Justification with Propositional Nominals

A Stepping Stone Towards Dynamic Justification Logic

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Motivation

Epistemic logic $\Box A$	$\longrightarrow$	Justification logic $t : A$
$\downarrow$		$\downarrow$
Dynamic epistemic logic $[A]\Box A$	$\longrightarrow$	Dynamic Justification Logic $[A](t_A : A)$

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Logic $\mathsf{JUP}_{\mathsf{CS}}$ by R. Kuznets and T. Studer (2013) provides:

- Belief expansion
- Minimal change
- Evidential Ramsey Axiom

The logic $\mathsf{JUP}_{\mathsf{CS}}$: Language

Definition 1 (Language of JUP).

- Evidence terms

$$\mathsf{Tm} := c_i \mid x_i \mid \mathsf{up}(A) \mid t \cdot_A s$$

where $t, s \in \mathsf{Tm}$, $A \in \mathsf{Fm}$

- Formulas

$$\mathsf{Fm} := P_i \mid \neg A \mid A \to B \mid t : A \mid [\Gamma]A$$

where $A, B \in \mathsf{Fm}$, $t \in \mathsf{Tm}$, Γ is a finite set of formulas

$t : A =$ "A is believed for reason t"
$[\Gamma]A =$ "A holds after an update with all formulas in Γ"

The logic $\mathsf{JUP}_{\mathsf{CS}}$: Updates and evidence

Terms of the form $\mathsf{up}(A)$ represent special evidence that becomes valid only after an update that contains A:

- $\vdash \neg \mathsf{up}(A) : B$ for any A, B
- $\vdash [\Gamma](\mathsf{up}(A) : A)$, if $A \in \Gamma$

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We have the principles of persistence and minimal change:

- $\vdash t : A \to [\Gamma]t : A$
- $\vdash [\Gamma]t : A \to t : A$ if t does not contain $\mathsf{up}(A)$ for $A \in \Gamma$

Semantics for $\mathsf{JUP_{CS}}$: Evidence relation

Semantics for $\mathsf{JUP_{CS}}$ use generated models, a class of basic modular models.

Definition 2 (Evidence closure).

Let basis $\mathcal{B} \subseteq \mathsf{ATm} \times \mathsf{Fm}$
Then, for $X \subseteq \mathsf{Tm} \times \mathsf{Fm}$, $\mathsf{cl}_{\mathcal{B}}(X)$ is defined by:

- If $(t, A) \in \mathcal{B}$, then $(t, A) \in \mathsf{cl}_{\mathcal{B}}(X)$
- If $(s, A) \in X$ and $(t, A \rightarrow B) \in X$, then $(t \cdot_A s, B) \in \mathsf{cl}_{\mathcal{B}}(X)$

Definition 3 (Evidence relation).

For any $\mathcal{B} \subseteq \mathsf{ATm} \times \mathsf{Fm}$, the *minimal* evidence relation $\mathcal{E}(\mathcal{B})$ is the least fixed point of $\mathsf{cl}_{\mathcal{B}}$.

Semantics for $\mathsf{JUP}_{\mathsf{CS}}$: Models

Definition 4 (Model, CS-model, initial model).

A model is a pair $\mathcal{M} = (\mathsf{v}, \mathcal{B})$, where $\mathsf{v} \subseteq \mathsf{Prop}$, $\mathcal{B} \subseteq \mathsf{ATm} \times \mathsf{Fm}$.
$\mathcal{M}$ is a *CS*-model if $\mathsf{CS} \subseteq \mathcal{B}$ for constant specification CS.
$\mathcal{M}$ is an initial model if $(\mathsf{up}(A), B) \notin \mathcal{B}$ for any formulas A, B.

Definition 5 (Updated model).

For a finite set of formulas Γ, let $\mathcal{U}_\Gamma := \{(\mathsf{up}(A), A) \mid A \in \Gamma\}$.
For a model $\mathcal{M} = (\mathsf{v}, \mathcal{B})$, the updated model $\mathcal{M}^\Gamma$ is $(\mathsf{v}, \mathcal{B} \cup \mathcal{U}_\Gamma)$.

