Probabilistic reasoning with lambda terms

Silvia Ghilezan[1,2] Jelena Ivetić[1] Simona Kašterović[1]
Zoran Ognjanović[2] Nenad Savić[3]

[1] Faculty of Technical Sciences, University of Novi Sad
[2] Mathematical Institute SANU, Belgrade, Serbia
[3] Institute of Computer Science, University of Bern, Switzerland

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3 Probabilistic logical system for simply typed lambda terms $\mathrm{P}\Lambda_{\rightarrow}$

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5 Completeness

- combination of simply typed lambda terms and probabilistic logic;
- probabilistic logic;
- simple type assignment.

1 Probabilistic logic

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  - Classical propositional formulas
  - Basic probabilistic formulas

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- infinitary axiomatization,
- every consistent set can be extended to the maximal consistent set
- canonical model.

Z. Ognjanović, M. Rašković, Z. Marković, Probability Logics: Probability-Based Formalization of Uncertain Reasoning. Springer, 2016.

- neither mixing of classical formulas and probabilistic formulas, nor nested probability operators is allowed,
- the following two expressions are *not* (well defined) formulas of the logic $\mathsf{P}\Lambda_{\rightarrow}$:

$$\alpha \wedge P_{\geq \frac{1}{2}} \beta, \qquad P_{\geq \frac{1}{3}} P_{\geq \frac{1}{2}} \alpha.$$

2 Simple type assignment $\Lambda_{\rightarrow}$

Let $\mathtt{V}_{\Lambda} = \{x, y, z, \ldots, x_1, \ldots\}$ be a countable set of λ-term variables.

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Simple types

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Definition [$=_\beta$]

The lambda term M is β-equal to the lambda term N (notion $M =_\beta N$) if and only if there is a sequence $M \equiv N_0, N_1, \dots, N_n \equiv N$, where $N_i \rightarrow_\beta N_{i+1}$ or $N_{i+1} \rightarrow_\beta N_i$ for all $i \in \{0, 1, \dots, n\}$.

Definition

The simple type assignment, $\Lambda_{\rightarrow}$, is defined as follows:

$$\frac{M : \sigma \rightarrow \tau \qquad N : \sigma}{MN : \tau}\ (\rightarrow_E)$$

$$\frac{\begin{array}{c}[x : \sigma] \\ \vdots \\ M : \tau\end{array}}{\lambda x.M : \sigma \rightarrow \tau}\ (\rightarrow_I)$$

$$\frac{M : \sigma \qquad M =_\beta N}{N : \sigma}\ (\mathrm{eq})$$

Term model $\mathcal{M} = \langle D, \cdot, [\![\ ]\!] \rangle$

Definition

(i) Domain of a term model is a set of all convertibility-classes of terms. For $M \in \Lambda$, the convertibility-class represented by M will be denoted by $[M]$, i.e., $[M] = \{N : N =_\beta M\}$.

(ii) If $\rho : \mathtt{V}_\Lambda \rightarrow D$ is the valuation of term variables in D, then $[\![M]\!]_\rho \in D$ is the interpretation of $M \in \Lambda$ in $\mathcal{M}$ via ρ.

(iii) Map $\cdot$ is defined by $[M] \cdot [N] = [MN]$, and $[\![\ ]\!]_\rho$ is defined by $[\![M]\!]_\rho = [M[N_1, \dots, N_n/x_1, \dots, x_n]]$, where $x_1, \dots, x_n$ are the free variables of M, and $\rho(x_i) = [N_i]$ and $[\cdots / \cdots]$ is simultaneous substitution.

(iv) Let $\xi : \mathtt{V}_{\mathtt{Type}} \rightarrow \mathcal{P}(D)$ be a valuation of type variables. The interpretation of $\sigma \in \mathtt{Type}$ in $\mathcal{M}$ via ξ, denoted by $[\![\sigma]\!]_\xi \in \mathcal{P}(D)$, is defined:

- $[\![a]\!]_\xi = \xi(a)$;
- $[\![\sigma \rightarrow \tau]\!]_\xi = \{d \in D \mid \forall e \in [\![\sigma]\!]_\xi,\ d \cdot e \in [\![\tau]\!]_\xi\}$.

The soundness and completeness of type assignment were proved with the notion of term model.

Theorem [Soundness]

$\Gamma \vdash M : \sigma \;\Rightarrow\; \Gamma \models M : \sigma$.

Theorem [Completeness]

$\Gamma \models M : \sigma \Rightarrow \Gamma \vdash M : \sigma$.

J.R. Hindley, The completeness theorem for typing lambda terms. *Theoretical computer Science*, 22: 1–17, 1983.

