The Lambek Calculus with Unary Connectives

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(partially based on joint work with M. Kanovich, A. Ščedrov, and N. Ryzhkova)

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Outline

Logic	**Applications**
fragments and variants of non-commutative linear logic	categorial grammars for fragments of natural language

The Lambek Calculus

$$\frac{}{A \to A}\ (\mathrm{id})$$

$$\frac{\Pi \to A \quad \Delta_1, B, \Delta_2 \to C}{\Delta_1, B \,/\, A, \Pi, \Delta_2 \to C}\ (/ \to) \qquad \frac{\Pi, A \to B}{\Pi \to B \,/\, A}\ (\to /)$$

$$\frac{\Pi \to A \quad \Delta_1, B, \Delta_2 \to C}{\Delta_1, \Pi, A \setminus B, \Delta_2 \to C}\ (\setminus \to) \qquad \frac{A, \Pi \to B}{\Pi \to A \setminus B}\ (\to \setminus)$$

$$\frac{\Delta_1, A, B, \Delta_2 \to C}{\Delta_1, A \cdot B, \Delta_2 \to C}\ (\cdot \to) \qquad \frac{\Pi_1 \to A \quad \Pi_2 \to B}{\Pi_1, \Pi_2 \to A \cdot B}\ (\to \cdot)$$

Lambek Grammar

John loves Mary

Lambek Grammar

John	loves	Mary
np	$(np \backslash s) / np$	np

Lambek Grammar

John	loves	Mary	
np	$(np \backslash s) / np$	np	$\rightarrow s$

Lambek Grammar

$$\mathbf{L} \vdash \underset{np}{\text{John}} \quad \underset{(np \backslash s) / np}{\text{loves}} \quad \underset{np}{\text{Mary}} \quad \rightarrow s$$

Lambek Grammar

	John	loves	Mary	
$\mathbf{L} \vdash$	np	$(np \backslash s) / np$	np	$\to s$

the girl whom John loves

Lambek Grammar

	John	loves	Mary	
$\mathbf{L} \vdash$	np	$(np \backslash s) / np$	np	$\rightarrow s$

the	girl	whom	John	loves
np / n	n	$(n \backslash n) / (s / np)$	np	$(np \backslash s) / np$

Lambek Grammar

$$\mathbf{L} \vdash \underset{np}{\text{John}} \quad \underset{(np \backslash s)/np}{\text{loves}} \quad \underset{np}{\text{Mary}} \quad \to s$$

$$\underset{np/n}{\text{the}} \quad \underset{n}{\text{girl}} \quad \underset{(n \backslash n)/(s/np)}{\text{whom}} \quad \underbrace{\underset{np}{\text{John}} \quad \underset{(np \backslash s)/np}{\text{loves}}}_{\to s/np}$$

Lambek Grammar

$$\mathbf{L} \vdash \underset{np}{\text{John}} \quad \underset{(np \backslash s) / np}{\text{loves}} \quad \underset{np}{\text{Mary}} \quad \rightarrow s$$

$$\mathbf{L} \vdash \underset{np / n}{\text{the}} \quad \underset{n}{\text{girl}} \quad \underset{\color{red}{(n \backslash n) / (s / np)}}{\text{whom}} \quad \underbrace{\underset{np}{\text{John}} \quad \underset{(np \backslash s) / np}{\text{loves}}}_{\rightarrow s / np} \quad \rightarrow np$$

