A probability logic for reasoning about quantum observations

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Outline

Quantum mechanics,quantum states

- Quantum mechanics is a fundamental branch of physics concerned with processes involving, for example, atoms and photons. Systems such as these which obey quantum mechanics can be in a quantum superposition of different states, unlike in classical physics.
- Quantum state can be considered as a maximal piece of information about quantum systems. - In classical mechanics state of a single particle can be represented as a $(r_1, \ldots, r_6)$ where $r_1, \ldots, r_3$, correspond to position and $r_4, \ldots, r_6$, correspond to momentum.
- The key idea is that wave function represent a quantum state of a particle.
- Quantum state describes potential outcomes of some measurement (position, momentum, energy...) so quantum state is a linear combination of all other quantum states that can be obtained as a result of a measurement.

Quantum mechanics,quantum states-Example

- Consider a subatomic particle on a line and suppose that it can only be detected in equally spaced points $\{x_1, \ldots, x_n\}$.
- In Quantum mechanics we use Dirac ket notation: x_i is denoted by $|x_i\rangle$.
- Let $|\psi\rangle$ be an arbitrary state. $|\psi\rangle = c_1|x_1\rangle + \ldots + c_n|x_n\rangle$, $c_i \in \mathbf{C}$, $\sum_{i=1}^{n} \|c_i\|^2 = 1$.
- $\|c_i\|^2$ represents a probability that after a measuring a particle will be found in a point x_i.
- After observing particle at a point x_i the state $|\psi\rangle$ "collapses"-passes to the state $|x_i\rangle$.

Why quantum logic?

- Logics for reasoning about real valued probability ($[0, 1]$).
- Logics for reasoning about p-adic valued probability.
- Logics for reasoning about complex valued probability. Complex numbers appear as quantum amplitudes.

Ortholattice(Orthomodularlattice)

Ortholattice is a structure $\mathbf{B} = (B, \sqsubseteq, ', 1, 0)$ where

- $\sqsubseteq$ is partial order;
- any pair of element a and b has an infimum $a \sqcap b$ and supremum $a \sqcup b$ such that:
  1. $a \sqcap b \sqsubseteq a, b$ and for every c: if $c \sqsubseteq a, b$ then $c \sqsubseteq a \sqcap b$;
  2. $a, b \sqsubseteq a \sqcup b$ and for every c: if $a, b \sqsubseteq c$ then $a \sqcup b \sqsubseteq c$;
  3. for every a, $0 \sqsubseteq a$, $a \sqsubseteq 1$.
- $'$ satisfied the following:
  1. $a'' = a$;
  2. if $a \sqsubseteq b$ then $b' \sqsubseteq a'$;
  3. $a \sqcap a' = 0$.

Orthomodularlattice is Ortholattice that satisfied orthomodular property: for every $a, b \in B$: $a \sqcap (a' \sqcup (a \sqcap b)) \sqsubseteq b$

Two approaches

- Minimal quantum logic OL "represented by Ortholattice"
- Quantum logic OQL "represented by Orthomodularlattice"

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Closed subsets of Hilbert space form Orthomodularlattice.

- Algebraic realization for OL is a pair $\mathcal{A} = \langle \mathcal{B}, v \rangle$ where
  - $\mathcal{B} = (B, \subseteq, ^c, 0, 1)$ is ortholattice;
  - $v : For_{OL} \rightarrow B$

  $\models_{\mathcal{A}} \alpha$ iff $v(\alpha) = 1$

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- Kripke realization for OL is a system $\mathcal{K} = \langle W, R, \Pi, \rho \rangle$ where
  - W is nonempty set of worlds;
  - R is reflexive and symmetric binary relation on W; $w \perp u$ denotes: not wRu.
  - $\Pi = \{X \subset W \mid (X^{\perp})^{\perp} = X\}$ such that $\emptyset, W \in \Pi$ and Π is closed for intersection and orthocomplement $\perp$.($X^{\perp} = \{u \in W \mid$ for every $w \in X, w \perp u\}$)
  - $\rho : For_{OL} \to \Pi$: $\rho(\neg\beta) = (\rho(\beta))^{\perp}$, $\rho(\alpha \wedge \beta) = \rho(\alpha) \cap \rho(\beta)$;

  $w \models \alpha$ iff $w \in \rho(\alpha)$; $\models_{\mathcal{K}} \alpha$ iff $\rho(\alpha) = W$.