3 Probabilistic logical system for simply typed lambda terms $P\Lambda_{\to}$

Syntax of $P\Lambda_{\rightarrow}$

Let S be the set of rational numbers from $[0, 1]$, i.e., $S = [0, 1] \cap \mathbb{Q}$. The *alphabet* of the logic $P\Lambda_{\rightarrow}$ consists of

- all symbols needed to define simply typed lambda terms,
- the classical propositional connectives $\neg$ and $\wedge$,
- the list of probability operators $P_{\geq s}$, for every $s \in S$.

Basic and Probabilistic Formulas

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$$\mathsf{For}_{\mathsf{P}} \quad \phi ::= P_{\geq s}\alpha \mid \phi \wedge \phi \mid \neg\phi.$$

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Example:

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- $x : \sigma \to \tau \wedge y : \sigma$;
- $x : \sigma \to \tau \wedge y : \sigma \Rightarrow xy : \tau$.

Probabilistic Formulas

$$\mathsf{For}_{\mathsf{P}} \quad \phi ::= P_{\geq s}\alpha \mid \phi \wedge \phi \mid \neg\phi.$$

Example:

- $P_{\geq \frac{1}{2}} x : \sigma$;
- $P_{=1}(x : \sigma \to \rho \wedge y : \sigma) \Rightarrow P_{=1}(xy : \rho)$.

Semantics of $P\Lambda_{\rightarrow}$

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Definition [$P\Lambda_{\rightarrow}$-structure]

A $P\Lambda_{\rightarrow}$-structure is a tuple $\mathcal{M} = \langle W, \rho, \xi, H, \mu \rangle$, where:

(i) W is a nonempty set of worlds, where each world is one term model, i.e., for every $w \in W$, $w = \langle \mathcal{L}(w), \cdot_w, [\![\;]\!]_w \rangle$;

(ii) $\rho : \mathtt{V}_{\Lambda} \times \{w\} \longrightarrow \mathcal{L}(w)$, $w \in W$;

(iii) $\xi : \mathtt{V}_{\mathtt{Type}} \times \{w\} \longrightarrow \mathcal{P}(\mathcal{L}(w))$, $w \in W$;

(iv) H is an algebra of subsets of W, i.e. $H \subseteq \mathcal{P}(W)$ such that
- $W \in H$,
- if $U, V \in H$, then $W \setminus U \in H$ and $U \cup V \in H$;

(v) μ is a finitely additive probability measure defined on H, i.e.,
- $\mu(W) = 1$,
- if $U \cap V = \emptyset$, then $\mu(U \cup V) = \mu(U) + \mu(V)$, for all $U, V \in H$.

We say that a lambda statement $M : \sigma$ holds in a world w, notation $w \models M : \sigma$, if and only if

$$[\![M]\!]^w_\rho \in [\![\sigma]\!]^w_\xi.$$

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Definition [Satisfiability relation]

The satisfiability relation $\models \subseteq P\Lambda^{\mathsf{Meas}}_{\rightarrow} \times \mathsf{For}_{P\Lambda_{\rightarrow}}$ is defined in the following way:

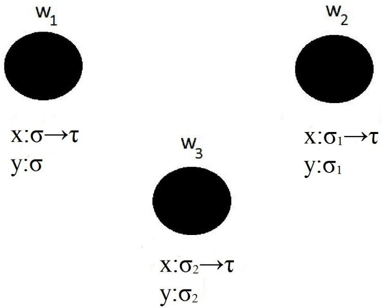
- $\mathcal{M} \models M : \sigma$ iff $w \models M : \sigma$, for all $w \in W$;
- $\mathcal{M} \models P_{\geq s}\alpha$ iff $\mu([\alpha]) \geq s$;
- $\mathcal{M} \models \neg\mathfrak{A}$ iff it is not the case that $\mathcal{M} \models \mathfrak{A}$;
- $\mathcal{M} \models \mathfrak{A}_1 \wedge \mathfrak{A}_2$ iff $\mathcal{M} \models \mathfrak{A}_1$ and $\mathcal{M} \models \mathfrak{A}_2$.

Example

Consider the following model with three worlds, i.e., let $\mathcal{M} = \langle W, \rho, \xi, H, \mu \rangle$, where:

- $W = \{w_1, w_2, w_3\}$,
- $H = \mathcal{P}(W)$,
- $\mu(\{w_j\}) = \frac{1}{3}$, $j = 1, 2, 3$,

and ρ and ξ are defined such that
$w_1 \models (x : \sigma \to \tau) \wedge (y : \sigma)$,
$w_2 \models (x : \sigma_1 \to \tau) \wedge (y : \sigma_1)$,
$w_3 \models (x : \sigma_2 \to \tau) \wedge (y : \sigma_2)$.


