Decidability and Complexity of Some Interpretability Logics

Luka Mikec\textsuperscript{1}  Tin Perkov\textsuperscript{2}  Mladen Vuković\textsuperscript{2}

\textsuperscript{1}University of Rijeka, Croatia

\textsuperscript{2}University of Zagreb, Croatia

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Basic interpretability logic **IL**

- Interpretability logics have a binary modal operator $\triangleright$.
- Basic interpretability logic **IL**:
  
  classically valid formulas (in the new language with $\Box$, $\Diamond$, $\triangleright$);

  $\begin{align*}
  K & \quad (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B); \\
  \text{Löb} & \quad (\Box A \rightarrow A) \rightarrow \Box A; \\
  J1 & \quad (A \rightarrow B) \rightarrow A \triangleright B; \\
  J2 & \quad (A \triangleright B) \land (B \triangleright C) \rightarrow A \triangleright C; \\
  J3 & \quad (A \triangleright C) \land (B \triangleright C) \rightarrow A \lor B \triangleright C; \\
  J4 & \quad A \triangleright B \rightarrow (\Diamond A \rightarrow \Diamond B); \\
  J5 & \quad \Diamond A \triangleright A.
  \end{align*}$

  - rules: modus ponens and necessitation $A/\Box A$.

(parentheses priority: $\neg$, $\Box$, $\Diamond$; $\land$, $\lor$; $\triangleright$; $\rightarrow$, $\leftrightarrow$)
Models

- Semantics: extend the usual relational (Kripke) model.
- **IL-frame** (Veltman frame): \( \mathcal{F} = \langle W, R, \{ S_w : w \in W \} \rangle \), where:
  1. \( W \neq \emptyset \);
  2. \( R^{-1} \) is well-founded (no \( x_0Rx_1Rx_2R \ldots \) chains);
  3. \( R \) is transitive;
  4. \( S_w \subseteq R(w)^2 \) is reflexive, transitive, contains \( R \cap R(w)^2 \)  
      \( (wRuRv \text{ implies } uS_w v) \);
- **IL-model** (Veltman model): \( \mathcal{M} = \langle W, R, \{ S_w : w \in W \}, V \rangle \), where:
  1. \( \langle W, R, \{ S_w : w \in W \} \rangle \) is a **IL-frame**;
  2. \( V \subseteq W \times Prop \) (or \( V : Prop \to \mathcal{P}(W) \)).
Models (2)

- Veltman model: $\mathcal{M} = \langle \mathcal{W}, R, \{S_w : w \in \mathcal{W}\}, V \rangle$.
- $w \vDash p$ if and only if $wVp$, for $p \in \text{Prop}$.
- Logical connectives have classical semantics.
- Truth of a formula $F \triangleright G$ ("$F$ interprets $G$") in a world $w \in \mathcal{M}$:
  
  \[ w \vDash F \triangleright G \iff \forall x \in R(w) : x \vDash F \Rightarrow \exists y \in S_w(x) : y \vDash G. \]

- Soundness and completeness:
  \[ \text{IL} \vdash F \iff \forall \mathcal{F} : \mathcal{F} \vDash F. \]
Some extensions of IL:

\[\text{ILM}_0\] \quad \text{IL} + A \rightarrow B \rightarrow \Diamond A \land \Box C \rightarrow B \land \Box C

\[\text{ILW}\] \quad \text{IL} + A \rightarrow B \rightarrow A \rightarrow B \land \Box \neg A

\[\text{ILW}^*\] \quad \text{IL} + A \rightarrow B \rightarrow B \land \Box C \rightarrow B \land \Box C \land \Box \neg A

\[\text{ILW}^* = \text{ILM}_0 W \subseteq \text{IL}(\text{All})\]

