Leftrightarrow \Gamma \models M : \sigma.$

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Let $S = [0,1] \cap \mathbb{Q}$. The *alphabet* of the logic $P\Lambda^{\cap}$ consists of

- all symbols needed to define lambda terms with intersection types,
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Remark: Using $P_{\geq s}\alpha$ we can define other inequalities:

$$\begin{array}{lll} P_{<s}\alpha & \text{stands for} & \neg P_{\geq s}\alpha, \\ P_{\leq s}\alpha & \text{stands for} & P_{\geq 1-s}\neg\alpha, \\ P_{>s}\alpha & \text{stands for} & \neg P_{\leq s}\alpha, \\ P_{=s}\alpha & \text{stands for} & P_{\geq s}\alpha \wedge \neg P_{>s}\alpha. \end{array}$$

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- $P_{=\frac{1}{3}} x : \sigma$, $P_{\geq\frac{1}{4}}(\lambda x.xy : \sigma \to \tau)$
- $P_{\leq 0.2}(x : \sigma) \vee P_{\geq 0.8}(y : \sigma \cap \tau)$

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- $P_{\leq 0.2}(x : \sigma) \vee P_{\geq 0.8}(y : \sigma \cap \tau)$, $\quad P_{=1}(x : \sigma \wedge y : \sigma \to \tau) \Rightarrow P_{=1}(yx : \tau)$

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- $P_{\geq \frac{1}{3}} P_{\geq \frac{1}{2}}(xy : \sigma)$

Kripke-style Semantics of $P\Lambda^{\cap}$

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Definition ($P\Lambda^{\cap}$-structure)

A $P\Lambda^{\cap}$-structure is a tuple $\mathcal{M} = \langle W, \rho, \xi, H, \mu \rangle$, where:

(i) W is a nonempty set of *worlds*, where each world is one lambda model, i.e. for every $w \in W$, $w = \langle \mathcal{L}(w), \cdot_w, [\![\;]\!]_w \rangle$;

(ii) $\rho : \mathtt{V}_\Lambda \times \{w\} \longrightarrow \mathcal{L}(w)$, $w \in W$;

(iii) $\xi : \mathtt{V}_{\mathtt{Type}} \times \{w\} \longrightarrow \mathcal{P}(\mathcal{L}(w))$, $w \in W$;

(iv) H is an *algebra of subsets* of W, i.e. $H \subseteq \mathcal{P}(W)$ such that
- $W \in H$,
- if $U, V \in H$, then $W \setminus U \in H$ and $U \cup V \in H$;

(v) μ is a *finitely additive probability measure* defined on H, i.e.
- $\mu(W) = 1$,
- if $U \cap V = \emptyset$, then $\mu(U \cup V) = \mu(U) + \mu(V)$, for all $U, V \in H$.

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$$[\![M]\!]_{\rho}^{w} \in [\![\sigma]\!]_{\xi}^{w}.$$

Definition (Satisfiability relation)

The satisfiability relation $\models \subseteq P\Lambda^{\cap}_{\mathrm{Meas}} \times \mathrm{For}_{P\Lambda^{\cap}}$ is defined in the following way:

- $\mathcal{M} \models M : \sigma$ iff $w \models M : \sigma$, for all $w \in W$;
- $\mathcal{M} \models P_{\geq s}\alpha$ iff $\mu([\alpha]) \geq s$;
- $\mathcal{M} \models \neg A$ iff it is not the case that $\mathcal{M} \models A$;
- $\mathcal{M} \models A_1 \wedge A_2$ iff $\mathcal{M} \models A_1$ and $\mathcal{M} \models A_2$.

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Consider following two models with three worlds, i.e., let $\mathcal{M}_i = \langle W_i, \rho_i, \xi_i, H_i, \mu_i \rangle$, $i = 1, 2$, where:

- $W_i = \{w_1, w_2, w_3\}$, $i = 1, 2$,
- $H_i = \mathcal{P}(W_i)$, $i = 1, 2$,
- $\mu_i(\{w_j\}) = \frac{1}{3}$, $j = 1, 2, 3$, $i = 1, 2$,

and ρ_i and ξ_i are defined such that $\mathcal{M}_i \models P_{=\frac{1}{3}}(x : \sigma \to \tau)$ and $\mathcal{M}_i \models P_{=\frac{2}{3}}(y : \sigma)$, but in the model $\mathcal{M}_1$: $w_1 \models x : \sigma \to \tau$, $w_2 \models y : \sigma$ and $w_3 \models y : \sigma$ (Figure 1), while in the model $\mathcal{M}_2$: $w_1 \models x : \sigma \to \tau$, $w_1 \models y : \sigma$ and $w_2 \models y : \sigma$ (Figure 2).
Considering the first model we obtain that $\mathcal{M}_1 \models P_{=0}(xy : \tau)$, but considering the second model we obtain that $\mathcal{M}_2 \models P_{=\frac{1}{3}}(xy : \tau)$.