Lambek Grammar

	John	loves	Mary	
$\mathbf{L} \vdash$	np	$(np \backslash s) / np$	np	$\rightarrow s$

	the	girl	whom	John	loves	
$\mathbf{L} \vdash$	np / n	n	$(n \backslash n) / (s / np)$	np	$(np \backslash s) / np$	$\rightarrow np$

$$\underbrace{np \quad (np \backslash s) / np}_{\rightarrow s / np}$$

the	boy	who	loves	Mary

Lambek Grammar

$$\mathbf{L} \vdash \underset{np}{\text{John}} \quad \underset{(np \backslash s) / np}{\text{loves}} \quad \underset{np}{\text{Mary}} \quad \rightarrow s$$

$$\mathbf{L} \vdash \underset{np / n}{\text{the}} \quad \underset{n}{\text{girl}} \quad \underset{\color{red}{(n \backslash n) / (s / np)}}{\text{whom}} \quad \underbrace{\underset{np}{\text{John}} \quad \underset{(np \backslash s) / np}{\text{loves}}}_{\rightarrow s / np} \quad \rightarrow np$$

$$\underset{np / n}{\text{the}} \quad \underset{n}{\text{boy}} \quad \underset{\color{red}{(n \backslash n) / (np \backslash s)}}{\text{who}} \quad \underset{(np \backslash s) / np}{\text{loves}} \quad \underset{np}{\text{Mary}}$$

Lambek Grammar

	John	loves	Mary	
$\mathbf{L} \vdash$	np	$(np \backslash s) / np$	np	$\to s$

	the	girl	whom	John	loves	
$\mathbf{L} \vdash$	np / n	n	$(n \backslash n) / (s / np)$	np	$(np \backslash s) / np$	$\to np$

$$\underbrace{np \quad (np \backslash s) / np}_{\to s / np}$$

the	boy	who	loves	Mary
np / n	n	$(n \backslash n) / (np \backslash s)$	$(np \backslash s) / np$	np

$$\underbrace{(np \backslash s) / np \quad np}_{\to np \backslash s}$$

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Lambek Grammar

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$\mathbf{L} \vdash$	np	$(np \backslash s) / np$	np	$\to s$

	the	girl	whom$_i$	John	loves	e_i
$\mathbf{L} \vdash$	np / n	n	$(n \backslash n) / (s / np)$	np	$(np \backslash s) / np$	$\to np$

$$\underbrace{np \quad (np \backslash s) / np}_{\to s / np}$$

	the	boy	who$_i$	e_i	loves	Mary	
$\mathbf{L} \vdash$	np / n	n	$(n \backslash n) / (np \backslash s)$		$(np \backslash s) / np$	np	$\to np$

$$\underbrace{(np \backslash s) / np \quad np}_{\to np \backslash s}$$

Lambek's Non-Emptiness Restriction

In the original Lambek calculus, all antecedents are forced to be non-empty. $\mathbf{L}^*$ stands for the Lambek calculus without this restriction.

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book	$\triangleright$	n	(noun)
interesting	$\triangleright$	$n \,/\, n$	(adjective = left noun modifier)
very	$\triangleright$	$(n \,/\, n) \,/ (n \,/\, n)$	(adverb = left adjective modifier)

very interesting book

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$$\underset{\text{very}}{(n\,/\,n)\,/(n\,/\,n)},\quad \underset{\text{interesting}}{n\,/\,n},\quad \underset{\text{book}}{n}$$

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$$\begin{array}{rcll} \text{book} & \rhd & n & \text{(noun)} \\ \text{interesting} & \rhd & n \,/\, n & \text{(adjective = left noun modifier)} \\ \text{very} & \rhd & (n \,/\, n) \,/ (n \,/\, n) & \text{(adverb = left adjective modifier)} \end{array}$$

$$\underset{\text{very}}{(n \,/\, n) \,/ (n \,/\, n)}, \quad \underset{\text{interesting}}{n \,/\, n}, \quad \underset{\text{book}}{n} \quad \to n$$

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\end{array}
$$

$$
\mathbf{L} \vdash \underbrace{(n \,/\, n) \,/ (n \,/\, n),}_{\text{very}} \quad \underbrace{n \,/\, n,}_{\text{interesting}} \quad \underbrace{n}_{\text{book}} \quad \to n
$$

$$
\mathbf{L}^* \vdash (n \,/\, n) \,/ (n \,/\, n), \quad n \quad \to n
$$

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In the original Lambek calculus, all antecedents are forced to be non-empty. $\mathbf{L}^*$ stands for the Lambek calculus without this restriction.

