Towards Relevant Justifications

(Ongoing Work)

N. Savić[1] T. Studer[1]

[1]Institute of Computer Science
University of Bern

LAP, September 2017

Outline

1. Justification Logics
   - Motivation
   - Semantics
   - Axiomatization
2. Relevant Logics
   - Motivation
   - Logic R: Semantics
   - Logic R: Axiomatization
   - Logic NR
3. Relevant Justification Logic (RJ)
   - Axiomatization
   - Semantics
   - Goal

Outline

Modal Logic

Classical propositional logic + $\Box\alpha$ = Modal Logic

Modal Logic

Classical propositional logic $+ \Box\alpha =$ Modal Logic

Two traditions:
Epistemic Logic:
$\Box\alpha$ means α is known / believed

Modal Logic

Classical propositional logic $+ \Box\alpha =$ Modal Logic

Two traditions:
Epistemic Logic:
$\Box\alpha$ means α is known / believed

Proof Theory:
$\Box\alpha$ means α is provable in system S

One problem: Proof-Theoretic Tradition

One problem: Proof-Theoretic Tradition

$\Box\bot \to \bot$ Axiom

One problem: Proof-Theoretic Tradition

$\Box\bot \to \bot$ Axiom

$\neg\Box\bot$ is provable

One problem: Proof-Theoretic Tradition

$\Box\bot \to \bot$ Axiom

$\neg\Box\bot$ is provable

$\Box\neg\Box\bot$ is provable

One problem: Proof-Theoretic Tradition

$\Box\bot \to \bot$ Axiom

$\neg\Box\bot$ is provable

$\Box\neg\Box\bot$ is provable

$\Box\bot$ means $\bot$ is provable in S

One problem: Proof-Theoretic Tradition

$\Box\bot \rightarrow \bot$ Axiom

$\neg\Box\bot$ is provable

$\Box\neg\Box\bot$ is provable

$\Box\bot$ means $\bot$ is provable in S

$\neg\Box\bot$ means $\bot$ is not provable in S (S is consistent)

One problem: Proof-Theoretic Tradition

$\Box\bot \to \bot$ Axiom

$\neg\Box\bot$ is provable

$\Box\neg\Box\bot$ is provable

$\Box\bot$ means $\bot$ is provable in S

$\neg\Box\bot$ means $\bot$ is not provable in S (S is consistent)

$\Box\neg\Box\bot$ means it is provable in S that S is consistent

One problem: Proof-Theoretic Tradition

$\Box\bot \to \bot$ Axiom

$\neg\Box\bot$ is provable

$\Box\neg\Box\bot$ is provable

$\Box\bot$ means $\bot$ is provable in S

$\neg\Box\bot$ means $\bot$ is not provable in S (S is consistent)

$\Box\neg\Box\bot$ means it is provable in S that S is consistent

Gödel: If S is consistent and has a certain strength it can not prove its own consistency.

Justification Logics

Justification Logics

Justification logics replace the $\Box$-operator of modal logic by explicit justifications.

Justification Logics

Justification logics replace the $\Box$-operator of modal logic by explicit justifications.

That is justification logics feature formulas of the form $t : A$ with the same inteded meaning.

Justification Terms and Formulas

Justification Terms and Formulas

Terms are built from countable sets of constants and variables as follows:

$$t ::= c \mid x \mid t \cdot t \mid t + t \mid\ !t,$$

where c is a constant and x is a variable.

Justification Terms and Formulas

Terms are built from countable sets of constants and variables as follows:

$$t ::= c \mid x \mid t \cdot t \mid t + t \mid !t,$$

where c is a constant and x is a variable.

Formulas: $\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \ t : \alpha$,
where t is a term and p is an atomic proposition.

