# Asterix calculus - classical computation in detail
(denoted $^*\mathcal{X}$)

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# Outline

Sequent calculus, proofs and programs

The $^*\mathcal{X}$ calculus : explicit erasure and duplication

Strong normalisation property

# Outline

# The framework

- Classical logic
- Sequent calculus of G.Gentzen (two important formalisms : *sequent calculus* and *natural deduction calculus*)
- The Curry-Howard correspondence (the relation between proof theory and the programming language theory)

The goal is to study classical computation, by assigning computational interpretation(s) to classical logic represented in the sequent calculus.

# The Curry-Howard correspondence

- The intuitionistic logic and simply typed $\lambda$-calculus

| | | |
|---|---|---|
| *Proofs* | $\Leftrightarrow$ | *Terms* |
| *Propositions* | $\Leftrightarrow$ | *Types* |
| *Normalization* | $\Leftrightarrow$ | *Reduction* |

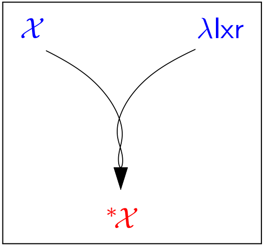
- Classical logic : T. Griffin (1990)
  - M. Parigot (1992) : $\lambda\mu$-calculus (natural deduction)
  - P. L. Curien, H. Herbelin (1995-2005) : $\bar{\lambda}\mu\tilde{\mu}$-calculus (sequent calculus)

# Related work

## The two predecessors

- *Classical logic* : the $\mathcal{X}$ calculus
  - C. Urban (2000)
  - S. van Bakel, S.Lengrand, P. Lescanne (2005)

- *Intuitionistic logic* : the $\lambda$lxr-calculus
  - D. Kesner, S. Lengrand (2005)




# G3 sequent system for classical logic $\boxed{\mathcal{X}\text{-calculus}}$

$$\frac{}{\Gamma, A \vdash A, \Delta}\ (\text{axiom})$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ (\to \text{left}) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to \text{right})$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\ (\text{cut})$$

- Contexts $\Gamma, \Delta$ are $\boxed{\text{sets}}$
- Context-sharing style
- Structural rules $\boxed{\text{implicit}}$

# G1 sequent system for classical logic $\boxed{{}^*\mathcal{X}\text{ calculus}}$

$$\frac{}{A \vdash A}\ \text{(axiom)}$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \to B \vdash \Delta, \Delta'}\ (\to \text{left}) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ (\to \text{right})$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \text{(cut)}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ (\underline{\text{left weakening}}) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ (\underline{\text{right weakening}})$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ (\underline{\text{left contraction}}) \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ (\underline{\text{right contraction}})$$

# Outline

- Sequent calculus, proofs and programs
  - Implicit structural rules – the $\mathcal{X}$ calculus
  - Explicit structural rules – the ${}^*\mathcal{X}$ calculus
- The ${}^*\mathcal{X}$ calculus : explicit erasure and duplication
  - Logical setting
  - From sequent proofs to terms
  - The syntax and reduction rules
  - Diagrammatic view
- Strong normalisation property

- Computational interpretation of classical proofs
- Sequent calculus with explicit structural rules (weakening and contraction)
- 1) Terms in ${}^*\mathcal{X}$-calculus are in fact annotations for proofs,
  2) Computation in ${}^*\mathcal{X}$-calculus corresponds to cut-elimination
- Weakening as an eraser / Contraction as a duplicator

# Names

The terms are built from *names*.