(M.L.D. Chiara, R. Giuntini)

Theorem

- *For any algebraic realization* $\mathcal{A}$ *there is a Kripke realization* $\mathcal{K}^{\mathcal{A}}$ *such that for every* α, $\models_{\mathcal{A}} \alpha$ *iff* $\models_{\mathcal{K}} \alpha$;
- *For any Kripke realization* $\mathcal{K}$ *there is an algebraic realization* $\mathcal{A}^{\mathcal{K}}$ *such that for every* α, $\models_{\mathcal{K}} \alpha$ *iff* $\models_{\mathcal{A}} \alpha$;

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Theorem

- *If $\mathcal{A}$ is orthomodular then $\mathcal{K}^{\mathcal{A}}$ is orthomodular.*
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Motivation

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- Simon Kramer gave meaning for use modal operator in quantum mechanics, emphasizing the difference between notions: "something is true " and "something is observed to be true ".
- He used the fact that $\Box$ do not distribute over $\vee$ in both directions and he suggests that using formulas $\Box$ with the intended meaning "p is observed to be true" is enough to capture subjectivity of observations in quantum experiment, so introducing non-distributive logic is not necessary.

Motivation

A half spin particle is prepared in eigenstate $|z^+\rangle$. Consider the following sentences:

- p: "The spin component in the z direction is up "
- q: "The spin component in the x direction is up "
- r: "The spin component in the x direction is down "

Below is a list of statement that lead to the conclusion that the quantum logic is not distributive:

1. If we measure the spin component in the x direction we obtain value up or value down. Therefore $q \vee r$ is true.
2. Since p is true and $q \vee r$ is true, $p \wedge (q \vee r)$ is true .
3. According to the uncertainty principle, the spin component in the z direction and the spin component in the x direction cannot be simultaneously well defined, therefore $p \wedge q$ is false and $p \wedge r$ is false and hence $(p \wedge r) \vee (p \wedge r)$ is false.

Comparing 2 and 3 we conclude that distributivity low is violated.

Motivation

- We know that p is true (before measurement is made) because the system is prepared in that state, so writing the formula $p \wedge (q \vee r)$, i.e. putting p and $(q \vee r)$ in the same sentence we say that before measurement particle posses either value *up* either value *down*.

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- The wave interpretation reject the assumption that particle posses any concrete value before the measurement is made.
- According to this interpretation $q \vee r$ is false, so the distributivity low is not violated.

Motivation

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- According this interpretation a single particle possesses simultaneously well defined values for spin components along x and z axes and the uncertainty principle is just consequence of impossibility of preparing identical microscopic states.

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- According this interpretation a single particle possesses simultaneously well defined values for spin components along x and z axes and the uncertainty principle is just consequence of impossibility of preparing identical microscopic states.
- Therefore, according this interpretations $(p \wedge r) \vee (p \wedge r)$ is true so the distributivity low is not violated.

Motivation

- Thus, ensemble interpretation is not strictly related to the notion of measurement, it propagates the very existence, while wave interpretation not recognize any value until they are measured.

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- Thus, ensemble interpretation is not strictly related to the notion of measurement, it propagates the very existence, while wave interpretation not recognize any value until they are measured.
- Therefore, logic for reasoning about quantum mechanics should make a distinction between these things. Let q means :" The spin component in the x direction is up" while $\Box q$ means :" It is measured that the spin component in the x direction is up" (similarly for p and r).

Motivation

Formula / interpretation	$\Box p \wedge \Box(q \vee r)$	$\Box p \wedge (\Box q \vee \Box r)$
wave	$\top$	$\bot$
corpuscular	/	/

$\Box(p \wedge q) \vee \Box(q \wedge r)$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
$\bot$	$\bot$	$\bot$
/	$\top$	$\top$

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- We express ontological possessed values with classical formulas and since from the point of view of corpuscular interpretation distributivity low is not violated, the laws of classical logic are suitable for this fragment;

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- We express ontological possessed values with classical formulas and since from the point of view of corpuscular interpretation distributivity low is not violated, the laws of classical logic are suitable for this fragment;
- We express the notion of measurement by modal operator and hence using formulas of the form $\Box\varphi$ we capture wave interpretation. Since $\Box$ not distributes over $\vee$ "it already catches non-distributity", so we do not need non-distributive structure (like non-distributive lattice) and we are satisfied with Boolean logic with some modal laws.
- We only need axiom of classical logic plus three modal lows instead of many axioms or rules which are normally used in quantum logics.