These logics are complete w.r.t. certain classes of frames:

\begin{itemize}
  \item \((M_0)\) \quad wRuRxS_w v \Rightarrow R(v) \subseteq R(u);
  \item \((W)\) \quad S_w \circ R \text{ is reverse well-founded for each } w;
  \item \((W^*)\) \quad (M_0) \text{ and } (W).
\end{itemize}

\[\text{ILW-frame is IL-frame that satisfies (W) etc.}\]
Proving decidability

- FMP: if $x \models F$, then there is finite $\mathcal{M}$ and $x' \in \mathcal{M}$ s.t. $x' \models F$.
- Decision procedure: simultaneously do two things:
  - Enumerate the (countable) set of all IL-proofs.
  - Enumerate the (countable) set of (descriptions of) finite IL-models.
- The usual way of proving FMP is by filtrations.
Filtrations on IL-models

- Let $\Gamma$ contain $A$, closed under subformulas.
- Assume $\sim$ is an equivalence relation on $W$, $\sim \subseteq \equiv_{\Gamma}$.
- For any $V \subseteq W$, define $\tilde{V} = \{[v] | v \in V\}$.
- We define the rest of $\tilde{M}$ as follows.
  - $\tilde{R} = \{([w], [u]) | wRu, \exists \Box C \in \Gamma : w \not\models \Box C, u \models \Box C\}$.
- Define $\models$ so that $x$ and $[x]$ agree on variables in $\Gamma$.
- Problem: how to define $S_{[w]}$.
  - “Generous” definitions do not preserve transitivity; while “strict” definitions lose $S_{[w]}$-witnesses for some $\triangleright$-formulas.
- Solution: a more fine-grained semantics, where $S_{[w]}$-witnesses are not complete sets of formulas.
Problems with filtrations on IL-models

(1. try) \([u][w][v]\) if and only if \([u], [v] \in \tilde{R}([w])\), and for all/some \(w' \in [w]\) and some \(u' \in [u]\) such that \(w' Ru'\) we have \(u' S_{w'} v'\) for some \(v' \sim v\).

- \(w \rightarrow \{ u \leadsto v_1 \sim v_2 \leadsto z \}, \ [w] \rightarrow \{ [u] \leadsto [v] \leadsto [z] \} \)
- But, we do not have \([u] \leadsto [z]\).
- If transitivity forced, some false \(\triangleright\)-formula might get its witness and become true.

(2. try) \([u][w][v]\) if and only if \([u], [v] \in \tilde{R}([w])\), and for all/some \(w' \in [w]\) and all \(u' \in [u]\) such that \(w' Ru'\) we have \(u' S_{w'} v'\) for some \(v' \sim v\).

- \(w \rightarrow \{ v_1 [X] \leftrightarrow u_1 \sim u_2 \leadsto v_2 [\neg X] \}, \ [w] \rightarrow \{ [u] \leadsto ? \} \)
- Problem: we lose \(S_w\)-successors that do not agree enough.
Generalized models

- Generalized IL-models (generalized Veltman models).

- \( \mathcal{M} = \langle W, R, \{S_w : w \in W\}, V \rangle \), where:
  1. \( W \neq \emptyset \);
  2. \( R^{-1} \) is well-founded (no \( x_0 R x_1 R x_2 R \ldots \) chains);
  3. \( R \) is transitive;
  4. \( S_w \subseteq R(w) \times (2^{R(w)} \setminus \{\emptyset\}) \) is:
     - quasi-reflexive \( uS_w\{u\} \);
     - quasi-transitive \( uS_w\{v_i | i \in I\} \) and \( v_i S_w Z_i \Rightarrow uS_w \cup \{Z_i | i \in I\} \);
     - contains \( R \cap R(w)^2 \) \( wRuRv \) implies \( uS_w\{v\} \);
     - is monotonous \( uS_w V \Rightarrow uS_w V', V \subseteq V' \)
  5. \( V \subseteq W \times \text{Prop} \) (or \( V : \text{Prop} \rightarrow \mathcal{P}(W) \)).