Two categories of names : $x, y, z...$ in-names
$\alpha, \beta, \gamma...$ out-names

Binders wear "hats" : $\widehat{x}, \widehat{y}, \widehat{z}... \ \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}...$

Examples :

$$\langle x.\alpha \rangle \qquad \widehat{x}\,\langle x.\beta\rangle\,\widehat{\beta}\cdot\alpha \qquad [P\rangle^{\widehat{\beta}}_{\widehat{\gamma}}{>}\alpha$$

# Name $\neq$ Variable

- In $\lambda$-calculus : variables
  - $(\lambda y.xyz)M \xrightarrow{\beta} xMz$
  - An arbitrary term M substitutes the variable y
- In ${}^{*}\mathcal{X}$-calculus : names
  - A name can never be substituted for a term
  - A name can only be renamed

# G1 sequent system for classical logic

$$\frac{}{A \vdash A} \text{ (axiom)}$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \to B \vdash \Delta, \Delta'} \ (\to \text{left}) \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta} \ (\to \text{right})$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{ (cut)}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ (left weakening)} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \text{ (right weakening)}$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ (left contraction)} \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \text{ (right contraction)}$$

# The terms correspond to proofs :

$$\frac{}{\langle x.\alpha\rangle :\cdot\; x : A \vdash \alpha : A}\;(\text{caps})$$

$$\frac{P :\cdot\; \Gamma \vdash \alpha : A, \Delta \qquad Q :\cdot\; \Gamma', x : B \vdash \Delta'}{P\widehat{\alpha}\;[y]\;\widehat{x}Q :\cdot\; \Gamma, \Gamma', y : A \to B \vdash \Delta, \Delta'}\;(\text{imp}) \qquad \frac{P :\cdot\; \Gamma, x : A \vdash \alpha : B, \Delta}{\widehat{x}\,P\,\widehat{\alpha}\cdot\beta :\cdot\; \Gamma \vdash \beta : A \to B, \Delta}\;(\text{exp})$$

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$$\frac{P :\cdot\; \Gamma \vdash \Delta}{x \odot P :\cdot\; \Gamma, x : A \vdash \Delta}\;(\text{left eraser}) \qquad \frac{P :\cdot\; \Gamma \vdash \Delta}{P \odot \alpha :\cdot\; \Gamma \vdash \alpha : A, \Delta}\;(\text{right eraser})$$

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# Linearity

- In ${}^{*}\mathcal{X}$-calculus only linear terms are considered :
  - ◇ every free name occurs only once
  - ◇ every binder does bind an occurrence of a free name (and therefore only one)
- Examples of non-linear terms :

$$\widehat{x}\,\langle y.\beta\rangle\,\widehat{\beta}\cdot\alpha \qquad \text{and} \qquad \langle x.\alpha\rangle \odot \alpha$$

- Every non-linear term has a linear representation :

$$\widehat{x}\,(x\odot\langle y.\beta\rangle)\,\widehat{\beta}\cdot\alpha \qquad \text{and} \qquad [\langle x.\alpha_1\rangle \odot \alpha_2\rangle\frac{\widehat{\alpha_1}}{\widehat{\alpha_2}}{>}\alpha$$

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# The ${}^*\mathcal{X}$ calculus

Grouping the rules :

- Logical rules (L-principal names involved)
- Structural rules (S-principal names involved)

- (Activation rules)
- (Deactivation rules)

- Propagation rules

# The ${}^{*}\mathcal{X}$ calculus

## The syntax

| | | | |
|---|---|---|---|
| $P, Q$ | $::=$ | $\langle x.\alpha \rangle$ | *capsule* |
| | $\mid$ | $\widehat{x}\, P\, \widehat{\beta} \cdot \alpha$ | *exporter* |
| | $\mid$ | $P\widehat{\alpha}\ [y]\ \widehat{x}Q$ | *importer* |
| | $\mid$ | $P\widehat{\alpha}\ \dagger\ \widehat{x}Q$ | *cut* |
| | $\mid$ | $x \odot P$ | *left-eraser* |
| | $\mid$ | $P \odot \alpha$ | *right-eraser* |
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- The notion of *principal name* of a term.
  1. L-principal name
  2. S–principal name

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# $^{*}\mathcal{X}$ : logical rules