Semantics for $\mathsf{JUP_{CS}}$: Truth

Definition 6 (Truth in model).

For a model $\mathcal{M} = (\mathsf{v}, \mathcal{B})$, relation $\mathcal{M} \Vdash A$ is defined by:

$$\begin{array}{rcl} \mathcal{M} \Vdash P & \textit{iff} & P \in \mathsf{v} \\ \mathcal{M} \Vdash \neg A & \textit{iff} & \mathcal{M} \nVdash A \\ \mathcal{M} \Vdash A \to B & \textit{iff} & \mathcal{M} \nVdash A \textit{ or } \mathcal{M} \Vdash B \\ \mathcal{M} \Vdash t : A & \textit{iff} & (t, A) \in \mathcal{E}(\mathcal{B}) \\ \mathcal{M} \Vdash [\Gamma]A & \textit{iff} & \mathcal{M}^{\Gamma} \Vdash A \end{array}$$

Definition 7 (Validity).

A formula A is valid with respect to constant specification CS if $\mathcal{M} \Vdash A$ for all initial CS-models $\mathcal{M}$.

$\mathsf{JUP}_{\mathsf{CS}}$: Nice things

Theorem 8 (Soundness).

$$\mathsf{JUP}_{\mathsf{CS}} \vdash A \quad \Rightarrow \quad A \textit{ is valid with respect to } \mathsf{CS}$$

Theorem 9 (Completeness).

$$A \textit{ is valid with respect to } \mathsf{CS} \quad \Rightarrow \quad \mathsf{JUP}_{\mathsf{CS}} \vdash A$$

And several other nice properties such as the principle of minimal change, an evidential analog of Ramsey axiom, and AGM postulates for expansion.

$\mathsf{JUP_{CS}}$: The subscript

Keeping track of the formula used in application ($t \cdot_A s : B$) allows to have equivalences in axioms (App), (MC.2), which in turn allows to keep the basis of models atomic and simplifies the completeness proof.

$$t : (A \to B) \wedge s : A \leftrightarrow t \cdot_A s : B \qquad \text{(App)}$$

$$[\Gamma] t \cdot_A s : B \leftrightarrow [\Gamma] t : (A \to B) \wedge [\Gamma] s : A \qquad \text{(MC.2)}$$

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$$[\Gamma] t \cdot_A s : B \leftrightarrow [\Gamma] t : (A \to B) \wedge [\Gamma] s : A \qquad \text{(MC.2)}$$

However, it is unnatural for justification logic. So, a possible direction for research is to try and get rid of it.

Modification: No subscript

Inability to decompose evidence terms leads to models with non-atomic bases:

$$\mathcal{M} = (\mathsf{v}, \mathcal{B}), \mathcal{B} \subseteq \mathsf{Tm} \times \mathsf{Fm}$$

Naively modified axioms are insufficient to preserve soundness and completeness.

So the task is to modify the axiom system and the class of models to obtain "good" soundness and completeness results.

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In order to reduce the complexity of the problem, we dropped the "Dynamic" part, looking at a state after a certain number of updates.

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Our goal is to obtain an axiomatization that both captures the essense of up(A) terms and has simple models, ideally — atomic models.

Capturing the updates: Nominals

Let's look back at the $\mathsf{JUP}_{\mathsf{CS}}$ and its update terms:

$$\mathsf{JUP}_{\mathsf{CS}} \vdash \neg up(A) : B \qquad \mathsf{JUP}_{\mathsf{CS}} \vdash \neg up(A) : A$$
$$\mathsf{JUP}_{\mathsf{CS}} \vdash \neg [A] up(A) : B \qquad \mathsf{JUP}_{\mathsf{CS}} \vdash [A] up(A) : A$$
$$\mathsf{JUP}_{\mathsf{CS}} \vdash \neg [C] up(A) : B \qquad \mathsf{JUP}_{\mathsf{CS}} \vdash \neg [C] up(A) : A$$

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After some updates, there is a finite set of formulas Γ such that for $A \in \Gamma$, $\mathcal{M}' \Vdash up(A) : A$ and $\mathcal{M}' \Vdash \neg up(A) : B$ for $B \neq A$.