- Truth of a formula \( F \triangleright G \) ("\( F \) interprets \( G \)") in a world \( x \in \mathcal{M} \):

  \[ w \vDash F \triangleright G :\iff \forall x \in R(w) : x \vDash F \Rightarrow \exists V \in S_w(x) : V \vDash G. \]

- \( V \vDash G \) stands for \( v \vDash G \) for all \( v \in V \).
Filtration property

- \( \tilde{M} = \langle \tilde{W}, \tilde{R}, \tilde{S}_{[w]}, \vdash \rangle \).
- \( \tilde{W} = \{ [w] \mid w \in W \} \).
- \( \tilde{R} = \{ ([w], [u]) \mid wRu, \exists \Box C \in \Gamma : w \nvdash \Box C, u \vdash \Box C \} \).
- \([u]\tilde{S}_{[w]} \tilde{V}\) if and only if \{[u]\}, \( \tilde{V} \subseteq R([w]) \), and for all \( w' \in [w] \) and all \( u' \in [u] \) such that \( w'Ru' \) we have \( u'S_{w'} V(w', u') \) for some \( V(w', u') \subseteq \tilde{V} \).
- Forcing relation compatible with \( M \).
- Assume \( \langle \tilde{W}, \tilde{R}, \tilde{S}, \vdash \rangle \) is a generalized model (depends on \( \sim \)).
- Do we have \( w \vdash F \iff [w] \vdash F \)?
Theorem

\[ w \vdash F \iff [w] \vdash F. \]

- So, if \( \langle \tilde{W}, \tilde{R}, \tilde{S}, \vdash \rangle \) is a model at all, then it is a filtration of \( \mathcal{M} = \langle W, R, S, \vdash \rangle \).

- Is it a model (does it satisfy quasi-transitivity etc.)? Depends on what \( \sim \) is.

- Ideally, \( x \) and \( [x] \) are structurally similar, so that quasi-transitivity etc. is preserved.

- So, each \( y \sim x \) should be structurally similar to \( x \).
A *bisimulation* between generalized \textbf{IL}-models
\[
\langle W, R, \{S_w : w \in W\}, \models \rangle \text{ and } \langle W', R', \{S'_w : w' \in W'\}, \models \rangle
\]
is any \(Z \subseteq W \times W', Z \neq \emptyset:\)

- (at) if \(wZw'\) then \(w \models p \iff w' \models p;\)
- (forth) if \(wZw'\) and \(wRu,\) then there exists \(u' \in R'(w')\) with \(uZu'\)
  and for all \(V' \in S'_{w'}(u')\) there is \(V \in S_w(u)\) such that for all
  \(v \in V\) there is \(v' \in V'\) with \(vZv';\)
- (back) if \(wZw'\) and \(w'R'u',\) then there exists \(u \in R(w)\) such that
  \(uZu'\) and for all \(V \in S_w(u)\) there is \(V' \in S'_{w'}(u')\) such that for
  all \(v' \in V'\) there is \(v \in V\) with \(vZv'.\)

- By induction on \(F,\) if \(x\) and \(y\) are bisimilar (w.r.t. any
  bisimulation), \(x \models F \iff y \models F.\)
- Union of bisimulations (over generalized models) is itself a
  bisimulation (*Vrgoč and Vuković, 2010*).
- In particular, there is a largest (auto)bisimulation \(Z \subseteq W^2.\)
Denote by $\sim$ the largest bisimulation on $W^2$.
(equivalently, denote $x \sim y$ if there is any bisimulation at all which equates $x$ and $y$)

Theorem

$\langle \tilde{W}, \tilde{R}, \tilde{S}, \models \rangle$ is a model.

Thus, if $\sim$ is the largest bisimulation on $W^2$, then $\langle \tilde{W}, \tilde{R}, \tilde{S}, \models \rangle$ is a model, and a filtration.

We were trying to prove finite model property; is this a finite model?

Each $\tilde{R}$-transition eliminates at least one $\diamond$-formula from $\Gamma$; so height is finite.