Formal language

- $\mathbf{Q}$ is the set of rational numbers;
- $\mathbf{Q}^+$ is the set of positive rational numbers;
- $\mathbf{C}$ is the set of complex numbers;
- $\|\cdot\|$ is the complex norm, $\|x + i\cdot y\| = \sqrt{x^2 + y^2}$.

and introduce the following sets:

- $\sqrt{\mathbf{Q}} = \{\pm\sqrt{q} \mid q \in \mathbf{Q}\}$
- $\mathbf{C}_{\sqrt{\mathbf{Q}}} = \{a + i\cdot b, a, b \in \sqrt{Q}\}$;
- $\mathbf{C_1} = \{a + i\cdot b, a, b \in \mathbf{C}, \sqrt{a^2 + b^2} \leq 1\}$;
- $\mathbf{C}_{\mathbf{1},\sqrt{\mathbf{Q}}} = \{a + i\cdot b, a, b \in \sqrt{\mathbf{Q}}, \sqrt{a^2 + b^2} \leq 1\}$;
- $\mathbf{C}_{\mathbf{1}}^{*} = \{(z_1, z_2, \ldots, z_k) \mid z_i \in \mathbf{C}, \sum_{i=1}^{k} \|z_i\|^2 \leq 1, k \leq n\}$;
- $\mathbf{C}_{\mathbf{1},\sqrt{\mathbf{Q}}}^{*} = \{(z_1, z_2, \ldots, z_k) \mid z_i \in \mathbf{C}_{\mathbf{1},\sqrt{\mathbf{Q}}}, \sum_{i=1}^{k} \|z_i\|^2 \leq 1, k \leq n\}$;

Formal language

- countable set $Var = \{p, q, r, \ldots\}$ of propositional letters, classical connectives $\neg$ and $\wedge$, modal operator $\Box$ and $CS_{z_1,\rho_1;z_2,\rho_2;\ldots z_m,\rho_m}$ where $(z_1, z_2, \ldots, z_m) \in \mathbf{C}^*_{\mathbf{1},\sqrt{\mathbf{Q}}}, \rho_i \in \mathbf{Q}^+, m \leq n$.
  The set of classical propositional formulas over Var will be denoted by For_{Cl}.

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  The set of classical propositional formulas over Var will be denoted by For_{Cl}.
- The set For_{mod} of classical-modal formulas is built over the set Var using connectives $\neg$ and $\wedge$ in the standard manner: if $p \in Var$ then $p \in For_{mod}$; if $\alpha, \beta \in For_{mod}$ then $\neg\alpha, \alpha \wedge \beta, \Box\alpha \in For_{mod}$.

Formal language

- ^{c} and $\cap$ to denote set theoretic relative complement and $\cap$ set theoretic intersection.
- R is binary relation defined on some non-empty set W.
- For $i, j \in W$, not Rij will be denoted by $i \perp j$.
- For any $X \subseteq W$ we use the following sets:
    - the *orthocomplement* $X^{\perp}$ defined as $X^{\perp} = \{i \mid \forall j(j \in X \hookrightarrow i \perp j)\}$
    - the set $\boxdot X = \{i \mid \forall j(Rij \to j \in X)\}$ which corresponds to the modal operator $\Box$ (this correspondences will be clear later). $\overline{\Diamond} X = \{i \mid \exists j(Rij \text{ and } j \in X)\}$ which corresponds to the modal operator $\Diamond$

Formal language

Note that

- $\boxdot X = \{i \mid \forall j(Rij \to j \in X)\} = \{i \mid \forall j(j \in X^c \to i \perp j)\} = (X^c)^\perp$.
- $\overline{\Diamond} X = (X^\perp)^c$
- We introduce addition of the sets of the form $\boxdot\overline{\Diamond}A$, $A \subseteq W$ in the following way:

$$\boxdot\overline{\Diamond}A + \boxdot\overline{\Diamond}B = \boxdot\overline{\Diamond}(A \cup B)$$

.

B-Semantics

Definition

An $QL_{prob,B}$-model is a structure $M_{\mathbf{B}} = \langle W, R, H, v \rangle$ where:

1. W is a nonempty set of elements called worlds;
2. R is a reflexive and symmetric binary relation.
3. H is a subset of the power-set of W such that:
   1. $\emptyset, W \in H$;
   2. H is closed under the set theoretic relative complement ^{c}, set theoretic intersection $\cap$ and the operator $\boxdot$ (if $X \in H$ then $\boxdot X \in H$).
4. $v : For_{mod} \to H$ is a function which associated with every formula an element from H such that:
   1. $v(\neg\alpha) = v(\alpha)^c$;
   2. $v(\alpha \wedge \beta) = v(\alpha) \cap v(\beta)$;
   3. $v(\Box\alpha) = \boxdot v(\alpha)$

Semantics

"Problem" in arbitrary **B** model sets of the form $v(\alpha)$ and $v(\Box\alpha)$ do not correspond to a closed subsets of a Hilbert space while sets $v(\Box\Diamond\alpha)$ do. We want to measure sets of the form $v(\Box\Diamond\alpha)$.