Capturing the updates: Nominals

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$$\mathsf{JUP_{CS}} \vdash \neg [A] up(A) : B \qquad \mathsf{JUP_{CS}} \vdash [A] up(A) : A$$
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After some updates, there is a finite set of formulas Γ such that for $A \in \Gamma$, $\mathcal{M}' \Vdash up(A) : A$ and $\mathcal{M}' \Vdash \neg up(A) : B$ for $B \neq A$.

We want to capture this property — a term that justifies one and only one formula — with an extension to terms, nominals.

Justification with (Propositional) Nominals

Justification Logic (with Nominals) $\longrightarrow$ Dynamic Justification Logic (with Updates)

Definition 10 (Language of $\mathsf{JN_V}$).

- Evidence terms

$$\mathsf{Tm} := c_i \mid x_i \mid n_P \mid t \cdot s$$

where $P \in \mathsf{V} \subseteq \mathsf{Prop}$, V is finite

Essense of nominals: they prove exactly one formula.

$$\vdash n_P : F \quad \Longleftrightarrow \quad F = P$$

Justification with Nominals: $\mathsf{JN_V}$

Definition 11 (Logic $\mathsf{JN_V}$).

All propositional tautologies.	(Taut)
$t : (F \to G) \wedge s : F \to t \cdot s : G$	(App)
$n_P : P \qquad P \in \mathsf{V}$	(N_+)
$\neg n_P : F \qquad P \neq F$	(N_-)
$\neg t : F \qquad$ for all t with $n_P \cdot s \in \mathrm{sub}(t)$	($\mathsf{N}_\triangleleft$)
$t \cdot n_P : F \to t : (P \to F)$	($\mathsf{N}_\triangleright$)
$t : F \wedge s : P \to \langle i, P, s \rangle t : F$	(N_σ)

$$\frac{F \qquad F \to G}{G}\,(\mathsf{MP}) \qquad \frac{(c, F) \in \mathsf{CS}}{c : F}\,(\mathsf{AN})$$

Semantics for $\mathsf{JN_V}$: Evidence relation

Definition 12 (Evidence closure).

Let basis $\mathcal{B} \subseteq \mathsf{Tm} \times \mathsf{Fm}$
Then, for $X \subseteq \mathsf{Tm} \times \mathsf{Fm}$, $\mathsf{cl}_{\mathcal{B}}(X)$ is defined by:

- If $(t, F) \in \mathcal{B}$, then $(t, F) \in \mathsf{cl}_{\mathcal{B}}(X)$
- If $(s, F) \in X$ and $(t, F \to G) \in X$, then $(t \cdot s, G) \in \mathsf{cl}_{\mathcal{B}}(X)$
- If $(t, F) \in X$ and $(s, P) \in X$ for $P \in V$, then $(\langle i, P, s\rangle t, F) \in \mathsf{cl}_{\mathcal{B}}(X)$

Definition 13 (Evidence relation).

For any $\mathcal{B} \subseteq \mathsf{Tm} \times \mathsf{Fm}$, the *minimal* evidence relation $\mathcal{E}(\mathcal{B})$ is the least fixed point of $\mathsf{cl}_{\mathcal{B}}$.

Semantics for $\mathsf{JN_V}$: Models

Definition 14 (Model, CS-model).

A model is a pair $\mathcal{M} = (\mathsf{v}, \mathcal{B})$, where $\mathsf{v} \subseteq \mathsf{Prop}$, $\mathcal{B} \subseteq \mathsf{Tm} \times \mathsf{Fm}$.
$\mathcal{M}$ is a *CS*-model if $\mathsf{CS} \subseteq \mathcal{B}$ for constant specification CS.

Definition 15 (Truth in model).