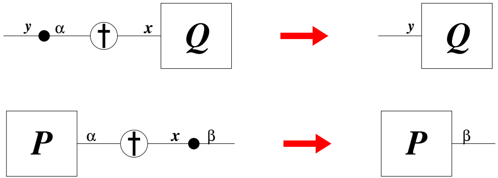
Renaming :

$$
\begin{array}{lcccc}
\text{(ren-L)} & : & \langle y.\alpha\rangle\widehat{\alpha} \dagger \widehat{x}Q & \rightarrow & Q\{y/x\} \\
\text{(ren-R)} & : & P\widehat{\alpha} \dagger \widehat{x}\langle x.\beta\rangle & \rightarrow & P\{\beta/\alpha\}
\end{array}
$$

Diagrammatically :

(ren-L) :

(ren-R) :

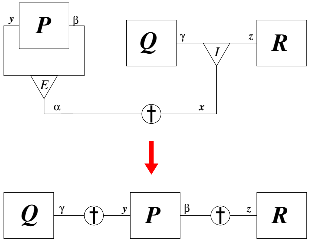


# $^*\mathcal{X}$ : logical rules

Inserting (ei-insert) :

$$(\widehat{y}\, P\, \widehat{\beta} \cdot \alpha)\widehat{\alpha} \;\dagger\; \widehat{x}(Q\widehat{\gamma}\; [x]\; \widehat{z}R) \;\rightarrow\; \text{either} \begin{cases} (Q\widehat{\gamma} \;\dagger\; \widehat{y}P)\widehat{\beta} \;\dagger\; \widehat{z}R \\ Q\widehat{\gamma} \;\dagger\; \widehat{y}(P\widehat{\beta} \;\dagger\; \widehat{z}R) \end{cases}$$

Diagrammatically :



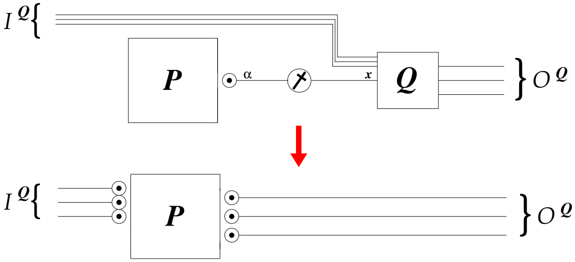

- Notice : logical rules define reducing when a cut binds L-principal names

# $^*\mathcal{X}$ : structural rules

Left erasure :

$$(\not\times\text{-eras}) \quad : \quad (P \odot \alpha)\widehat{\alpha} \not\times \widehat{x}Q \quad \rightarrow \quad \mathcal{I}^{Q} \odot \; P \; \odot \mathcal{O}^{Q},$$
$$\text{where } \; \mathcal{I}^{Q} = \overline{I}(Q) \setminus x, \;\; \mathcal{O}^{Q} = \overline{O}(Q)$$

Diagrammatically :

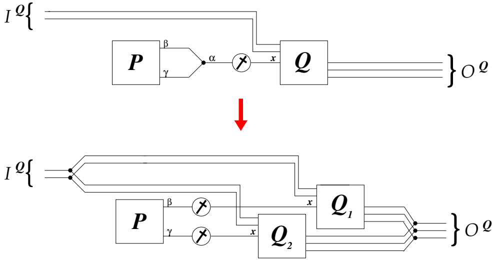


# ${}^{*}\mathcal{X}$ : structural rules

Left duplication ($\dagger$-dupl) :

