Some Notes on
Finite and Hyperfinite model theory

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Three seminal papers

1. Gaifman: Concerning measures on first order calculi, Israel J. Math 2 (1964), 1-18


Specifying probabilities on $\text{Sent}_L$

Gaifman: A probability on $\text{Sent}_L$ is a function $P : \text{Sent}_L \rightarrow [0, 1]$ such that:

- $P(\alpha \lor \beta) + P(\alpha \land \beta) = P(\alpha) + P(\beta)$
- $P(\neg \alpha) = 1 - P(\alpha)$
- $P(\alpha) = P(\beta)$, if $\vdash \alpha \leftrightarrow \beta$
- $P(\alpha) = 1$, if $\vdash \alpha$.

A probability on $L$ can be conceived as a non-trivial, non-negative, finitely-additive probability measure on the Lindenbaum-Tarski algebra of the sentences of $L$. 
Specifying probabilities on $\text{Sent}_L$

- Let $(M, R)$ be a classical structure:

$$P_M : \text{Sent}_L \to [0,1], P_M(\varphi) = \begin{cases} 1, & M \models \varphi \\ 0, & M \not\models \varphi \end{cases}$$

- Let $\mathcal{M}$ be a set of classical structures with a probability $m$ on the algebra of subsets of $\mathcal{M}$ generated by $[\varphi]_{\mathcal{M}} = \{M \in \mathcal{M} \mid M \models \varphi\}$:

$$P_{\mathcal{M}} : \text{Sent}_L \to [0,1], P_{\mathcal{M}}(\varphi) = \int_{\mathcal{M}} P_M([\varphi]_{\mathcal{M}}) \, dm(M)$$
Specifying probabilities on \( \text{Sent}_L \)

**Example 1.** \( L = \{B\}, \text{ar}(B) = 2 \)

\[
\begin{pmatrix}
\frac{5}{8} & \frac{1}{4} & \frac{1}{8}
\end{pmatrix}
\]

\[
P(\forall x \exists y \ Bxy) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}
\]

\[
P(\exists x \exists y \ (x \neq y \rightarrow Bxy)) = 1
\]
\( M \neq \emptyset, L(M) = \{B\} \cup \{c_a \mid a \in M\} \)

**Classical model:**
\[
B^M : M \times M \to \{0, 1\}
\]
\[
c^M_a = a, a \in M
\]

**Probabilistic models**
\[
\mathcal{P} : \text{Atomic}_{L(M)} \to [0, 1]
\]

\( \mathcal{P} \) is not sufficient (the values assigned to atomic sentences do not determine unique values for Boolean combinations of these sentences)!
Probabilistic models

\[ M \neq \emptyset, L(M) = \{B\} \cup \{c_a \mid a \in M\} \]

Classical model:
\[ B^M : M \times M \to \{0, 1\} \]
\[ c_a^M = a, a \in M \]

Probabilistic models
\[ P : \text{Bool}_{L(M)} \to [0, 1] \]

\[ I : \text{Atomic}_{L(M)} \to \{0, 1\} \]

\[ \hat{J} : \text{Sent}_{L(M)} \to \{0, 1\} \]
Gaifman’s condition

Theorem. If \( P : \text{Sent}_{L(M)} \to [0,1] \) is a probability, then
\[
(G) \quad P(\exists x \alpha(x)) = \sup \{ P(\alpha(a_1) \lor \cdots \lor \alpha(a_k)) \mid k \in \mathbb{N}, a_1, \ldots, a_k \in M \}.
\]

Theorem. Let \((M, \mathcal{P})\) be a probabilistic model (i.e., \( \mathcal{P} : \text{Bool}_{L(M)} \to [0,1] \)). Then there is a unique probability \( \mathcal{P}^* \) which extends \( \mathcal{P} \) to \( \text{Sent}_{L(M)} \) and satisfies (G).

Theorem. Every probability on \( \text{Sent}_L \) has a probabilistic model whose power is \( \aleph_0 + |\text{Sent}_L| \).
Probabilistic models

Example 2. $M = \{0,1\}$

$\mathcal{P}(B_{00} \land B_{01} \land B_{10} \land B_{11}) = p_1$

$\mathcal{P}(B_{00} \land B_{01} \land B_{10} \land \neg B_{11}) = p_2$

$\vdots$

$\mathcal{P}(\neg B_{00} \land \neg B_{01} \land \neg B_{10} \land \neg B_{11}) = p_{32}$

$p_1 + p_2 + \cdots + p_{32} = 1$

$\mathcal{P} : \text{Sent}_{L(M)} \to [0,1]$
Example 2. $M = \{0, 1\}$

\[
\begin{align*}
    p_1 + p_2 + \cdots + p_{32} &= 1 \\
    \mathcal{P} : \text{Sent}_{L(M)} &\to [0, 1]
\end{align*}
\]
Example 3. $B : \{0,1\}^2 \to [0,1]$, \[
\begin{pmatrix}
B_{00} & B_{01} & B_{10} & B_{11} \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{3} & \frac{1}{6}
\end{pmatrix}
\]