For a model $\mathcal{M} = (\mathsf{v}, \mathcal{B})$, relation $\mathcal{M} \Vdash A$ is defined by:

$$\begin{aligned} \mathcal{M} \Vdash P \quad & \textit{iff} \quad P \in \mathsf{v} \\ \mathcal{M} \Vdash \neg A \quad & \textit{iff} \quad \mathcal{M} \nVdash A \\ \mathcal{M} \Vdash A \to B \quad & \textit{iff} \quad \mathcal{M} \nVdash A \textit{ or } \mathcal{M} \Vdash B \\ \mathcal{M} \Vdash t : A \quad & \textit{iff} \quad (t, A) \in \mathcal{E}(\mathcal{B}) \end{aligned}$$

$\mathsf{JN_V}$: Intermediate models

Definition 16 (Intermediate model).

A model $\mathcal{M} = (\mathsf{v}, \mathcal{B})$ is called intermediate if the following conditions on $\mathcal{B}$ hold:

(R.1) $(n_P, F) \in \mathcal{B} \Leftrightarrow F = P$ (for $P \in \mathsf{V}$)

(R.2) $n_P \cdot s \notin \mathrm{sub}(t)$ for all $(t, F) \in \mathcal{B}$

(R.3) $(t \cdot n_P, F) \notin \mathcal{B}$

A formula F is called valid w.r.t. intermediate CS-models iff $\mathcal{M} \Vdash F$ for all intermediate CS-models $\mathcal{M}$ (notation: $\Vdash_i F$).

$\mathsf{JN_V}$: Soundness

Theorem 17 (Soundness).

$$\mathsf{JN_V} \vdash F \quad \Rightarrow \quad \Vdash_i F$$

Proof highlights.

Proof proceeds by induction on the derivation.

- Cases for (App) and (N_σ) follow from l.f.p. properties of $\mathcal{E}(\mathcal{B})$.
- $\mathcal{M} \Vdash t : F \Leftrightarrow (t, F) \in \mathcal{B}$ for atomic t.
  This covers cases for (N_+) and (N_-), and used in other branches.
- The property $(n_P \cdot s) \notin \mathrm{sub}(t)$ is preserved by substitution.
  This is used to prove the case for ($N_\triangleleft$)
- ($N_\triangleright$) is the most complex case due to substitutions.

□

$\mathsf{JN_V}$: Induced models

Definition 18 (Induced model).

For a maximal consistent set of formulas Φ, its induced model $\mathcal{M}_\Phi = (\mathsf{v}_\Phi, \mathcal{B}_\Phi)$ is defined by:

- $\mathsf{v}_\Phi := \Phi \cap \mathsf{Prop}$
- $\mathcal{B}_\Phi := \{(t, F) \mid t : F \in \Phi, t \neq s \cdot n_P \text{ for any } s, P\}$

Lemma 19.

For any maximal consistent set, $\mathcal{M}_\Phi$ is an intermediate model.

Lemma 20 (Canonical evidence).

$$(t, F) \in \mathcal{E}(\mathcal{B}_\Phi) \quad \Leftrightarrow \quad t : F \in \Phi$$

The direction from right to left notably uses $(\mathsf{N}_\triangleright)$.

$\mathsf{JN_V}$: Completeness

Lemma 21 (Truth lemma).

$$\mathcal{M}_\Phi \Vdash F \quad \Leftrightarrow \quad F \in \Phi$$

Theorem 22 (Completeness).

$$\Vdash_i F \quad \Rightarrow \quad \mathsf{JN_V} \vdash F$$

JN_V: Atomic models, finite models

Definition 23 (Atomic model).

A model $\mathcal{M}_a = (\mathsf{v}, \mathcal{B}_a)$ is called atomic if it satisfies (R.1) and $\mathcal{B}_a \subseteq \mathsf{ATm} \times \mathsf{Fm}$.
A formula F is called valid w.r.t. atomic CS-models iff $\mathcal{M} \Vdash F$ for all atomic CS-models $\mathcal{M}$ (notation: $\Vdash_a F$).

Remark.

An atomic model is also an intermediate model.

Definition 24 (Finite model).

A model $\mathcal{M}_f = (\mathsf{v}_f, \mathcal{B}_f)$ is called finite if both v_f and $\mathcal{B}_f$ are finite.

JN_V: The plan

The plan is to show that JN_V is sound and complete w.r.t. atomic CS-models.