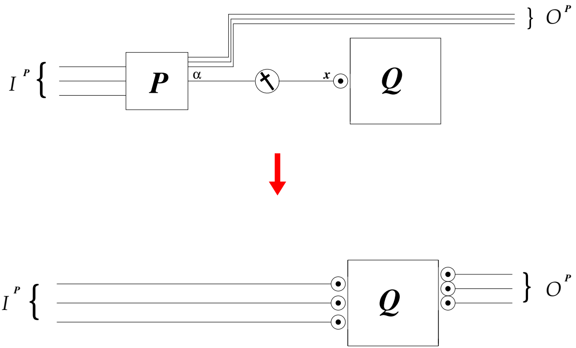
$$([P\rangle \overset{\widehat{\alpha_1}}{\underset{\widehat{\alpha_2}}{}} {>} \alpha)\widehat{\alpha} \dagger \widehat{x} Q \quad \rightarrow \quad \mathcal{I}^Q {<} \overset{\widehat{\mathcal{I}_1^Q}}{\underset{\widehat{\mathcal{I}_2^Q}}{}} \left\langle (P\widehat{\alpha_1} \dagger \widehat{x_1}\, Q_1)\widehat{\alpha_2} \dagger \widehat{x_2}\, Q_2 \right\rangle \overset{\widehat{\mathcal{O}_1^Q}}{\underset{\widehat{\mathcal{O}_2^Q}}{}} {>} \mathcal{O}^Q,$$

Diagrammatically :



# Symmetry...

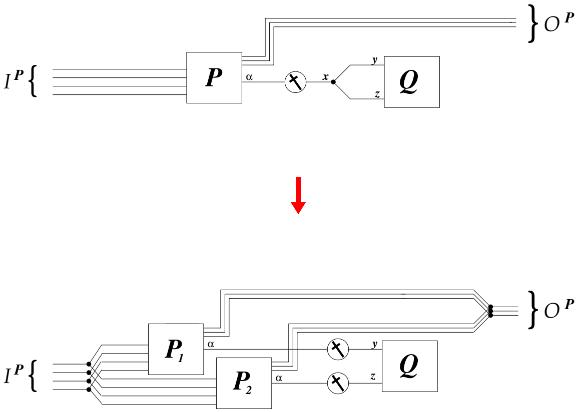
## The ⤪-erasure




- An illustration of the symmetry...

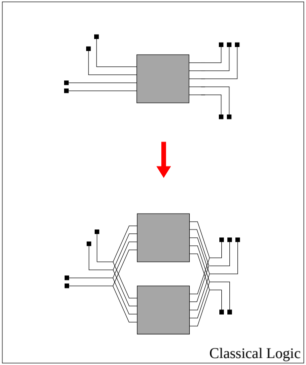
# Symmetry...

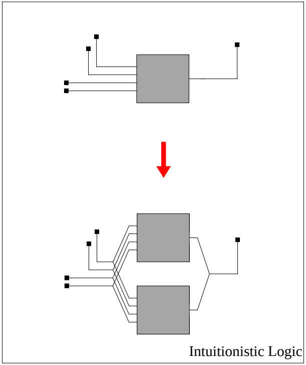
## The ⫰-duplication




- An illustration of the symmetry...

# Duplication, informally




- Component duplicated, interface preserved.
- Similarly for erasure

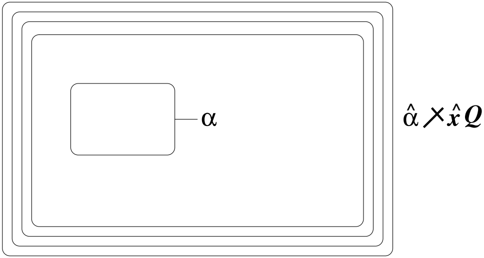
# ${}^*\mathcal{X}$ : propagation rules (left subgroup)