$B_{00} \land B_{01} \land B_{10} \land B_{11} \iff \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{6}$

$B_{00} \land B_{01} \land B_{10} \land \neg B_{11} \iff \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{5}{6}$

$\vdots$

$\neg B_{00} \land \neg B_{01} \land \neg B_{10} \land \neg B_{11} \iff \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{5}{6}$
Example 4.

\[ P(B00 \land \neg B01 \land \neg B10 \land B11) = P(B00) \cdot P(\neg B01 \mid B00) \cdot P(\neg B10) \cdot P(B11 \mid B00 \land \neg B01) = 0.15 \cdot (1 - 0.85) \cdot (1 - 0.25) \cdot 0.9 \]
Random structure $M(n, p)$


$M = \{0, 1, \ldots, n - 1\}$, $0 \leq p \leq 1$

Define $\mathcal{P} : \text{Bool}_{L(M)} \rightarrow [0,1]$ by:

$\mathcal{P}(Bab) = p$, $a, b \in M$, and these events are mutually independent.

$$\mathcal{P} \left( \bigwedge_{i=1}^{k} Ba_i b_i \land \bigwedge_{j=k+1}^{n} \neg Ba_j b_j \right) = p^k (1 - p)^{n-k},$$

where sentences $Ba_i b_i$ and $Ba_j b_j$ are not identical.
Random structure $M(8,0.5)$

Example 5.

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Example 6: Let $M$ an infinite set, and $p$ a fixed number, $0 < p < 1$.

Define $\mathcal{P} : \text{Bool}_{L(M)} \to [0,1]$ by:

- $\mathcal{P}(Bab) = p$, $a, b \in M$,
- $\mathcal{P}\left(\bigwedge_{i=1}^{k} Ba_i b_i \land \bigwedge_{j=k+1}^{n} \neg Ba_j b_j\right) = p^k (1 - p)^{n-k}$, where sentences $Ba_i b_i$ and $Ba_j b_j$ are not identical,
- $\mathcal{P}(a = b) = 0$, $a \neq b$.

Let $\mathcal{P}^* : \text{Sent}_{L(M)} \to [0,1]$ be the extension of $\mathcal{P}$ satisfying (G).
Gaifman's simple example

**Lemma.** If $\alpha, \beta \in \text{Sent}_{L(M)}$ and no constant occurs both in $\alpha$ and $\beta$ then
\[ \mathcal{P}^*(\alpha \land \beta) = \mathcal{P}^*(\alpha) \cdot \mathcal{P}^*(\beta). \]

In particular, for every $\sigma \in \text{Sent}_L$
\[ \mathcal{P}^*(\sigma) = \mathcal{P}^*(\sigma \land \sigma) = (\mathcal{P}^*(\sigma))^2, \]
and hence $\mathcal{P}^*(\sigma)$ is either 0 or 1.

$(M, \mathcal{P})$ determines a complete theory!
The complete theory

Let $V_n = \{v_0, v_1, \ldots, v_{n-1}\}$ be a set of distinct variables. A complete diagram $\sigma(V_n)$ is a conjunction s.t. for every pair $(v_i, v_j) \in V_n \times V_n$ either $Bv_i v_j$ or $\neg Bv_i v_j$ is a conjunct.

$$\sigma(V_3) = Bv_0 v_0 \land \neg Bv_0 v_1 \land Bv_0 v_2 \land$$
$$\quad \land \neg Bv_1 v_0 \land Bv_1 v_1 \land Bv_1 v_2 \land$$
$$\quad \land Bv_2 v_0 \land \neg Bv_2 v_1 \land \neg Bv_2 v_2$$
The complete theory

A complete diagram \( \sigma(V_{n+1}) \) extends another complete diagram \( \sigma'(V_n) \) if \( \vdash \sigma'(V_n) \rightarrow \sigma(V_{n+1}) \).

\[
\sigma(V_3) = Bv_0v_0 \land \neg Bv_0v_1 \land Bv_0v_2 \land \\
\land \neg Bv_1v_0 \land Bv_1v_1 \land Bv_1v_2 \land \\
\land Bv_2v_0 \land \neg Bv_2v_1 \land \neg Bv_2v_2
\]

extends

\[
\sigma'(V_2) = Bv_0v_0 \land \neg Bv_0v_1 \land \\
\land \neg Bv_1v_0 \land Bv_1v_1
\]
The complete theory

Let $E$ be the set of all sentences $\varepsilon_n$:

$$\forall v_0 \ldots v_{n-1} \left( \bigwedge_{i \neq j} v_i \neq v_j \land \sigma(V_n) \right)$$

$$\rightarrow \exists v_n \bigwedge_i v_i \neq v_n \land \sigma'(V_{n+1})$$

**Theorem.** $E$ is consistent and complete.
The complete theory

The theory $E$ has exactly one, up to isomorphism, countable model – the Rado graph.