$y : \sigma$ holds in w_2 and w_3:

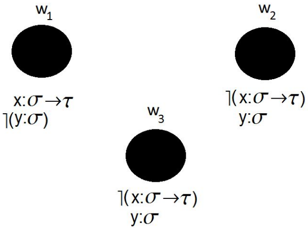

Figure:

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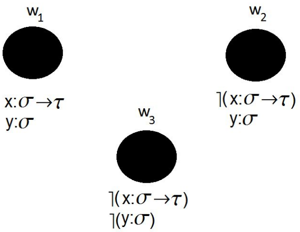

Figure:

Axiom schemes

(1) all instances of the classical propositional tautologies, (atoms are λ-statements or any $\mathrm{P}\Lambda^{\cap}$-formulas),

(2) $P_{\geq 0}\alpha$,

(3) $P_{\leq r}\alpha \Rightarrow P_{<s}\alpha$, $s > r$,

(4) $P_{<s}\alpha \Rightarrow P_{\leq s}\alpha$,

(5) $(P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge P_{\geq 1}(\neg\alpha \vee \neg\beta)) \Rightarrow P_{\geq min\{1,r+s\}}(\alpha \vee \beta)$,

(6) $(P_{\leq r}\alpha \wedge P_{<s}\beta) \Rightarrow P_{<r+s}(\alpha \vee \beta)$, $r + s \leq 1$,

(7) $P_{\geq 1}(\alpha \Rightarrow \beta) \Rightarrow (P_{\geq s}\alpha \Rightarrow P_{\geq s}\beta)$.

Inference Rules I

$$\frac{M : \sigma \to \tau \qquad N : \sigma}{MN : \tau}\ (\to_E)$$

$$\frac{\begin{array}{c}[x:\sigma]\\ \vdots \\ M : \tau\end{array}}{\lambda x.M : \sigma \to \tau}\ (\to_I)$$

$$\frac{M : \sigma \cap \tau}{M : \sigma}\ (\cap_E)$$

$$\frac{M : \sigma \cap \tau}{M : \tau}\ (\cap_E)$$

$$\frac{M : \sigma \qquad M : \tau}{M : \sigma \cap \tau}\ (\cap_I)$$

$$\frac{}{M : \omega}\ (\omega)$$

$$\frac{M : \sigma \qquad \sigma \leq \tau}{M : \tau}\ (\leq)$$

Inference Rules II

(1) From A_1 and $A_1 \Rightarrow A_2$ infer A_2,

(2) from α infer $P_{\geq 1}\alpha$,

(3) from the set of premises

$$\{\phi \Rightarrow P_{\geq s - \frac{1}{k}}\alpha \mid k \geq \frac{1}{s}\}$$

infer $\phi \Rightarrow P_{\geq s}\alpha$.

Soundness and Strong Completeness

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Theorem (Soundness)

The axiomatic system $Ax_{P\Lambda^{\cap}}$ is sound with respect to the class of $P\Lambda^{\cap}_{\mathrm{Meas}}$-models.

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We need a few auxiliary lemmas in order to prove the strong completeness theorem:

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We need a few auxiliary lemmas in order to prove the strong completeness theorem:

Theorem

Every consistent set can be extended to a maximal consistent set.

Construction of the canonical model

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Definition

If $T^\star$ is the maximally consistent set of formulas, then a tuple $\mathcal{M}_{T^\star} = \langle W, \rho, \xi, H, \mu \rangle$ is defined:

- $W = \{w = \langle \mathcal{F}(w), \cdot_w, [\![\ ]\!]_w \rangle \mid w \models Cn_B(T)\}$ contains all **filter** lambda models that satisfy the set $Cn_B(T)$,
- $\rho_w(x) = \{\sigma \in Type \mid w \models x : \sigma\}$,
- $\xi_w(\sigma) = \{d \in \mathcal{F}(w) \mid \sigma \in d\}$,
- $H = \{[\alpha] \mid \alpha \in \mathrm{For_B}\}$, where $[\alpha] = \{w \in W \mid w \models \alpha\}$,
- $\mu([\alpha]) = \sup\{s \mid P_{\geq s}\alpha \in T^\star\}$.

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Theorem (Strong completeness)

Every consistent set of formulas T is $\mathrm{P}\Lambda^{\cap}_{\mathrm{Meas}}$-satisfiable.

Further Work

- Intuitionistic instead of classical propositional calculus

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- Restriction to the finite case

References

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