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$$\mathbf{L} \vdash \underbrace{(n / n) / (n / n)}_{\text{very}}, \underbrace{n / n}_{\text{interesting}}, \underbrace{n}_{\text{book}} \to n$$

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In the original Lambek calculus, all antecedents are forced to be non-empty. **L*** stands for the Lambek calculus without this restriction.

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$$\mathbf{L} \vdash \underbrace{(n \,/\, n) \,/ (n \,/\, n)}_{\text{very}},\ \underbrace{n \,/\, n}_{\text{interesting}},\ \underbrace{n}_{\text{book}} \to n$$

$$\mathbf{L}^* \vdash \underbrace{(n \,/\, n) \,/ (n \,/\, n)}_{\text{very}},\ \underbrace{n}_{\text{book}} \to n$$

But, Lambek's restriction sometimes leads to problems...

Properties of the Lambek calculus

Theorem (C. Gaifman 1960; M. Pentus 1992)

Lambek grammars generate precisely the class of context-free languages.

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L-models: $w(A \cdot B) = \{uv \mid u \in w(A), v \in w(B)\}$,
$w(A \setminus B) = \{u \mid (\forall v \in w(A))\, vu \in w(B)\}$,
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Theorem (M. Pentus 1995)

The Lambek calculus is sound and complete w.r.t. L-models, i.e., $A \to B$ is derivable iff $w(A) \subseteq w(B)$ for any w.

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The derivability problem for the Lambek calculus is NP-complete. The derivability problem for the Lambek calculus with only one operation ($/$) is decidable in polynomial time ($O(n^3)$).

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This all works both for **L** and $\mathbf{L}^*$.

$\mathbf{L}^{\mathrm{R}}$: the Lambek Calculus with the Reversal

$$(\mathrm{id}) \quad (\backslash \to) \quad (\to \backslash) \quad (/ \to) \quad (\to /) \quad (\cdot \to) \quad (\to \cdot)$$

$$\frac{A_1, \ldots, A_n \to C}{A_n^{\mathrm{R}}, \ldots, A_1^{\mathrm{R}} \to C^{\mathrm{R}}}\ ({}^{\mathrm{R}}{\to}^{\mathrm{R}}) \qquad \frac{\Delta_1, A^{\mathrm{RR}}, \Delta_2 \to C}{\Delta_1, A, \Delta_2 \to C}\ ({}^{\mathrm{RR}}{\to})_{\mathrm{E}} \qquad \frac{\Gamma \to C^{\mathrm{RR}}}{\Gamma \to C}\ ({\to}^{\mathrm{RR}})_{\mathrm{E}}$$

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L-interpretation: $w(A^{\mathrm{R}}) = \{a_n \ldots a_1 \mid a_1 \ldots a_n \in w(A)\}$.

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Theorem (S. K. 2012)

- $\mathbf{L}^{\mathrm{R}}$ *is sound and complete w.r.t. L-models.*

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Theorem (S. K. 2012)

- $\mathbf{L}^{\mathrm{R}}$ *is sound and complete w.r.t. L-models.*
- $\mathbf{L}^{\mathrm{R}}$*-grammars generate precisely the class of context-free languages.*
- *The derivability problem in* $\mathbf{L}^{\mathrm{R}}$ *(even with only one division) is NP-complete.*