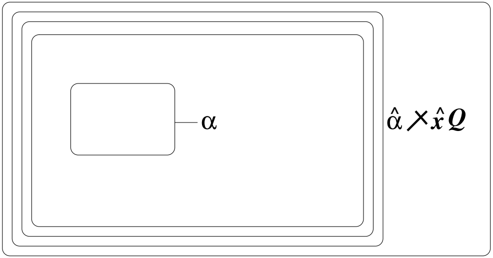
$$
\begin{array}{llll}
(exp\nearrow - prop) & : (\widehat{x}\,P\,\widehat{\gamma}\cdot\alpha)\widehat{\beta}\nearrow\widehat{y}R & \rightarrow \widehat{x}\,(P\widehat{\beta}\nearrow\widehat{y}R)\,\widehat{\gamma}\cdot\alpha & \\
(imp\nearrow - prop_1) & : (P\widehat{\alpha}\,[x]\,\widehat{z}Q)\widehat{\beta}\nearrow\widehat{y}R & \rightarrow (P\widehat{\beta}\nearrow\widehat{y}R)\widehat{\alpha}\,[x]\,\widehat{z}Q, & \beta\in O(P) \\
(imp\nearrow - prop_2) & : (P\widehat{\alpha}\,[x]\,\widehat{z}Q)\widehat{\beta}\nearrow\widehat{y}R & \rightarrow P\widehat{\alpha}\,[x]\,\widehat{z}(Q\widehat{\beta}\nearrow\widehat{y}R), & \beta\in O(Q) \\
(cut(c)\nearrow - prop) & : (P\widehat{\alpha}\dagger\widehat{x}\langle x.\beta\rangle)\widehat{\beta}\nearrow\widehat{y}R & \rightarrow P\widehat{\alpha}\dagger\widehat{y}R & \\
(cut\nearrow - prop_1) & : (P\widehat{\alpha}\dagger\widehat{x}Q)\widehat{\beta}\nearrow\widehat{y}R & \rightarrow (P\widehat{\beta}\nearrow\widehat{y}R)\widehat{\alpha}\dagger\widehat{x}Q, & \beta\in O(P),\ Q\neq\langle x.\beta\rangle \\
(cut\nearrow - prop_2) & : (P\widehat{\alpha}\dagger\widehat{x}Q)\widehat{\beta}\nearrow\widehat{y}R & \rightarrow P\widehat{\alpha}\dagger\widehat{x}(Q\widehat{\beta}\nearrow\widehat{y}R), & \beta\in O(Q),\ Q\neq\langle x.\beta\rangle \\
(L\text{–}eras\nearrow - prop) & : (x\odot M)\widehat{\beta}\nearrow\widehat{y}R & \rightarrow x\odot(M\widehat{\beta}\nearrow\widehat{y}R) & \\
(R\text{–}eras\nearrow - prop) & : (M\odot\alpha)\widehat{\beta}\nearrow\widehat{y}R & \rightarrow (M\widehat{\beta}\nearrow\widehat{y}R)\odot\alpha, & \alpha\neq\beta \\
(L\text{–}dupl\nearrow - prop) & : (x{<}^{\widehat{x_1}}_{\widehat{x_2}}\langle M])\widehat{\beta}\nearrow\widehat{y}R & \rightarrow x{<}^{\widehat{x_1}}_{\widehat{x_2}}\langle M\widehat{\beta}\nearrow\widehat{y}R] & \\
(R\text{–}dupl\nearrow - prop) & : ([M\rangle^{\widehat{\alpha_1}}_{\widehat{\alpha_2}}{>}\alpha)\widehat{\beta}\nearrow\widehat{y}R & \rightarrow [M\widehat{\beta}\nearrow\widehat{y}R\rangle^{\widehat{\alpha_1}}_{\widehat{\alpha_2}}{>}\alpha, & \alpha\neq\beta
\end{array}
$$

