Generalized Veltman models

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The language of the interpretability logic contains:

- propositional letters $p_0,\ p_1, \ldots,$
- logical connectives $\wedge,\ \vee,\ \rightarrow,\ \leftrightarrow$ and $\neg$,
- unary modal operator $\Box$
  (we use modal operator $\Diamond$ for abbreviation $\neg\Box\neg$)
- binary modal operator $\rhd$.

Here are the **axioms and deduction rules of the system** $\mathtt{IL}$.

(L0)–(L3) *axioms of the system* $\mathtt{GL}$ *(Gödel–Löb)*

(J1) $\Box(A \to B) \to (A \rhd B)$

(J2) $((A \rhd B) \wedge (B \rhd C)) \to (A \rhd C)$

(J3) $((A \rhd C) \wedge (B \rhd C)) \to ((A \vee B) \rhd C)$

(J4) $(A \rhd B) \to (\Diamond A \to \Diamond B)$

(J5) $\Diamond A \rhd A$

The deduction rules of $\mathtt{IL}$ are modus ponens and necessitation.

Principles of interpretability (1)

Montagna's principle

$$M \equiv A \rhd B \rightarrow (A \wedge \Box C) \rhd (B \wedge \Box C)$$

Denote by ILM the system which results from the IL by adding the Montagna's principle.

A. Berarducci and V. Shavrukov (1990) showed that ILM is complete for arithmetical interpretation over PA.

Principles of interpretability (2)

The principle of persistency

$$P \equiv A \rhd B \rightarrow \Box(A \rhd B)$$

The system ILP results from the IL by adding the persistence principle.

A. Visser (1988) obtained the arithmetical completeness for ILP over any finitely axiomatizable theory.

Principles of interpretability (3)

$F, \; W, \; M_0, \; W^*, \; P_0, \; R, \; R_n, \; n \in \omega, \; \dots$

Open problem: What is interpretability logic of all "reasonable" theories?

Basic semantics for interpretability logic

An ordered quadruple $(W, R, \{S_w : w \in W\}, \Vdash)$ is called **Veltman model**, if it satisfies the following conditions:

a) (W, R) is a GL–frame, i.e. W is a non empty set, and the relation R is transitive and reverse well–founded relation on W

b) $S_w \subseteq W[w] \times W[w]$, where $W[w] = \{x : wRx\}$

c) The relation S_w is reflexive

d) The relation S_w is transitive

e) If $wRvRu$ then $vS_w u$

f) $\Vdash$ is a forcing relation. We emphasize only the definition: $w \Vdash A \rhd B$ if and only if

$$\forall u((wRu \ \& \ u \Vdash A) \Rightarrow \exists v(uS_w v \ \& \ v \Vdash B)).$$

D. de Jongh and F. Veltman proved in 1988 the **modal completeness** of the systems IL, ILM and ILP with respect to Veltman semantics.

Theorem. For each formula F of interpretability logic we have:

$$\vdash_{IL} F \quad \text{if and only if} \quad W \models F, \text{ for each Veltman model } W$$

$$\vdash_{ILM} F \quad \text{if and only if} \quad W \models F, \text{ for each ILM–model } W$$

$$\vdash_{ILP} F \quad \text{if and only if} \quad W \models F, \text{ for each ILP–model } W$$

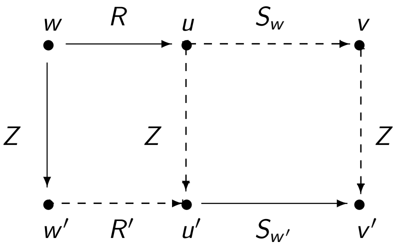
Bisimulation between two Veltman models W and W' is a nonempty binary relation $Z \subseteq W \times W'$ such that the following conditions hold:

(at) If wZw' then $W, w \Vdash p$ if and only if $W', w' \Vdash p$, for all propositional variables p;

(forth) If wZw' and wRu, then there exists $u' \in W'$ with $w'R'u'$, uZu' and for all $v' \in W'$ if $u'S_{w'}v'$ there is $v \in W$ such that uS_wv;

(back) If wZw' and $w'Ru'$, then there exists $u \in W$ with wRu, uZu' and for all $v \in W$ if uS_wv there is $v' \in W'$ such that $u'S_{w'}v'$.

(forth)



By using a bisimulation A. Visser (1998) proved that **Craig interpolation lemma is not valid** for systems between $\mathtt{ILM}_0$ and $\mathtt{ILM}$.

D. Vrgoč and M. Vuković (Reports Math. Logic, 2011) consider **bisimulation quotients of Veltman models**.

T. Perkov and M. Vuković (Ann. Pure and App. Logic, 2014) prove that a first–order formula is equivalent to standard translation of some formula of interpretability logic with respect to Veltman models if and only if it is invariant under bisimulations between Veltman models. (**Van Benthem's characterisation theorem for interpretability logic**)

Bisimulations between different kinds of models

A. Visser (1988) proved that every Veltman model with some special property can be bisimulated by a finite Friedman model.

(This fact and de Jongh–Veltman's theorem imply completeness of the system ILP w.r.t. finite Friedman models).

A. Berarducci (1990) used a bisimulation for the proof of completeness of system ILM w.r.t. simplified Veltman models.

An ordered quadruple $(W', R', \{S'_w : w \in W'\}, \Vdash)$ is called **generalized Veltman model**, if it satisfies the following conditions:

a) (W', R') is a GL–frame
b) For every $w \in W'$ is $S'_w \subseteq W'[w] \times \mathcal{P}(W'[w])$
c) The relations S'_w are quasi–reflexive
d) If $wR'uR'v$ then $uS'_w\{v\}$

e) The relations S'_w are **quasi–transitive**, i.e. $uS'_w V$ and $(\forall v \in V)S'_w Z_v$ imply $uS'_w \cup Z_v$

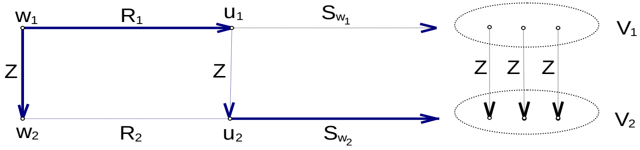
f) The relations S'_w are monotone
g) $\Vdash$ is a forcing relation.