B-Semantics

Lemma

1. $v(\Box\Diamond(\alpha \vee \beta)) = \boxdot\overline{\Diamond}v(\alpha) + \boxdot\overline{\Diamond}v(\beta)$;
2. $v(\Box\Diamond\alpha \vee \Box\Diamond\beta) = \boxdot\overline{\Diamond}v(\alpha) \cup \boxdot\overline{\Diamond}v(\beta)$;
3. $v(\Box\Diamond(\alpha \vee \beta)) \neq v(\Box\Diamond\alpha \vee \Box\Diamond\beta)$;
4. $v(\Box\Diamond\neg\alpha) = (v(\Box\alpha))^{\perp}$.

B-Satisfiability

Definition

In the model $M_{\mathbf{B}} = \langle W, R, H, v \rangle$ the satisfiability relation is defined for For_{mod} formulas in the following way: If $\alpha \in For_{mod}, w \in W$ then

- $M_{\mathbf{B}}, w \models \alpha$ iff $w \in v(\alpha)$;

Definition

Let the model $M_{\mathbf{B}}$ be defined as above. Then for every formula $\alpha \in For_{Cl}$ we introduce integer $K(\Box\Diamond\alpha)$ such that:

1. $1 \leq K(\Box\Diamond\alpha) \leq n$;
2. if for every $w \in W$, $M_{\mathbf{B}}, w \models \Box\Diamond\alpha \Leftrightarrow \Box\Diamond\beta$ then $K(\Box\Diamond\alpha) = K(\Box\Diamond\beta)$;

The intended meaning of $K(\Box\overline{\Diamond}\alpha)$ is dimension of subspace related to measuring α.

Pre-Probabilistic formulas

The set For_{PProb} of *pre-probabilistic* formulas is defined as the least set that satisfies the following:

- If $\alpha \in For_{Cl}$, $m = K(\Box\alpha)$, $(z_1, z_2, \ldots, z_m) \in \mathbf{C}^*_{\mathbf{1},\sqrt{\mathbf{Q}}} =$, $\rho_i \in \mathbf{Q}^+$, $i = \overline{1, m}$, then $CS_{z_1,\rho_1;z_2,\rho_2;\ldots z_m,\rho_m}\Box\Diamond\alpha$ is pre-probabilistic formula.
- if φ and ψ are pre-probabilistic formulas then $\neg\varphi$ and $\varphi \wedge \psi$ are pre-probabilistic formulas.

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- if φ and ψ are pre-probabilistic formulas then $\neg\varphi$ and $\varphi \wedge \psi$ are pre-probabilistic formulas.
- The set For_{Qv} of all QL_{prob}-formulas is a union of For_{mod} and For_{PProb}.

Pre-Probabilistic formulas

Pre-Probabilistic formulas have the following meaning:

- Let A be a subspace related to measuring α and $w_1, \ldots, w_m$ be the chosen base of eigenvectors. The meaning of the formula $CS_{z_1,\rho_1;z_2,\rho_2;\ldots z_m,\rho_m}\Box\Diamond\alpha$ is always related to some world (vector) w and it is the following: $w = c_1 \cdot w_1 +, \ldots, +c_m \cdot w_m$ for some $c_i \in \mathbf{C}$ such that $\|c_1 - z_1\| \leq \rho_1$, $\ldots \|c_m - z_m\| \leq \rho_m$ and hence the probability of measuring α in a state w is equal $\|c_1\|^2 + \ldots + \|c_m\|^2$.

Semantics

Definition

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Definition

Given a model $M_{\mathbf{B}} = \langle W, R, H, v \rangle$ we set

$$\boxminus H = v(\Box\Diamond\alpha)\ ,\ \alpha \in For_{mod}.$$

Let $\mathbf{B}^* = \langle W, R, \boxminus H, v \rangle$

Theorem

- *For any* $\mathbf{B}^* = \langle W, R, \boxminus H, v \rangle$ *there is a Kripke realization* $\mathcal{K}^{\mathcal{B}^*}$ *for OQL such that for every* α, $\models_{\mathcal{K}^{\mathcal{B}^*}} \alpha$ *iff* $\models_{\mathbf{B}^*} \Box\Diamond\alpha$;
- *For any Kripke realization* $\mathcal{K}$ *for OQL there is an model* $\mathbf{B}^{*K} = \langle W, R, \boxminus H, v \rangle$ *such that for every* α, $\models_{\mathcal{K}} \alpha$ *iff* $\models_{\mathbf{B}^{*K}} \Box\Diamond\alpha$;