- Propagation rules do not have a diagrammatic representation

# The ${}^*\mathcal{X}$ calculus

## Propagation rules

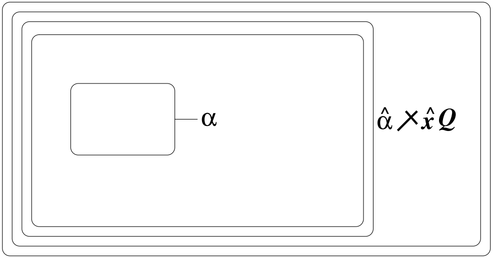




α
α̂ × x̂ Q

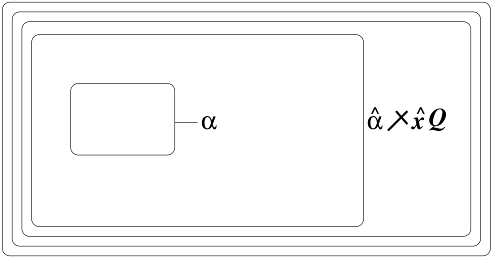
# The $^{*}\mathcal{X}$ calculus

## Propagation rules



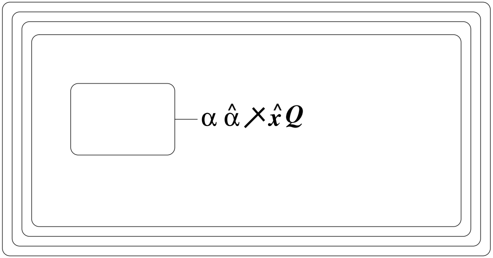
# The ${}^{*}\mathcal{X}$ calculus

## Propagation rules



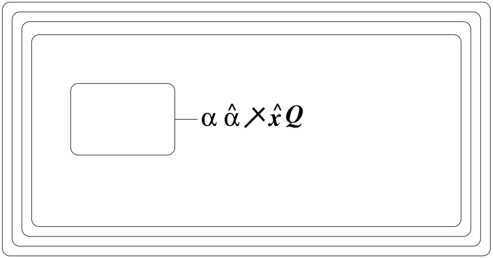
# The $^*\mathcal{X}$ calculus

## Propagation rules



# The $^*\mathcal{X}$ calculus

## Propagation rules




- We may think of $\widehat{\alpha} \not\dagger \widehat{x}Q$ as an explicit substitution

# ${}^*\mathcal{X}$ : the basic properties

1. Linearity preservation

   If $P$ is linear and $P \to Q$ then $Q$ is linear

2. Free names ("interface") preservation

   If $P \to Q$ then $N(P) = N(Q)$

3. Type preservation – computation ~~can be~~ (has to be) seen as proof-transformation

   If $P$ :· $\Gamma \vdash \Delta$ and $P \to P'$, then $P'$ :· $\Gamma \vdash \Delta$

4. Strong normalisation (termination of reduction)

# Peirce's law in $^*\mathcal{X}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\langle x.\alpha_1\rangle \;:\!\cdot\; x : A \vdash \alpha_1 : A}\,(capsule)}{\langle x.\alpha_1\rangle \odot \beta \;:\!\cdot\; x : A \vdash \alpha_1 : A, \beta : B}\,(R-eraser)}{\widehat{x}\,(\langle x.\alpha_1\rangle \odot \beta)\,\widehat{\beta}\cdot\gamma \;:\!\cdot\; \vdash \alpha_1 : A, \gamma : A \to B}\,(exporter) \quad \dfrac{}{\langle y.\alpha_2\rangle \;:\!\cdot\; y : A \vdash \alpha_2 : A}\,(capsule)}{(\widehat{x}\,(\langle x.\alpha_1\rangle \odot \beta)\,\widehat{\beta}\cdot\gamma)\,\widehat{\gamma}\,[z]\,\widehat{y}\,\langle y.\alpha_2\rangle \;:\!\cdot\; z : (A \to B) \to A \vdash \alpha_1 : A, \alpha_2 : A}\,(importer)}{[(\widehat{x}\,(\langle x.\alpha_1\rangle \odot \beta)\,\widehat{\beta}\cdot\gamma)\,\widehat{\gamma}\,[z]\,\widehat{y}\,\langle y.\alpha_2\rangle\rangle \frac{\widehat{\alpha_1}}{\widehat{\alpha_2}} > \alpha \;:\!\cdot\; z : (A \to B) \to A \vdash \alpha : A}\,(R-duplicator)}{\widehat{z}\,([(\widehat{x}\,(\langle x.\alpha_1\rangle \odot \beta)\,\widehat{\beta}\cdot\gamma)\,\widehat{\gamma}\,[z]\,\widehat{y}\,\langle y.\alpha_2\rangle\rangle \frac{\widehat{\alpha_1}}{\widehat{\alpha_2}} > \alpha)\,\widehat{\alpha}\cdot\delta \;:\!\cdot\; \vdash \delta : ((A \to B) \to A) \to A}\,(exporter)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(axiom)}{A \vdash A, B}\,(R-weakening)}{\vdash A, A \to B}\,(\to R) \quad \dfrac{}{A \vdash A}\,(axiom)}{(A \to B) \to A \vdash A, A}\,(\to L)}{(A \to B) \to A \vdash A}\,(R-contraction)}{\vdash ((A \to B) \to A) \to A}\,(\to R)$$