Outline

1. Justification Logics
   - Motivation
   - Semantics
   - Axiomatization
2. Relevant Logics
   - Motivation
   - Logic R: Semantics
   - Logic R: Axiomatization
   - Logic *NR*
3. Relevant Justification Logic (*RJ*)
   - Axiomatization
   - Semantics
   - Goal

Semantics

A basic evaluation is a function $\nu : \mathit{Prop} \to \{0,1\}$ together with a function $\spadesuit : \mathit{Term} \to \mathcal{P}(\mathit{For})$ such that for arbitrary $s, t \in \mathit{Term}$ and any formula F

$s^\spadesuit \cdot t^\spadesuit \subseteq (s \cdot t)^\spadesuit$
$s^\spadesuit \cup t^\spadesuit \subseteq (s + t)^\spadesuit$
$t : (t^\spadesuit) \subseteq (!t)^\spadesuit$
$F \in t^\spadesuit$ *if* $(t, F) \in \mathsf{CS}$

where for sets of formulas X and Y, we write

$$X \cdot Y := \{F \mid G \to F \in X \text{ and } G \in Y, \text{ for some formula } G\}$$
$$X \wedge Y := \{F \mid F = G \wedge H, \text{ for some } G \in X \text{ and } H \in Y\}$$
$$t : X := \{t : F \mid F \in X\}.$$

Semantics

A basic evaluation is a function $\nu : Prop \to \{0,1\}$ together with a function $\spadesuit : Term \to \mathcal{P}(For)$ such that for arbitrary $s, t \in Term$ and any formula F

$s^\spadesuit \cdot t^\spadesuit \subseteq (s \cdot t)^\spadesuit$
$s^\spadesuit \cup t^\spadesuit \subseteq (s + t)^\spadesuit$
$t : (t^\spadesuit) \subseteq (!t)^\spadesuit$
$F \in t^\spadesuit$ *if* $(t, F) \in \mathsf{CS}$

where for sets of formulas X and Y, we write

$$X \cdot Y := \{F \mid G \to F \in X \text{ and } G \in Y, \text{ for some formula } G\}$$
$$X \wedge Y := \{F \mid F = G \wedge H, \text{ for some } G \in X \text{ and } H \in Y\}$$
$$t : X := \{t : F \mid F \in X\}.$$

Truth under basic evaluation:

$\Vdash p$ iff $\nu(p) = 1$, for $p \in Prop$
$\Vdash F \to G$ iff $\nVdash F$ or $\Vdash G$
$\Vdash \neg F$ iff $\nVdash F$
$\Vdash t : F$ iff $F \in t^\spadesuit$

Outline

1. Justification Logics
   - Motivation
   - Semantics
   - Axiomatization
2. Relevant Logics
   - Motivation
   - Logic R: Semantics
   - Logic R: Axiomatization
   - Logic *NR*
3. Relevant Justification Logic (*RJ*)
   - Axiomatization
   - Semantics
   - Goal

The Correspodence Theorem (Realization Theorems)

The Correspodence Theorem (Realization Theorems)

The Correspodence Theorem is a cumulative result stating that for each of major epistemic modal logics K, T, K4, S4, K45, KD45, S5, there is a system of justification terms and a corresponding Justification Logic system (called J, JT, J4, LP, J45, JD45, and JT45) capable of recovering explicit justifications for modalities in any theorem of the original modal logic.

Axiomatization of J4

Axiomatization of J4

(1) $t : (A \to B) \to (s : A \to (t \cdot s) : B)$

(2) $t : A \to !t : t : A$

(3) $t : A \to (t + s) : A$ and $t : A \to (s + t) : A$

Axiomatization of J4

(1) $t : (A \to B) \to (s : A \to (t \cdot s) : B)$

(2) $t : A \to !t : t : A$

(3) $t : A \to (t + s) : A$ and $t : A \to (s + t) : A$

To introduce the rules of our logic, we need the following notion: a constant specification is a set
$\mathsf{CS} \subseteq \{(c, A) \mid c \text{ is a constant and } A \text{ is an axiom}\}$.