The Exponential

$$(\text{id}) \quad (\backslash \to) \quad (\to \backslash) \quad (/ \to) \quad (\to /) \quad (\cdot \to) \quad (\to \cdot) \quad (\mathbf{1} \to) \quad (\to \mathbf{1})$$

$$\frac{\Delta_1, A, \Delta_2 \to C}{\Delta_1, !A, \Delta_2 \to C}\ (! \to) \qquad \frac{!A_1, \ldots, !A_n \to C}{!A_1, \ldots, !A_n \to !C}\ (\to !)$$

$$\frac{\Delta_1, !A, \Delta_2, \Delta_3 \to C}{\Delta_1, \Delta_2, !A, \Delta_3 \to C}\ (\text{perm}_1) \qquad \frac{\Delta_1, \Delta_2, !A, \Delta_3 \to C}{\Delta_1, !A, \Delta_2, \Delta_3 \to C}\ (\text{perm}_2)$$

$$\frac{\Delta_1, \Delta_2 \to C}{\Delta_1, !A, \Delta_2 \to C}\ (\text{weak})$$

$$\frac{\Delta_1, !A, !A, \Delta_2 \to C}{\Delta_1, !A, \Delta_2 \to C}\ (\text{contr})$$

The Exponential

(id) $(\backslash \to)$ $(\to \backslash)$ $(/ \to)$ $(\to /)$ $(\cdot \to)$ $(\to \cdot)$ $(\mathbf{1} \to)$ $(\to \mathbf{1})$

$$\frac{\Delta_1, A, \Delta_2 \to C}{\Delta_1, !A, \Delta_2 \to C}\ (! \to) \qquad \frac{!A_1, \ldots, !A_n \to C}{!A_1, \ldots, !A_n \to !C}\ (\to !)$$

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DANGER!

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Undecidability

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The Lambek calculus with the exponential is undecidable.

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- Encode Lambek *theories* using !: for an extra non-logical axioms of the form $A \to B$ add $!(B / A)$ to the antecedent.
- Encode *semi-Thue systems* (type-0 grammars) as Lambek theories: for rewriting rule $u_1 \dots u_k \Rightarrow v_1 \dots v_m$ add non-logical axiom $v_1, \dots, v_m \to u_1 \cdot \dots \cdot u_k$.

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- (W. Buszkowski 1982) A trick allows to use only / in this encoding.
- (M. Kanovich 1994) A substitution that reduces to the one-variable fragment (needs checking for the Lambek calculus...)
- Corollary: derivability from finite theories is undecidable even in the one-variable fragment.

Issues with Lambek's Restriction

1. If $\mathbf{L} \vdash \Pi \to A$, then $\mathbf{EL}^{\dagger} \vdash \Pi \to A$.
2. $$\frac{\Pi \to A \quad \Delta_1, A, \Delta_2 \to C}{\Delta_1, \Pi, \Delta_2 \to C}\ (\mathrm{cut})$$
3. $$\frac{\Pi \to A}{\Pi[q := Q] \to A[q := Q]}\ (\mathrm{subst})$$
4. $$\frac{A_1 \to A_2 \quad B_1 \to B_2}{B_1 \,/\, A_2 \to B_2 \,/\, A_1}\ (\mathrm{mon}_{/}) \qquad \frac{A_1 \to A_2 \quad B_1 \to B_2}{A_2 \setminus B_1 \to A_1 \setminus B_2}\ (\mathrm{mon}_{\setminus})$$
5. The rules (weak), (contr), and $(\mathrm{perm}_{1,2})$ are admissible in $\mathbf{EL}^{\dagger}$.
6. The rules $(/ \to)$, $(\setminus \to)$, $(\cdot \to)$, and $(\to \cdot)$ are admissible in $\mathbf{EL}^{\dagger}$ without restrictions.
7. If Π contains a formula without occurrences of ! (and therefore is non-empty) and B does not contain occurrences of !, than the rules $(\to /)$ and $(\to \setminus)$ are admissible in $\mathbf{EL}^{\dagger}$.

Theorem

If $\mathbf{EL}^{\dagger}$ *satisfies 1–7, then*

$$\mathbf{L}^{*} \vdash \Pi \to C \Rightarrow \mathbf{EL}^{\dagger} \vdash\, !q, \Pi \to C.$$

Subexponentials

Leave only some of the structural rules.