***X :**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(axiom)}{A \vdash A, B}\,(R - weakening)}{\vdash A, A \to B}\,(\to R) \qquad \dfrac{}{A \vdash A}\,(axiom)}{(A \to B) \to A \vdash A, A}\,(\to L)}{(A \to B) \to A \vdash A}\,(R - contraction)}{\vdash ((A \to B) \to A) \to A}\,(\to R)$$

**X :**

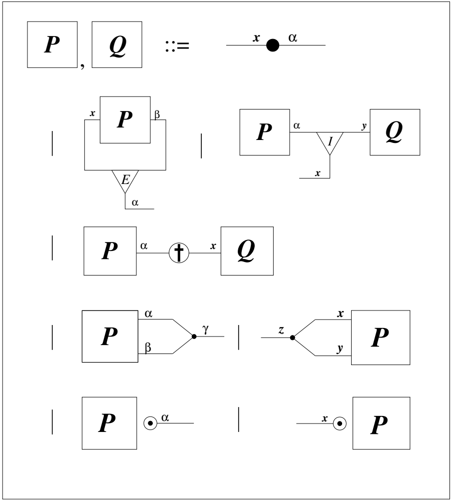
$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A, B}\,(axiom)}{\vdash A, A \to B}\,(\to R) \qquad \dfrac{}{A \vdash A}\,(axiom)}{(A \to B) \to A \vdash A}\,(\to L)}{\vdash ((A \to B) \to A) \to A}\,(\to R)$$

The terms of ${}^*\mathcal{X}$ are convenient for two-dimensional representation, due to

- the presence of erasers and duplicators
- but also the linearity constraints

# Diagrammatic calculus

## Syntax



# Outline

# Encoding $^*\mathcal{X}$ in $\mathcal{X}$ calculus

**Definition 39 (Encoding $^*\mathcal{X}$ into $\mathcal{X}$).** The encoding of $^*\mathcal{X}$-terms in $\mathcal{X}$ calculus is defined inductively as shown by Figure 6.2.

$$
\begin{array}{rcl}
\llbracket \langle x.\alpha\rangle \rrbracket^{\mathcal{X}} & := & \langle x.\alpha\rangle \\
\llbracket \widehat{x}\, P\, \widehat{\beta}\cdot\alpha \rrbracket^{\mathcal{X}} & := & \widehat{x}\, \llbracket P \rrbracket^{\mathcal{X}}\, \widehat{\beta}\cdot\alpha \\
\llbracket P\,\widehat{\alpha}\,[x]\,\widehat{y}\,Q \rrbracket^{\mathcal{X}} & := & \llbracket P \rrbracket^{\mathcal{X}}\,\widehat{\alpha}\,[x]\,\widehat{y}\,\llbracket Q \rrbracket^{\mathcal{X}} \\
\llbracket P\widehat{\alpha} \dagger \widehat{x}Q \rrbracket^{\mathcal{X}} & := & \llbracket P \rrbracket^{\mathcal{X}}\widehat{\alpha} \dagger \widehat{x}\llbracket Q \rrbracket^{\mathcal{X}} \\
\llbracket x{<}^{\widehat{y}}_{\widehat{z}}\langle P] \rrbracket^{\mathcal{X}} & := & \llbracket P \rrbracket^{\mathcal{X}}\{x/y\}\{x/z\} \\
\llbracket [P\rangle^{\widehat{\beta}}_{\widehat{\gamma}}{>}\alpha \rrbracket^{\mathcal{X}} & := & \llbracket P \rrbracket^{\mathcal{X}}\{\alpha/\beta\}\{\alpha/\gamma\} \\
\llbracket x \odot P \rrbracket^{\mathcal{X}} & := & \llbracket P \rrbracket^{\mathcal{X}} \\
\llbracket P \odot \alpha \rrbracket^{\mathcal{X}} & := & \llbracket P \rrbracket^{\mathcal{X}}
\end{array}
$$