Semantics

Definition

An QL_{prob}-model is a structure $M = \langle W, R, H, \rho, \upsilon \rangle$ where:

1. $\langle W, R, H, \upsilon \rangle$ is $M_{\mathbf{B}}$ model.
2. $\rho : W \times \boxminus H \to \mathbf{C}_{\mathbf{1}}^{*}$ is a function that satisfies the following:
   1. for every $w \in W$, $\rho(w, \emptyset) = (\underbrace{0, 0; \ldots, 0, 0}_{n-tuples})$;
   2. for every $A \subseteq \boxminus H$ if $w \in A$, and $\rho(w, A) = (c_1, c_2, \ldots, c_k)$, then $\sum_{i=1}^{k} ||c_i||^2 = 1$
   3. if $\upsilon(\Box\Diamond\beta_1), \ldots, \upsilon(\Box\Diamond\beta_l) \in \boxminus H$, $\rho(w, \upsilon(\Box\Diamond\beta_i)) = (a_{i1}, a_{i2}, \ldots, a_{im_i})$, $\upsilon(\Box\Diamond\beta_i) \perp \upsilon(\Box\Diamond\beta_j)$, for every $i, j = \overline{1, l}$ $\rho(w, \upsilon(\Box\Diamond\beta_1) + \ldots + \upsilon(\Box\Diamond\beta_l)) = (d_1, \ldots, d_f)$ then

$$\sum_{i=1}^{f} ||d_i||^2 = \sum_{i=1}^{l} \sum_{j=1}^{m_i} ||a_{ij}||^2$$

Satisfiability

Definition

Let $M = \langle W, R, H, \rho, v \rangle$ be an QL_{prob}-model. The satisfiability relation is inductively defined as follows:

- If $\alpha \in For_{mod}$ then the satisfiability is defined as above.
- If $\alpha \in For_{Cl}$, $(z_1, z_2, \ldots, z_k) \in \mathbf{C}^*_{1,\sqrt{\mathbf{Q}}}$, $\rho_i \in \mathbf{Q}^+$, $i = \overline{1, k}$, $k \leq n$
  then
  $$M, w \models CS_{z_1,\rho_1;z_2,\rho_2;\ldots z_k,\rho_k} \Box\Diamond\alpha \text{ iff}$$
  $$\rho(w, v(\Box\Diamond\alpha)) = (c_1, c_2, \ldots, c_k) \text{ such that } \|c_1 - z_1\| \leq \rho_1, \|c_2 - z_2\| \leq \rho_2, \ldots \|c_k - z_k\| \leq \rho_k.$$
- If $\varphi \in For_{PProb}$, then $M, w \models \neg\varphi$ iff it is not $M, w \models \varphi$.
- If $\varphi, \psi \in For_{PProb}$ then $M, w \models \varphi \wedge \psi$ iff $M, w \models \varphi$ and $M, w \models \psi$. $\Box$

Axiomatization

The axiomatic system $AX_{L_{QM}}$ consist of:

AQLm1. Substitutional instances of classical tautologies;

AQLm2. $\Box(\alpha \Rightarrow \beta) \Rightarrow (\Box\alpha \Rightarrow \Box\beta)$;

AQLm3. $\Box\alpha \Rightarrow \alpha$;

AQLm4. $\alpha \Rightarrow \Box\Diamond\alpha$;

RQLm1. from α infer $\Box\alpha$;

RQLm2. from α and $\alpha \Rightarrow \beta$ infer β;

AQLprob1. $CS_{z_1,\rho_1;,\dots z_i,\rho_i;,\dots z_m,\rho_m}\Box\Diamond\alpha \Rightarrow CS_{z_1,\rho_1;,\dots z_i,\rho'_i;,\dots z_m,\rho_m}\Box\Diamond\alpha$ if $\rho'_i \geq \rho$;

AQLprob2. $CS_{z_1,\rho_1;,\dots z_i,\rho_i;,\dots z_m,\rho_m}\Box\Diamond\alpha \Rightarrow \neg CS_{z_1,\rho_1;,\dots z'_i,\rho'_i;,\dots z_m,\rho_m}\Box\Diamond\alpha$ if $\|z_i - z'_i\| > \rho_i + \rho'_i$;

AQLprob3. $CS_{z_1,\rho_1;,\dots z_i,\rho_i;,\dots z_m,\rho_m}\Box\Diamond\alpha \Rightarrow CS_{z_1,\rho_1;,\dots z'_i,\rho_i+\rho'_i;,\dots z_m,\rho_m}\Box\Diamond\alpha$ if $\|z_i - z'_i\| \leq \rho'_i$;

Inference rules

RQL1. From A and $A \Rightarrow B$ infer B where $A, B \in For_{PProb}$;