Still, branching factor might be infinite.
Definition

A n-bisimulation between IL-models \( \langle W, R, \{S_w : w \in W\}, \models \rangle \) and \( \langle W', R', \{S'_w : w' \in W'\}, \models \rangle \) is any sequence \( Z_n \subseteq \cdots \subseteq Z_0 \subseteq W \times W' \):

- (at) if \( wZ_0 w' \) then \( w \models p \iff w' \models p \);
- (forth) if \( wZ_n w' \) and \( wRu \), then there exists \( u' \in R'(w') \) with \( uZ_{n-1} u' \) and for all \( V' \in S'_{w'}(u') \) there is \( V \in S_w(u) \) such that for all \( v \in V \) there is \( v' \in V' \) with \( vZ_{n-1} v' \);
- (back) if \( wZ_n w' \) and \( w'R'u \), then there exists \( u \in R(w) \) such that \( uZ_{n-1} u' \) and for all \( V \in S_w(u) \) there is \( V' \in S'_{w'}(u') \) such that for all \( v' \in V' \) there is \( v \in V \) with \( vZ_{n-1} v' \).

Since height of \( M \) is bounded by \( |\Gamma| \), worlds are \( |\Gamma| \)-bisimilar iff bisimilar.
• Put $u \equiv_n v$ if $u$ and $v$ agree on all formulas with at most $n$ nested modalities.
• From now on, assume $\text{Prop} := \text{Prop} \cap \Gamma$.
• Now there are only finitely many formulas of modal depth up to $|\Gamma|$ (finitely many up to local equivalence).
• Denote $Th_n w$ the set of all formulas $F$ with modal depth up to $|\Gamma|$ and $w \models F$. 
Lemma

\[ u \sim_n v \iff u \equiv_n v. \]

- Denote \( \mathcal{N} = \widetilde{\mathcal{M}}. \)
- For \( x, y \in \mathcal{N}, \) we now have \( x \sim y \iff x \sim_{|\Gamma|} y \iff x \equiv_\Gamma y. \)
- There are obviously only finitely many worlds in \( \mathcal{M}/ \equiv_\Gamma. \)
- Since \( \equiv_\Gamma = \sim_{|\Gamma|}, \) \( \widetilde{\mathcal{N}} \) (that is, \( \widetilde{\mathcal{M}} \)) has only finitely many worlds.
- Thus we have FMP for \( \mathbf{IL}. \)
Extending to \textbf{ILX}

- To prove FMP, given \textbf{ILX} that is complete w.r.t. class of Veltman frames that satisfy property $C$, we need to fill in the following:
  1. What is the (generalized) frame condition $\mathcal{G}$ of $X$?
  2. Is \textbf{ILX} complete w.r.t. to the class of $\mathcal{G}$-frames?
  3. Does $\bar{M}$ have $\mathcal{G}$ if $M$ has $\mathcal{G}$?

- For popular choices of $X$ (except for $W, W^*$), 1 is known; and 2 usually reduces to completeness w.r.t. $C$ (for each VM take the natural GVM, i.e. $uS_w v \Rightarrow uS_w \{v\}$).
Logic $\mathbf{ILM}_0$

- $\mathbf{ILM}_0$ is $\mathbf{IL} + A \triangleright B \rightarrow \Diamond A \land \square C \triangleright B \land \square C$.
- Frame condition ($M_0$):
  \[
  wRuRxS_w vRz \Rightarrow uRz.
  \]
- Frame condition ($M_0$)$_{gen}$:
  \[
  wRuRxS_w V \Rightarrow (\exists V' \subseteq V)(uS_w V' \land R(V') \subseteq R(u)).
  \]
- For each VM with ($M_0$), there is a natural GVM (put $xS_w \{y\}$ whenever $xS_w y$) with ($M_0$)$_{gen}$.
- Remains to prove $\widetilde{M}$ preserves ($M_0$)$_{gen}$.

**Theorem**

*If $\mathcal{M}$ has property ($M_0$)$_{gen}$, then $\widetilde{\mathcal{M}}$ has property ($M_0$)$_{gen}$.*
Logic ILW

- **ILW** is **IL** + \( A \wedge B \rightarrow A \wedge B \wedge \Box \neg A \).
- Frame condition \((W)\):
  \[ S_w \circ R \text{ is reverse well-founded for each } w \]
- Frame condition \((W)_{gen}\)?
  \[
  (\forall w \in W)(\forall X \subseteq R[w])(\forall z \in W) \\
  (zS_w X \Rightarrow (\exists V \subseteq X)(zS_w V \& (\forall v \in V)(R[v] \cap S_w^{-1}[X] = \emptyset)))
  \]
- For each VM with \((W)\), there is a natural GVM (put \(xS_w \{y\}\) whenever \(xS_w y\)) with \((W)_{gen}\).