This will not be shown directly; instead, we use the soundness and completeness w.r.t. intermediate CS-models, followed by a procedure to “atomize” any model satisfying a given formula.

Term complexity

Definition 25 (Term complexity).

We define the complexity of a term $\mathrm{cmp}(t)$ as the number of applications in it.
We define the basis complexity of a basis $\mathcal{B}$ as $\mathrm{cmp}(\mathcal{B}) := \sum_{(t,F)\in\mathcal{B}} \mathrm{cmp}(t)$, or ∞ if this sum is not finite.

Lemma 26 (Term complexity properties).

Term complexity has the following properties:

1. $t \in ATm \Leftrightarrow \mathrm{cmp}(t) = 0$
2. $\mathrm{cmp}(\sigma t) \geqslant \mathrm{cmp}(t)$ *for any term* t *and any basic substitution* σ

$\mathsf{JN_V}$: Relevant evidence

Definition 27 (Relevant evidence and propositional valuation).

For a formula F, we define

- relevant evidence $\mathcal{E}_F := \{(t, G) \mid t : G \in \mathrm{subf}(F)\}$
- relevant propositional valuation $\mathsf{v}_F := \{P \mid P \in \mathrm{subf}(F)\}$

We use "shallow" subformulas: $\mathrm{subf}(t : F) = \{t : F\}$.

Lemma 28.

If two models $\mathcal{M} = (\mathsf{v}, \mathcal{B})$ and $\mathcal{M} = (\mathsf{v}', \mathcal{B}')$

1. *agree on relevant propositional valuation for F, i.e. $\mathsf{v} \cap \mathsf{v}_F = \mathsf{v}' \cap \mathsf{v}_F$, and*
2. *agree on relevant evidence for F, i.e. $\mathcal{E}(\mathcal{B}) \cap \mathcal{E}_F = \mathcal{E}(\mathcal{B}') \cap \mathcal{E}_F$,*

then they agree on the valuation of F: $\mathcal{M} \Vdash F \Leftrightarrow \mathcal{M}' \Vdash F$.

Atomization: Overview

The idea of atomization is the following procedure:

1. Start with an arbitrary intermediate CS-model $\mathcal{M}$ that satisfies a formula F.
   Then, while maintaining satisfaction of F, perform the following steps:
2. Reduce $\mathcal{M}$ to a finite intermediate model $\mathcal{M}_0$ (potentially without full CS).
3. Step by step remove applications from the basis: $\mathcal{M}_i \mapsto \mathcal{M}_{i+1}$, until there are none left. The result is a finite atomic model.
4. Take the final model $\mathcal{M}_n$ and add back the whole CS, making it an atomic CS-model.

This entails soundness and completeness w.r.t. atomic CS-models.

Atomization: Restriction on CS

To be able to add back CS without disrupting satisfaction of F, we require the following restriction on CS:

Definition 29 (Locally finite CS).

We call a constant specification CS locally finite if for every term constant c, $\{(c, F) \mid (c, F) \in \mathsf{CS}\}$ is finite.

This still allows for "strong enough" constant specifications: one can have, for example, axiomatically appropriate locally finite CS.

The following will assume a locally finite constant speficiation CS.

Atomization: Obtaining a finite model

Lemma 30.

For any basis $\mathcal{B}$ and any (t, F) such that $(t, F) \in \mathcal{E}(\mathcal{B})$, there is a finite subset $\mathcal{B}_{(t,F)} \subseteq \mathcal{B}$ such that $(t, F) \in \mathcal{E}(\mathcal{B}_{(t,F)})$.

Assume that we have an intermediate CS-model $\mathcal{M} = (\mathrm{v}, \mathcal{B})$ and a formula F such that $\mathcal{M} \Vdash F$.

First, take all relevant evidence from $\mathcal{M}$: $\mathcal{E}' := \mathcal{E}(\mathcal{B}) \cap \mathcal{E}_F$. This set is finite. Then, for each $(t, G) \in \mathcal{E}'$ take the finite subset of the basis from the above lemma: $\mathcal{B}_{(t,G)}$.