Axiomatization of J4

(1) $t : (A \to B) \to (s : A \to (t \cdot s) : B)$

(2) $t : A \to !t : t : A$

(3) $t : A \to (t + s) : A$ and $t : A \to (s + t) : A$

To introduce the rules of our logic, we need the following notion: a constant specification is a set
$\mathsf{CS} \subseteq \{(c, A) \mid c \text{ is a constant and } A \text{ is an axiom}\}$.
Given a constant specification CS, the deductive system is given by the axioms and the rules

$$\frac{F \qquad F \to G}{G} \qquad\qquad \frac{(c, A) \in \mathsf{CS}}{c : A}$$

One problem of the Logic $J4$

One problem of the Logic $J4$

Consider a person A visiting a foreign town, which she doesn't know well. In order to get to a certain restaurant, she asks two persons B and C for the way. Person B says that A can take path P to the restaurant whereas person C replies that P does not lead to the restaurant and A should take another way. Person A now has a reason s to believe P and a reason t to believe $\neg P$. We can formalize this in justification logic by saying that both

$$s : P \qquad \text{and} \qquad t : \neg P \tag{1}$$

hold. However, then there exists a justification $r(s, t)$ such that

$$r(s, t) : (P \wedge \neg P)$$

holds. Now this implies that for any formula F, there is a justification u such that

$$u : F \tag{2}$$

Why we obtain that problem?

Why we obtain that problem?

$$A \wedge \neg A \rightarrow B$$

is a theorem of classical propositional logic.

Outline

1. Justification Logics
   - Motivation
   - Semantics
   - Axiomatization
2. Relevant Logics
   - Motivation
   - Logic R: Semantics
   - Logic R: Axiomatization
   - Logic *NR*
3. Relevant Justification Logic (*RJ*)
   - Axiomatization
   - Semantics
   - Goal

Paradoxes of Material and Strict Implication

Paradoxes of Material and Strict Implication

Material implication:

Paradoxes of Material and Strict Implication

Material implication:

M1 $A \to (B \to A)$

M2 $\neg A \to (A \to B)$

M3 $(A \to B) \vee (B \to A)$

M4 $(A \to B) \vee (B \to C)$

Paradoxes of Material and Strict Implication

Material implication:

M1 $A \rightarrow (B \rightarrow A)$

M2 $\neg A \rightarrow (A \rightarrow B)$

M3 $(A \rightarrow B) \vee (B \rightarrow A)$

M4 $(A \rightarrow B) \vee (B \rightarrow C)$

Strict implication ($A \rightarrow B := \Box(A \supset B)$, where $\supset$ is a material implication):

Paradoxes of Material and Strict Implication

Material implication:

M1 $A \rightarrow (B \rightarrow A)$

M2 $\neg A \rightarrow (A \rightarrow B)$

M3 $(A \rightarrow B) \vee (B \rightarrow A)$

M4 $(A \rightarrow B) \vee (B \rightarrow C)$

Strict implication ($A \rightarrow B := \Box(A \supset B)$, where $\supset$ is a material implication):

S1 $A \rightarrow (B \rightarrow B)$

S2 $A \rightarrow (B \vee \neg B)$

S3 $(A \wedge \neg A) \rightarrow B$

Outline

R-frame

R-frame: $\langle K, 0, R, * \rangle$, where:

- K is a non-empty set
- $0 \in K$
- R is a ternary relation on K
- $* : K \to K$

R-frame

R-frame: $\langle K, 0, R, * \rangle$, where:

- K is a non-empty set
- $0 \in K$
- R is a ternary relation on K
- $* : K \to K$

such that:

- $R0aa$
- $Rabc \Rightarrow Rbac$
- $R^2(ab)cd \Rightarrow R^2a(bc)d$
- $Raaa$
- $a \leq b \wedge Rbcd \Rightarrow Racd$
- $Rabc \Leftrightarrow Rac^*b^*$
- $a^{**} = a$

where $a \leq b := R0ab$.

Satisfiability relation

Satisfiability relation

Valuation is a function $\nu : K \to \mathcal{P}(Prop)$ such that if $a \leq b$ and $p \in \nu(a)$ then $p \in \nu(b)$.

Satisfiability relation

Valuation is a function $\nu : K \to \mathcal{P}(Prop)$ such that if $a \leq b$ and $p \in \nu(a)$ then $p \in \nu(b)$.Also, we say that for $p \in Prop$, $a \models p$ iff $p \in \nu(a)$.