**Theorem**

*If \(\mathcal{M}\) has property \((W)_{gen}\), then \(\widehat{\mathcal{M}}\) has property \((W)_{gen}\).*
Logic $\text{ILW}^*$

- $\text{ILW}^*$ is $\text{IL} + A \triangleright B \rightarrow B \land \Box C \triangleright B \land \Box C \land \Box \neg A$.
- $\text{ILW}^* = \text{ILWM}_0$.
- Frame condition $(W^*)_{\text{gen}}$?
- Each $\text{ILW}^*$-frame is $\text{ILW}$-frame ($\text{ILWM}_0 \supseteq \text{ILW}$) and $\text{ILM}_0$-frame ($\text{ILWM}_0 \supseteq \text{ILM}_0$).
- Conversely, if $\mathcal{F}$ is both an $\text{ILW}$-frame and an $\text{ILM}_0$-frame, then it is an $\text{ILWM}_0$-frame (induction on proof length).
- So, the frame condition is:
  \[(W)_{\text{gen}} \text{ and } (M_0)_{\text{gen}}.\]
- If $\text{ILW}^* \nvdash F$, there is a $\text{ILM}_0$-, $\text{ILW}$-$\text{VM}$ $\mathcal{M}$, $w \in \mathcal{M}$, s.t. $w \nvdash F$. Then there is a natural GVM $\mathcal{N}$ with similar properties. Then $\tilde{\mathcal{N}}$ is an $\text{ILM}_0$-, $\text{ILW}$-$\text{GVM}$, and so an $\text{ILW}^*$-$\text{GVM}$. 
Complexity

- Given $X$, what is comp. complexity of $\{F \mid ILX \vdash F\}$?
- Since $GL \subseteq IL$, at least PSPACE for any natural choice of $X$.
- The only (?) known result: $IL_0$ is PSPACE-hard.

- Our goal is to prove that $IL$ is in PSPACE. The result might generalize to various $ILX$. 
Let $F$ be any non-theorem of $\text{ILX}$. By completeness, there is $\mathcal{M}, w \in \mathcal{M}$ s.t. $w \not\models F$.

1. Show that $\mathcal{M}$ can be transformed to a certain model $\mathcal{M}^f$ with some desirable properties:
   - accessibility relation ($R$) is a tree;
   - polynomial height;
   - polynomial branching factor;
   - $S_w$-transitions should be “separated” or “factorized”.

2. Show that there is an algorithm that verifies the existence of all models with such properties. For $\text{ILX}$, ensure the resulting model has $\text{ILX}$. 

Complexity (3)

- Transforming the model (step 1):
  1. Unravel the accessibility relation \( R \).
  2. Recursively apply the following operation to all nodes \( x \in W \), starting from leaves:
     2.1 Denote \( N = \{ A \triangleright B : w \not\models A \triangleright B \} \).
     2.2 Make \(|N|\) copies of \( R[x] \), label them with formulas of \( N \).
     2.3 For all \( A \triangleright B \in N \), select a witness of falseness of \( A \triangleright B \) in the corresponding copy.
     2.4 In each copy, also select witnesses of true formulas.
  3. Remove worlds that do not witness anything.

- This leaves us with a model of polynomial height and branching factor; with easy to manage \( S_w \) relations.

- Step 2 is proving correctness and completeness of the algorithm that creates models resembling the ones from step 1.
Papers


- L. Mikec, T. Perkov, M. Vuković. Decidability of interpretability logics $\text{ILM}_0$ and $\text{ILW}^*$. Logic Journal of the IGPL, Volume 25, Issue 5, 1 October 2017, Pages 758–772,

- (complexity paper - work in progress)