RQL2. If $\alpha \in For_{Cl}$, $m = K(\Box\Diamond\alpha)$, $(z_1, z_2, \ldots, z_m) \in \mathbf{C}^*_{\mathbf{1},\sqrt{\mathbf{Q}}}$, $k \in \mathbf{N}$, $\sum_{i=1}^{m} \|z_i\|^2 > \frac{m}{k^2} + \frac{2m}{k} + 1$ or $\sum_{i=1}^{m} \|z_i\|^2 + \frac{2}{k} \sum_{i=1}^{m} \|z_i\| + \frac{m}{k^2} < 1$, then from $\Box\Diamond\alpha$ infer $CS_{z_1,\frac{1}{k};z_2,\frac{1}{k};,\ldots z_m,\frac{1}{k}}\Box\Diamond\alpha \Rightarrow \bot$;

RQL3. If $\alpha \in For_{Cl}$, $k \in \mathbf{N}$, from $\varphi \Rightarrow \neg CS_{z_1,\frac{1}{k};,z_2,\frac{1}{k};,\ldots z_{K(\Box\Diamond\alpha)},\frac{1}{k}}\Box\Diamond\alpha$ for every $(z_1, z_2, \ldots, z_{K(\Box\Diamond\alpha)}) \in \mathbf{C}^*_{\mathbf{1},\sqrt{\mathbf{Q}}}$, infer $\varphi \Rightarrow \bot$;

RQL4. From $\Box\Diamond\neg\alpha$ infer $CS_{\underbrace{0,\,0;\,0,\,0;\,,\ldots 0,\,0}_{K(\Box\Diamond\alpha)-tuples}}\Box\Diamond\alpha$;

RQL5. If $\alpha \in For_{Cl}$, $m = K(\Box\Diamond\alpha)$, $(z_1, z_2, \ldots, z_m) \in \mathbf{C}^*_{\mathbf{1},\sqrt{\mathbf{Q}}}$, $\rho_i \in \mathbf{Q}^+$ from $\varphi \Rightarrow CS_{z_1,\rho_1;,\ldots z_j,\rho_j+\frac{1}{k};,\ldots z_m,\rho_m}\Box\Diamond\alpha$ for every $k \in \mathbf{N}$, infer $\varphi \Rightarrow CS_{z_1,\rho_1;,\ldots z_j,\rho_j;,\ldots z_m,\rho_m}\Box\Diamond\alpha$;

Inference rules

RQL6. from $\Box\Diamond\alpha \Leftrightarrow \Box\Diamond\beta$ infer
$CS_{z_1,\rho_1;z_2,\rho_2;,\ldots z_m,\rho_m}\Box\Diamond\alpha \Leftrightarrow CS_{z_1,\rho_1;z_2,\rho_2;,\ldots z_m,\rho_m}\Box\Diamond\beta$.

RQL7. Let $(z_{i1},\ldots,z_{im_i})$, $i=\overline{1,s}$, $(z_1,\ldots,z_f)\in \mathbf{C}^*_{\mathbf{1},\sqrt{\mathbf{Q}}}$, $\frac{2}{l}<\|z_{ij}\|,\|z_u\|$ $i=\overline{1,s}, j=\overline{1,m_i}$, $u=\overline{1,f}$. If the following inequality does not hold:
$\sum_{i=1}^{f}\|z_i\|^2-\frac{2(m_1+\ldots+m_s+f)}{l}+\frac{m_1+\ldots+m_s-f}{l^2}\le$
$\sum_{i=1}^{s}\sum_{j=1}^{m_i}\|z_{ij}\|^2\le\sum_{i=1}^{f}\|z_i\|^2+\frac{2(m_1+\ldots+m_s+f)}{l}+\frac{m_1+\ldots+m_s-f}{l^2}$
then from

$$\varphi\Rightarrow\bigwedge_{i=1}^{s}CS_{z_{i1},\frac{1}{l};,\ldots,\ldots,z_{im_i},\frac{1}{l}}\Box\Diamond\alpha_i\wedge\bigwedge_{i,j=1}^{s}\Box\Diamond\alpha_i\Rightarrow$$
$$\Box\Diamond\neg\Diamond\alpha_j\wedge CS_{z_1,\frac{1}{l};,\ldots,\ldots,z_f,\frac{1}{l}}\Box\Diamond(\alpha_1\vee\ldots\vee\alpha_s)$$

infer $\varphi\Rightarrow\bot$;

Completeness

Theorem

Every consistent set of formulas T can be extended to a maximal consistent set.