Merge them together, the result being a finite set:

$$\mathcal{B}' := \bigcup_{(t,G) \in E'} \mathcal{B}_{(t,G)}$$

Atomization: Obtaining a finite model

Finally, add CS restricted to constants that occur in terms of $\mathcal{B}'$:

$$\mathcal{B}_0 := \mathcal{B}' \cup \left(\bigcup_{(t,G)\in\mathcal{B}'} \{(c, G) \mid (c, G) \in \mathsf{CS}, c \in \mathrm{sub}(t)\} \right)$$

This is again finite for a locally finite CS.

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This is again finite for a locally finite CS.

Take $\mathcal{M}_0 := (\mathsf{v} \cap \mathsf{v}_F, \mathcal{B}_0)$.

By construction, $\mathcal{M}$ and $\mathcal{M}_0$ agree on relevant evidence and propositional valuation for F; therefore, $\mathcal{M}_0 \Vdash F$.

Atomization: Obtaining a finite model

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This is again finite for a locally finite CS.

Take $\mathcal{M}_0 := (\mathsf{v} \cap \mathsf{v}_F, \mathcal{B}_0)$.

By construction, $\mathcal{M}$ and $\mathcal{M}_0$ agree on relevant evidence and propositional valuation for F; therefore, $\mathcal{M}_0 \Vdash F$.

Lemma 31.

For any intermediate model $\mathcal{M}$ and any F such that $\mathcal{M} \Vdash F$, there exists a finite intermediate model $\mathcal{M}_0$ such that $\mathcal{M}_0 \Vdash F$.

Atomization: Reducing an application

Suppose we have a finite intermediate model $\mathcal{M}_i$ that satisfies F and it's not atomic.

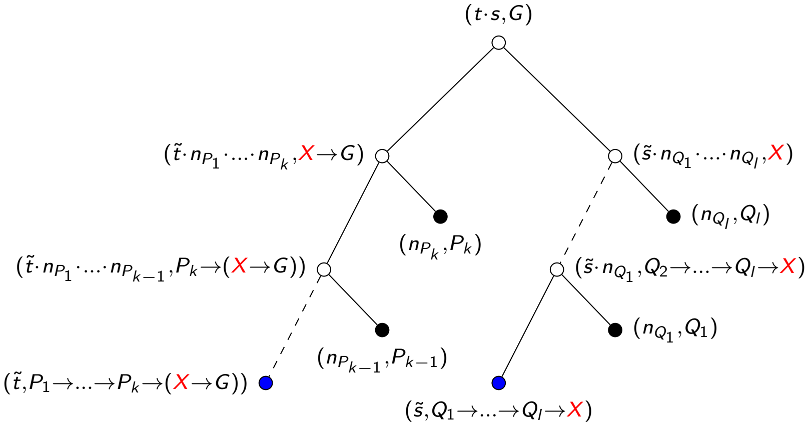
Then there is an evidence pair of the form $(t \cdot s, G) \in \mathcal{B}_i$.

- Neither t (by (R.2)) nor s (by (R.3)) can be n_P for any P.
- By (R.2), t can be uniquely represented as $\tilde{t} \cdot n_{P_1} \cdot \ldots \cdot n_{P_k}$, and similarly for s.

Since $\mathcal{B}_i$ is finite, we can select a "fresh" propositional variable X that does not occur in $\mathcal{B}_i$, F or V.
We use it to replace $(t \cdot s, G)$ with $(t, X \to G)$ and (s, X).

Atomization: Reducing an application

Take $\mathcal{B}_{i+1} := (\mathcal{B}_i \setminus \{(t \cdot s, G)\}) \cup \{$ blue dots $\}$



Atomization: Reducing an application

Collect all the evidence pairs containing X from the picture into a set $\mathcal{E}'$. Then the following can be proven:

$$(r, H) \in \mathcal{E}(\mathcal{B}_{i+1}) \quad \Rightarrow \quad \begin{cases} \text{either } (r, H) \in \mathcal{E}(\mathcal{B}_i), \\ \text{or } (r, H) = (\Sigma(r'), H) \\ \qquad \text{for some } (r', H) \in \mathcal{E}', \Sigma \in \mathcal{S}^*(\mathcal{B}_i) \end{cases}$$

The second case implies that H contains X, and therefore cannot be relevant to F.