The logic of interpretability IL is **complete** w.r.t. generalized Veltman models.

We use generalized Veltman models for proving **independence** of principles of interpretability (Glasnik matematički 1996, Notre Dame Journal of Formal Logic 1999).

E. Goris and J. Joosten (Logic Jou. IGPL, 2008) use some kind of generalized semantics for proving **independence** of principles of interpretability.

$(\textbf{forth})_{gen \to gen}$



We (BSL, 2005) define bisimulations between generalized Veltman models and prove Hennessy–Milner theorem for generalized Veltman semantics.

D. Vrgoč and M. Vuković (Log. Jou. IGPL, 2010) consider several notions of bisimulation between generalized Veltman models and determined connections between them.

T. Perkov and M. Vuković (MLQ, 2016) by using generalized Veltman model obtain fmp for logic $\mathtt{ILM}_0$ (and decidability).

We (Math. Log. Quarterly, 2008) prove that for a complete image–finite generalized Veltman model W' there exists a Veltman model W that is bisimular to W'.

(We use Hennessy–Milner theorem)

It is an open problem if there is a bisimulation between generalized Veltman model and some Veltman model.

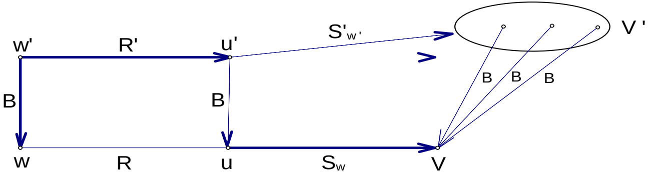
Bisimulation between generalized Veltman model W' and Veltman model W is a relation $B \subseteq W' \times W$ such that:

(at) *if* $w'Bw$ *then* $w' \Vdash p$ *if and only if* $w \Vdash p$;

$(\text{forth})_{gen \to o}$ *if* $w'Bw$ *and* $w'R'u'$ *then there exists* $u \in W$ *such that* wRu, $u'Bu$ *and for each* $v \in W$ *such that* $uS_w v$ *there exists* $V' \subseteq W'$ *such that* $u'S'_{w'}V'$ *and* $(\forall\, v' \in V')v'Bv$;

$(\text{back})_{gen \to o}$ .... $(\exists\, v' \in V')v'Bv$.

(forth)$_{gen \to o}$



Proposition 1. Let $B \subseteq W' \times W$ be a bisimulation between generalized Veltman model W' and Veltman model W. If $w'Bw$ then $w' \equiv w$.

Propostion 2. Let $\{B_i \subseteq W' \times W : i \in I\}$ be a set of bisimulations between generalized Veltman model W' and Veltman model W. Then the relation $\bigcup_{i \in I} B_i$ is a bisimulation.

Bisimulation between Veltman model W and generalized Veltman model W' is a relation $\beta \subseteq W \times W'$ such that:

(at) *if* $w \beta w'$ *then* $w \Vdash p$ *if and only if* $w' \Vdash p$;

$(\text{forth})_{o \to gen}$ *if* $w \beta w'$ *and* wRu *then there exists* $u' \in W'$ *such that* $w'R'u'$, $u \beta u'$ *and for each* V' *such that* $u' S'_{w'} V'$ *there exists* $v \in W$ *such that* $uS_w v$ *and* $(\exists\, v' \in V') v \beta v'$;

$(\text{back})_{o \to gen}$ .... $(\forall\, v' \in V') v \beta v'$.

Proposition 3. Let $\beta \subseteq W \times W'$ be a bisimulation between Veltman model W and generalized Veltman model W'. If $w \beta w'$ then $w \equiv w'$.

Propostion 4. Let $\{\beta_i \subseteq W \times W' : i \in I\}$ be a set of bisimulations between Veltman model W and generalized Veltman model W'. Then the relation $\bigcup_{i \in I} \beta_i$ is a bisimulation.

Proposition 5. Let $\beta \subseteq W \times W'$ and $B \subseteq W' \times W$ be bisimulations, where W is Veltman model and W' is generalized Veltman model.

Then the relations β^{-1} and B^{-1} are also bisimulations.

Further, the relations $\beta \circ B$ and $B \circ \beta$ are bisimulations.

Proposition 6. Let $(W, R, \{S_w : w \in W\}, \Vdash)$ be Veltman model. We define:

$$uS'_w V \text{ if and only if } (\exists v \in V) uS_w v,$$

for each $w, u \in W$ and $V \subseteq W[w]$. Then the relation $\{(w, w) : w \in W\}$ is a bisimulation between Veltman model W and generalized Veltman model $(W, R, \{S'_w : w \in W\}, \Vdash)$.

Goris and Joosten (2004) define two new kinds of generalized models: $\mathtt{IL}_{set}$ and $\mathtt{IL}_{set^*}$ models.

It is not necessery to consider bisimulation with $\mathtt{IL}_{set}$ and $\mathtt{IL}_{set^*}$ models, because we do not use quasi–transitivity in the definition of bisimulation (and in proofs of the properites of bismulation).

For example, the condition $(\textit{forth})_{set\to o}$ is the same as the condition $(\textit{forth})_{gen\to o}$.