Completeness

- Let $T_1 \subseteq For_{Qv}$ and T_1^* maximal consistent set obtained from T_1.
- T_1^* has the next property: For every formula $\alpha \in For_{Cl}$ and every $k \in \mathbf{N}$ there is at least one $(z_1, z_2, \ldots, z_m) \in \mathbf{C}^*_{1,\sqrt{\mathbf{Q}}}$ where $m = K(\Box\Diamond\alpha)$ such that $CS_{z_1,\frac{1}{k};,\ldots z_j,\frac{1}{k};,\ldots z_m,\frac{1}{k}}\Box\Diamond\alpha \in T_1^*$.
- For each formula $\alpha \in For_{Cl}$ we make a sequence of m-tuples $Z(\Box\Diamond\alpha) = (Z^k)_{k\in\mathbf{N}}$, $Z^k = (z_1^k, z_2^k, \ldots, z_m^k)$ in the following way:
  - For every $k \in \mathbf{N}$ we arbitrarily chose any $(z_1^k, z_2^k, \ldots, z_m^k) \in \mathbf{C}^*_{1,\sqrt{\mathbf{Q}}}$ such that $CS_{z_1^k,\frac{1}{k};,\ldots z_j^k,\frac{1}{k};,\ldots z_m^k,\frac{1}{k}}\Box\Diamond\alpha \in T_1^*$ and this $(z_1^k, z_2^k, \ldots, z_n^k)$ will be the k-th number of the sequence, i.e., $Z^k = (z_1^k, z_2^k, \ldots, z_m^k)$.

Completeness

- For $Z(\Box\Diamond\alpha)$ we also consider m sequences of all first coordinates, sequence of all second coordinates and so on. Precisely, if $Z(\Box\Diamond\alpha) = (Z^k)_{k\in\mathbf{N}}$ and

$$\begin{array}{ccccccccccccc} Z^1 & = & (& z_1^1 & , & z_2^1 & , & z_3^1 & , & \dots & , & z_m^1 &) \\ Z^2 & = & (& z_1^2 & , & z_2^2 & , & z_3^2 & , & \dots & , & z_m^2 &) \\ Z^3 & = & (& z_1^3 & , & z_2^3 & , & z_3^3 & , & \dots & , & z_m^3 &) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

  we consider m sequences $Z_1, Z_2 \dots Z_m$ where
  - $Z_1 = z_1^1, , z_1^2, z_1^3 \dots$
  - $Z_2 = z_2^1, , z_2^2, z_2^3 \dots$
  - $\dots$
  - $Z_m = z_m^1, , z_m^2, z_m^3 \dots$

Completeness

Lemma

Let $Z(\Box\Diamond\alpha)$ be defined as above. Then, for every $j = \overline{1, m}$, Z_j is Cauchy (with respect to the norm $|| \cdot ||$).

Lemma

Let $\alpha \in T_1^*$, $\alpha \in For_{Cl}$. Suppose that $(Z^k)_{k\in\mathbf{N}}$, and $(Z'^k)_{n\in\mathbf{N}}$ are two different sequences obtained by the above given construction (i.e., for at least one k_0 and least one j_0, $z'^{k_0}_{j_0} \neq z^{k_0}_{j_0}$). Then for every $j = \overline{1, n}$, $\lim_{k\to\infty} z^k_j = \lim_{k\to\infty} z'^k_j$.

Completeness-canonical model

Definition

A canonical $QL_{prob,B}$-model is a structure $M_{\mathbf{B}}^{T^*} = \langle W, R, H, \upsilon \rangle$ defined in the following way:

- W is the set of all maximal consistent set of formulas;
- R is syntactically compatibility relation;
- for every $\alpha \in For_{mod}$, $\upsilon(\alpha) = \{w \in W \mid \alpha \in w\}$;
- $H = \{\upsilon(\alpha) \mid For_{mod}\}$;

Completeness-canonical model

Definition

A canonical model $M^{T^*} = \langle W, R, H, \rho, v \rangle$ is defined such that:

- $\langle W, R, H, v \rangle$ is canonical $QL_{prob,B}$-model;
- If $\boxminus H = \{v(\Box\Diamond\alpha) \mid \alpha \in For_{mod}\}$ then for every $\alpha \in For_{mod}$, $\boxdot\overline{\Diamond}v(\alpha) \in \boxminus H$ and hence we define: for $w \in W$, $\alpha \in For_{mod}$ and $Z(\Box\Diamond\alpha)$ constructed relative to w,

$$\rho(w, \boxdot\overline{\Diamond}v(\alpha)) = (\lim_{k\to\infty} z_1^k, \lim_{k\to\infty} z_2^k, \ldots \lim_{k\to\infty} z_m^k)$$

Completeness

Theorem

A set of For_{Qv}-formulas T is consistent iff has an QL_{prob}-model.