Figure 18: Encoding the $^*\mathcal{X}$-terms into $\mathcal{X}$

# Encoding preserves types

**Lemma 43 (Preservation of types).** *For an arbitrary* $^*\mathcal{X}$*-term* $P$ *such that* $P :\cdot\ \Gamma \vdash \Delta$*, it stands*

$$[\![P]\!]^{\mathcal{X}} :\cdot\ \Gamma \vdash \Delta$$

# Simulating $^*\mathcal{X}$ reduction

**Theorem 42 (Simulating $^*\mathcal{X}$-reduction).** *Let $P$ and $P'$ be $\mathcal{X}$-terms. Then the following holds:*

$$\textit{If } P \xrightarrow{^*\mathcal{X}} P' \textit{ then } [\![P]\!]^{\mathcal{X}} \xrightarrow{\mathcal{X}}{}^{+} [\![P']\!]^{\mathcal{X}}$$

PROOF. The proof goes by inspecting the reduction rules and by induction on the structure of terms. We provide the proof for several reduction rules.

# Strong normalisation of ${}^*\mathcal{X}$-calculus

**Theorem 44 (Strong Normalisation).** *The reduction system of ${}^*\mathcal{X}$ is strongly normalising on simply-typed terms.*

PROOF. Let $P :\cdot\ \Gamma \vdash \Delta$. Assume that $P$ is not strongly normalising, which means that there is an infinite reduction starting with $P$

$$P \xrightarrow{{}^*\mathcal{X}} P_1 \xrightarrow{{}^*\mathcal{X}} \ldots \xrightarrow{{}^*\mathcal{X}} P_n \xrightarrow{{}^*\mathcal{X}} \ldots$$

then by Theorem 42,

$$\llbracket P \rrbracket^{\mathcal{X}} \xrightarrow{\mathcal{X}}{}^{+} \llbracket P_1 \rrbracket^{\mathcal{X}} \xrightarrow{\mathcal{X}}{}^{+} \ldots \xrightarrow{\mathcal{X}}{}^{+} \llbracket P_n \rrbracket^{\mathcal{X}} \xrightarrow{\mathcal{X}}{}^{+} \ldots$$

On the other hand according to Lemma 43,

$$\llbracket P \rrbracket^{\mathcal{X}} :\cdot\ \Gamma \vdash \Delta$$

and the fact that $\mathcal{X}$ calculus is strongly normalising on typed terms ([36]), we conclude that $\llbracket P \rrbracket^{\mathcal{X}}$ is strongly normalising, which contradicts the assumption. Hence, $P$ is strongly normalising. $\square$

# Future work

- define equivalent terms, the $^{©}\mathcal{X}$ calculus
- concurrency - which process calculus corresponds to this classical cut-elimination ?
- a 3D computational model...

# Some references


- ◇ S. Ghilezan, P. Lescanne, D. Žunić
  Comp. interpretation of classical logic with expl. struct. rules, draft.
- ◇ S. Ghilezan, J. Ivetic, P. Lescanne, D. Žunić
  Intuitionistic sequent-style calculus with explicit struct. rules, 2010.
- ◇ P. L. Curien, H. Herbelin
  The duality of computation, 2000.
- ◇ D. Kesner and S. Lengrand
  Explicit operators for $\lambda$-calculus, 2006.
- ◇ C. Urban and G. M. Bierman
  Strong normalisation of cut-elimination in classical logic, 1999.
- ◇ P. Wadler
  Propositions as sessions, 2012.
- ◇ E. Robinson
  Proof nets for classical logic, 2005.

Thanks..! 

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