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Non-local contraction:

$$\frac{\Delta_1, !A, \Delta_2, !A, \Delta_3 \to C}{\Delta_1, \Delta_2, !A, \Delta_3 \to C}\ (\mathrm{ncontr}_1) \qquad \frac{\Delta_1, !A, \Delta_2, !A, \Delta_3 \to C}{\Delta_1, !A, \Delta_2, \Delta_3 \to C}\ (\mathrm{ncontr}_2)$$

Subexponentials

Leave only some of the structural rules.

	L	$\mathbf{L}_{/}$
(weak), (contr), $(\mathrm{perm}_{1,2})$	undecidable	
(contr), $(\mathrm{perm}_{1,2})$	undecidable	
$(\mathrm{ncontr}_{1,2})$	undecidable	**?**
no (sub)exponentials	**NP**-complete (Pentus 2006)	polynomial (Savateev 2007)
$(\mathrm{perm}_{1,2})$	**NP**-complete	
(weak), $(\mathrm{perm}_{1,2})$	**NP**-complete	**?**
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NB: the **1** constant breaks L-completeness!

- **Conjecture.** One could eliminate **1** by means of substitution $\mathbf{1} := q \,/\, q$, $p_i := (q \,/\, p_i) \,/ (q \,/\, q)$ (cf. S.K. 2011 for $\mathbf{L}^{\mathbf{1}}_{/}$)

Subexponentials in Lambek Grammar: Medial Extraction

the girl whom John met yesterday

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$$\text{the girl} \qquad \text{whom}_i \qquad \underbrace{\text{John met } e_i \text{ yesterday}}_{\to\, s \,/\, !np}$$

$$\frac{\Delta_1, !A, \Delta_2, \Delta_3 \to C}{\Delta_1, \Delta_2, !A, \Delta_3 \to C}\ (\mathrm{perm}_1) \qquad \frac{\Delta_1, A, \Delta_2 \to C}{\Delta_1, !A, \Delta_2 \to C}\ (! \to)$$

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Subexponentials in Lambek Grammar: “Parasitic” Extraction

the paper that John signed without reading

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A conservative fragment of **Db!** by Morrill and Valentín (2015).

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- The latter system is actually implemented in CatLog (exponential-time proof search algorithm using the focusing technique).

The Kleene Star in Lambek Grammar: Iterated Coordination

John,	Bill,	Mary,	and	Suzy	
np	np	np	$np^* \setminus np / np$	np	$\rightarrow np$

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John,	Bill,	Mary,	and	Suzy	
np	np	np	$np^* \setminus np \, / \, np$	np	$\to np$

Kleene star:

$$\frac{\Gamma_1 \to A \quad \ldots \quad \Gamma_n \to A}{\Gamma_1, \ldots, \Gamma_n \to A^*} \; (\to^*)_n$$

The Lambek Calculus with the Kleene Star

Along with $(\rightarrow {}^*)_n$, we need a left rule for *.

The Lambek Calculus with the Kleene Star

$$\frac{\Gamma, A^n, \Delta \to C \qquad \text{for all } n \geq 0}{\Gamma, A^*, \Delta \to C}\ ({}^*\!\to)_\omega \qquad \frac{\Gamma_1 \to A \quad \dots \quad \Gamma_n \to A}{\Gamma_1, \dots, \Gamma_n \to A^*}\ (\to{}^*)_n$$

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L-interpretation: $w(A^*) = \{a_1 \dots a_n \mid n \geq 0, a_i \in w(A)\}$.

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For the case with Lambek's nonemptiness restriction, consider the Kleene plus instead of the Kleene star.

Complexity

! and *	Π_2^0-hard (Π_1^1-complete?)
!	r.e.-complete [P. Lincoln, J. Mitchell, A. Scedrov, N. Shankar 1992; M. Kanovich, S. K., A. Scedrov 2016]
*, $\cap$, and $\cup$	Π_1^0-hard [W. Buszkowski, E. Palka 2008]
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Technique for * and !: encode Kozen's complexity results about Horn theories on Kleene algebras.

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- Cut elimination.

$!(\text{SKNAHT}^R)^*$