That model is `most general’ in the sense that:

• every possible finite model is realized there as a submodel, and

• for every finite submodel every possible finite extension of it is realized.
What is the probability that a certain sentence holds for a randomly chosen finite structure?

$\mathcal{M}_n$ the class of all $n$-structures (on \{0,1, ..., $n-1$\}).

$\mathcal{M}_n[\sigma] = \{M \in \mathcal{M}_n \mid M \models \sigma\}$

$$P_n(\sigma) = \frac{|\mathcal{M}_n[\sigma]|}{|\mathcal{M}_n|}$$
Example 7.

\[ P_2(\forall x \exists y \ B_{xy}) = ? \]
Specifying probabilities on $\text{Sent}_L$

**Example 7.**

\[
P_2(\forall x \exists y \ B_{xy}) = ?
\]

\[
P_2(\forall x \exists y \ B_{xy}) = \frac{9}{16} \approx 0.56
\]
Example 7.

$P_3(\forall x \exists y \ Bxy) =$?
The zero-one law.

\[ P_n(\sigma) = \frac{|m_n[\sigma]|}{|m_n|} \rightarrow ? , n \rightarrow \infty \]
Random structure $M = (8,0.5)$

Example 8.

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Random graph

\[ \sigma(v_0, v_1) = (Bv_0v_0 \lor Bv_0v_1) \land (Bv_1v_0 \lor Bv_1v_1) \]

\[
P\{(a, b) \in M \times M \mid M \models \sigma[a, b]\} = ?
\]
Random graph

\[ \sigma(v_0, v_1) = (Bv_0v_0 \lor Bv_0v_1) \land (Bv_1v_0 \lor Bv_1v_1) \]

\[ P\{(a, b) \in M \times M \mid M \vDash \sigma[a, b]\} = ? \]

\[ \vDash \forall x \exists y \; Bxy \]

\[ = \frac{16}{\binom{8}{2}} = \frac{16}{28} \approx 0.57 \]
Count models!  

\[ P_2(\forall x \exists y Bxy) \approx 0.56 \]

Count pairs!  

\[ P\{\sigma(v_0, v_1)\} \approx 0.57 \]
Count models vs. count tuples

Intuition: If $H \gg n$, then $(H, 0.5)$ contains (almost) all $n$-structures as its substructures. Moreover, the distribution of $n$-structures inside $(H, 0.5)$ is (almost) uniform.
Hyperfinite model theory

Keisler: The purpose ofhyperfinite model theory is to study and classify a type of models which arises in applied mathematics.

⋯ [These] models have usually been either countable sequences of finite models or structures built upon the real numbers. Hyperfinite models provide a better source of infinite models which closely approximate large finite phenomena.

⋯ Hyperfinite models deal with limiting behavior of finite models.
Hyperfinite model theory

**Definition.** A hyperfinite probability space is a pair \((M, \mu)\) where \(A\) is a nonempty hyperfinite set and \(\mu\) is an internal function \(\mu : M \to *[0,1]\) such that \(\sum_{a \in M} \mu(a) = 1\).

\[\langle M_1, M_2, M_3, \ldots \rangle_u - \text{a hyperfinite set}\]
\[\langle M_1[\sigma], M_2[\sigma], M_3[\sigma], \ldots \rangle_u \subseteq \langle M_1, M_2, M_3, \ldots \rangle_u\]
\[\left\langle \frac{|M_1[\sigma]|}{|M_1|}, \frac{|M_2[\sigma]|}{|M_2|}, \frac{|M_3[\sigma]|}{|M_3|}, \ldots \right\rangle_u \in *[0,1]\]
Probability logic $L_{\omega P}$

Probability logic $L_{\omega P}$ is like first-order logic, but instead of $\forall$ and $\exists$ it has probability quantifiers $(P\bar{x} > r)$, $r \in \mathbb{Q}$.

$M \models (P x > r)\varphi(x)$ iff $\mu\{x \in M \mid M \models \varphi(\bar{x})\} > r$
## Logic and Probability

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<tr>
<th>First-order logic</th>
<th>Probability logic</th>
<th>Probability theory</th>
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<tr>
<td>Downward Löwenheim-Skolem theorem</td>
<td>Elementary submodel theorem</td>
<td>Weak law of large numbers</td>
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<td>The zero-one law</td>
<td>Elementary subsequence theorem</td>
<td>Strong law of large numbers</td>
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Thank you for your attention