Therefore, $\mathcal{M}_{i+1} := (\mathsf{v}_i, \mathcal{B}_{i+1})$ satisfies F.

Atomization: Achieving atomization

Overall number of applications in $\mathcal{B}_i$ is reduced by at least 1 with each step.

Lemma 32.

For every finite intermediate model $\mathcal{M}_i$ and formula F such that $\mathcal{M}_i \Vdash F$, if $\mathrm{cmp}(\mathcal{B}_i) > 0$, there exists a finite intermediate model $\mathcal{M}_{i+1}$ such that $\mathcal{M}_{i+1} \Vdash F$ and $\mathrm{cmp}(\mathcal{B}_{i+1}) < \mathrm{cmp}(\mathcal{B}_i)$

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Therefore, the process terminates with a (finite) atomic model $\mathcal{M}_n$ that satisfies F.

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Therefore, the process terminates with a (finite) atomic model $\mathcal{M}_n$ that satisfies F.

Final step is to merge CS back into the model:

$$\tilde{\mathcal{M}} := (\mathsf{v}_n, \mathcal{B}_n \cup \mathsf{CS})$$

Atomization: Achieving atomization

$$\tilde{\mathcal{M}} := (\mathsf{v}_n, \mathcal{B}_n \cup \mathsf{CS})$$

Again, we can prove a clear separation:

$$(r, H) \in \mathcal{E}(\mathcal{B}_n \cup \mathsf{CS}) \quad \Rightarrow \quad \begin{cases} \text{either } (r, H) \in \mathcal{E}(\mathcal{B}_n), \\ \text{or } r \text{ contains a ``fresh'' } \tilde{c} \end{cases}$$

The second case implies that (r, H) cannot be relevant to F.

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The second case implies that (r, H) cannot be relevant to F.

Lemma 33.

For every intermediate CS-model $\mathcal{M}$ and formula F such that $\mathcal{M} \Vdash F$, there exists an atomic CS-model $\tilde{\mathcal{M}}$ such that $\tilde{\mathcal{M}} \Vdash F$.

Atomization: Achieving atomization

$$\tilde{\mathcal{M}} := (\mathsf{v}_n, \mathcal{B}_n \cup \mathsf{CS})$$

Again, we can prove a clear separation:

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The second case implies that (r, H) cannot be relevant to F.

Lemma 33.

For every intermediate CS*-model* $\mathcal{M}$ *and formula* F *such that* $\mathcal{M} \Vdash F$*, there exists an atomic* CS*-model* $\tilde{\mathcal{M}}$ *such that* $\tilde{\mathcal{M}} \Vdash F$*.*

Theorem 34.

Logic JN_V *is sound and complete w.r.t. atomic* CS*-models.*

JN_V: Where to go from here?

We have the logic JN_V that emulates the state after some updates with atomic propositional statements. It has the desired class of models and desired restrictions on the language (no subscript).

What next?

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What next?

Justification Logic
(with Prop. Nominals)

↓

Justification Logic
(with Nominals) ⟶ Dynamic Justification Logic
(with Updates)

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We have the logic JN_V that emulates the state after some updates with atomic propositional statements. It has the desired class of models and desired restrictions on the language (no subscript).

What next?

Justification Logic (with Prop. Nominals)	$\longrightarrow$	Dynamic Justification Logic (with Prop. Updates)
$\downarrow$		$\downarrow$
Justification Logic (with Nominals)	$\longrightarrow$	Dynamic Justification Logic (with Updates)

JN_V: Where to go from here?

We have the logic JN_V that emulates the state after some updates with atomic propositional statements. It has the desired class of models and desired restrictions on the language (no subscript).

What next?

Justification Logic (with Prop. Nominals)	$\longrightarrow$	Dynamic Justification Logic (with Prop. Updates)
$\downarrow$		$\downarrow$
Justification Logic (with Nominals)	$\longrightarrow$	Dynamic Justification Logic (with Updates)

Also, perhaps the local finiteness restriction on CS can be relaxed.

Thank you for listening.