w_1
$x:\sigma\rightarrow\tau$
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w_3
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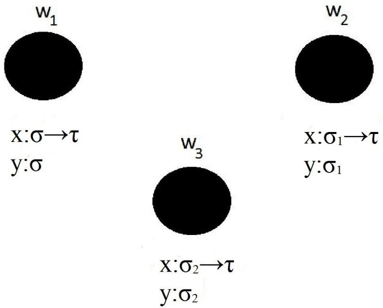
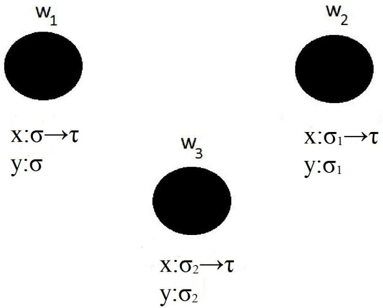
$y:\sigma_2$



$\mathcal{M} \models P_{=\frac{1}{3}}(x : \sigma \rightarrow \tau)$, $\mathcal{M} \models P_{=\frac{1}{3}}(y : \sigma)$,



$\mathcal{M} \models P_{=\frac{1}{3}}(x : \sigma \rightarrow \tau)$, $\mathcal{M} \models P_{=\frac{1}{3}}(y : \sigma)$,
$\mathcal{M} \models P_{=1}(xy : \tau)$.

4 The axiomatization $Ax_{\mathrm{P}\Lambda_{\rightarrow}}$

Axiom schemes

(1) all instances of the classical propositional tautologies, (atoms are any $\mathrm{P}\Lambda_{\to}$-formulas),

(2) $P_{\geq 0}\alpha$,

(3) $P_{\leq r}\alpha \Rightarrow P_{<s}\alpha$, $s > r$,

(4) $P_{<s}\alpha \Rightarrow P_{\leq s}\alpha$,

(5) $(P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge P_{\geq 1}(\neg\alpha \vee \neg\beta)) \Rightarrow P_{\geq min\{1,r+s\}}(\alpha \vee \beta)$,

(6) $(P_{\leq r}\alpha \wedge P_{<s}\beta) \Rightarrow P_{<r+s}(\alpha \vee \beta)$, $r + s \leq 1$,

(7) $P_{\geq 1}(\alpha \Rightarrow \beta) \Rightarrow (P_{\geq s}\alpha \Rightarrow P_{\geq s}\beta)$.

Inference Rules I

$$(1) \quad \frac{M : \sigma \rightarrow \tau \qquad N : \sigma}{MN : \tau} \, (\rightarrow_E)$$

$$(2) \quad \frac{\begin{array}{c}[x : \sigma] \\ \vdots \\ M : \tau\end{array}}{\lambda x.M : \sigma \rightarrow \tau} \, (\rightarrow_I)$$

$$(3) \quad \frac{M : \sigma \qquad M =_\beta N}{N : \sigma} \, (\mathrm{eq})$$

Inference Rules II

(1) From $\mathfrak{A}_1$ and $\mathfrak{A}_1 \Rightarrow \mathfrak{A}_2$ infer $\mathfrak{A}_2$,

(2) from α infer $P_{\geq 1}\alpha$,

(3) from the set of premises

$$\{\phi \Rightarrow P_{\geq s-\frac{1}{k}}\alpha \mid k \geq \frac{1}{s}\}$$

infer $\phi \Rightarrow P_{\geq s}\alpha$.

Soundness

Theorem [Soundness]

The axiomatic system $Ax_{P\Lambda_{\rightarrow}}$ is sound with respect to the class of $P\Lambda_{\rightarrow}^{\mathsf{Meas}}$-models.

5 Completeness

Theorem

Every consistent set can be extended to a maximal consistent set.

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Definition [Canonical model]

If $T^\star$ is the maximally consistent set of formulas, then a tuple $\mathcal{M}_{T^\star} = \langle W, \rho, \xi, H, \mu \rangle$ is defined:

- $W = \{w = \langle \mathcal{L}(w), \cdot_w, [\![\ ]\!]_w \rangle \mid w \models T\}$ contains all term models that satisfy the set T,
- $\rho_w(x) = [x]$,
- $\xi_w(a) = \{[M] \in \mathcal{L}(w) \mid w \models M : a\}$,
- $H = \{[\alpha] \mid \alpha \in \mathrm{For_B}\}$, where $[\alpha] = \{w \in W \mid w \models \alpha\}$,
- $\mu([\alpha]) = \sup\{s \mid P_{\geq s}\alpha \in T^\star\}$.

Strong completeness

Theorem

Every consistent set of formulas T is $\mathsf{P}\Lambda_{\rightarrow}^{\mathsf{Meas}}$-satisfiable.

Further work

- intuitionistic propositional calculus,
- typed lambda calculus with intersection types.