Decidability

Theorem

The satisfiability and validity problems for For_{Qv} formulas are decidable.

Description of the experiment

- The cat is enclosed in a box with one single Bi-212 atom, a Geiger counter, a hammer and flask of poison. When the atom decays, the Geiger counter will detect the radioactive radiation, the hammer will smash the flask releasing the poison and the cat will die. Bi-212 atom, has a a *half life* of 66.55 minutes which means that half of Bi-212 atoms will decay after this period
- When Geinger counter interact with the atom its state gets *entangled* with the state of atom.Let $[ready\rangle$ denotes initial state of Geinger counter. If the counter detect the radiation $[ready\rangle$ evolves to the state $[click\rangle$ and otherwise it evolves to the state $[noclick\rangle$.
- According to Schrodinger's equation after one hour the state of "atom-Geinger counter" evolves to the state $\frac{1}{\sqrt{2}}[undecayed\rangle[noclick\rangle + \frac{1}{\sqrt{2}}[decayed\rangle[click\rangle$.

Description of the experiment

- Since the hammer, the flask and the cat will all become entangled with atom and Geinger counter, the state of the whole system evolves to

$$\frac{1}{\sqrt{2}}[\varphi_1\rangle + \frac{1}{\sqrt{2}}[\varphi_2\rangle$$

where

$$[\varphi_1\rangle = \frac{1}{\sqrt{2}}[undecayed\rangle[noclick\rangle_{counter}[notfallen\rangle_{hammer}[notsmashed\rangle$$

and

$$[\varphi_2\rangle \frac{1}{\sqrt{2}}[decayed\rangle[click\rangle_{counter}[fallen\rangle_{hammer}[smashed\rangle_{flask}[dead\rangle_{cat}$$

Description of the experiment

- According to von Neumann, measurement is performed when conscious observer read the result of measuring device. In that case superposition $\frac{1}{\sqrt{2}}[\varphi_1\rangle + \frac{1}{\sqrt{2}}[\varphi_2\rangle$ will collapse on one of the states φ_1 and φ_2 whenever an observation is made. Since decaying of an atom will instantly led to click of Geinger counter, falling the hammer, smashing the flask and death of the cat, for simplicity we consider only states of the atom, i.e. $[undecayed\rangle$ and $[decayed\rangle$.

Logic L_{QM} and the Cat

- $\Box\Diamond\alpha$ means “it is observed (measured) that it possible that atom has decayed”,
- $\Box\Diamond\neg\alpha$ means “it is observed (measured) that it possible that atom has not decayed”

Logic L_{QM} and the Cat

- $\Box\Diamond\alpha$ means "it is observed (measured) that it possible that atom has decayed",
- $\Box\Diamond\neg\alpha$ means "it is observed (measured) that it possible that atom has not decayed"

Let w denotes current state of the system. Then

$$w \models CS_{\frac{1}{\sqrt{2}},0}\Box\Diamond\alpha \wedge CS_{\frac{1}{\sqrt{2}},0}\Box\Diamond\neg\alpha$$

can be interpreted as:

- in the state w cat is simultaneously death and alive, or or
- we only know what is possible, i.e., what can we obtain if we perform the measurement.

Logic L_{QM} and the Cat

From

$$w \models CS_{\frac{1}{\sqrt{2}},0}\Box\Diamond\alpha \wedge CS_{\frac{1}{\sqrt{2}},0}\Box\Diamond\neg\alpha$$

we obtain that

$$w \models \Diamond\alpha \wedge \Diamond\neg\alpha$$

This means that if we perform measurement in a state w it is possible to obtain that the atom has decayed (cat is dead) and equally that the atom has not decayed (cat is alive).
Note that it does not mean that it is possible in w that the cat is simultaneously death and alive: $w \not\models \Diamond(\alpha \wedge \neg\alpha)$

Logic L_{QM} and the Cat

Now, we finally perform the experiment and there are two possibilities:

- $\Box\Diamond\alpha$ it is observed that it is possible that the atom has decayed;
- $\Box\Diamond\neg\alpha$ it is observed that it is possible that the atom has not decayed;

Logic L_{QM} and the Cat

Now, we finally perform the experiment and there are two possibilities:

- $\Box\Diamond\alpha$ it is observed that it is possible that the atom has decayed;
- $\Box\Diamond\neg\alpha$ it is observed that it is possible that the atom has not decayed;

The state w collapses into one of the states w_1 and w_2, where $w_1 \models \Box\Diamond\alpha$ and $w_2 \models \Box\Diamond\neg\alpha$.