Satisfiability relation

Valuation is a function $\nu : K \to \mathcal{P}(Prop)$ such that if $a \leq b$ and $p \in \nu(a)$ then $p \in \nu(b)$.Also, we say that for $p \in Prop$, $a \models p$ iff $p \in \nu(a)$.

R-model: $\langle K, 0, R, *, \models \rangle$, where $\langle K, 0, R, * \rangle$ is an $\mathrm{R} - \mathit{frame}$ and $\models \subseteq K \times \mathit{Formulas}(\mathrm{R})$ with:

- If $a \models p$, for $p \in Prop$, and $a \leq b$, then $b \models p$
- $a \models A \wedge B$ iff $a \models A$ and $a \models B$
- $a \models A \vee B$ iff $a \models A$ or $a \models B$
- $a \models A \to B$ iff $Raxy$ and $x \models A$ imply $y \models B$, for all $x, y \in K$
- $a \models \neg A$ iff $a* \not\models A$

Outline

Logic R: Axiom schemes

(A1) $A \to A$

(A2) $A \to ((A \to B) \to B)$

(A3) $(A \to B) \to ((B \to C) \to (A \to C))$

(A4) $(A \to (A \to B)) \to (A \to B)$

(A5) $A \wedge B \to A$

(A6) $A \wedge B \to B$

(A7) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$

(A8) $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$

(A9) $\neg\neg A \to A$

(A10) $(A \to \neg B) \to (B \to \neg A)$

(A11) $A \vee B \leftrightarrow \neg(\neg A \wedge \neg B)$

(A12) $A \circ B \leftrightarrow \neg(A \to \neg B)$

Inference Rules

Inference Rules

(MP) From A and $A \to B$ infer B

(ADJ) From A and B infer $A \wedge B$

Outline

Logic *NR*

Relevant logic R + $S4$-style of necessity.

Logic NR

Relevant logic R + $S4$-style of necessity.

(A1) $A \to A$
(A2) $A \to ((A \to B) \to B)$
(A3) $(A \to B) \to ((B \to C) \to (A \to C))$
(A4) $(A \to (A \to B)) \to (A \to B)$
(A5) $A \wedge B \to A$
(A6) $A \wedge B \to B$
(A7) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
(A8) $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
(A9) $\neg\neg A \to A$
(A10) $(A \to \neg B) \to (B \to \neg A)$
(A11) $A \vee B \leftrightarrow \neg(\neg A \wedge \neg B)$
(A12) $A \circ B \leftrightarrow \neg(A \to \neg B)$

Logic NR

Relevant logic R + $S4$-style of necessity.

(A1) $A \to A$

(A2) $A \to ((A \to B) \to B)$

(A3) $(A \to B) \to ((B \to C) \to (A \to C))$

(A4) $(A \to (A \to B)) \to (A \to B)$

(A5) $A \wedge B \to A$

(A6) $A \wedge B \to B$

(A7) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$

(A8) $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$

(A9) $\neg\neg A \to A$

(A10) $(A \to \neg B) \to (B \to \neg A)$

(A11) $A \vee B \leftrightarrow \neg(\neg A \wedge \neg B)$

(A12) $A \circ B \leftrightarrow \neg(A \to \neg B)$

(A13) $\Box A \to A$

(A14) $\Box(A \to B) \to (\Box A \to \Box B)$

(A15) $\Box A \to \Box\Box A$

(A16) $\Box A \wedge \Box B \to \Box(A \wedge B)$

(A17) If A is an axiom, $\Box A$

Logic NR

Relevant logic R + $S4$-style of necessity.

(A1) $A \to A$

(A2) $A \to ((A \to B) \to B)$

(A3) $(A \to B) \to ((B \to C) \to (A \to C))$

(A4) $(A \to (A \to B)) \to (A \to B)$

(A5) $A \wedge B \to A$

(A6) $A \wedge B \to B$

(A7) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$

(A8) $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$

(A9) $\neg\neg A \to A$

(A10) $(A \to \neg B) \to (B \to \neg A)$

(A11) $A \vee B \leftrightarrow \neg(\neg A \wedge \neg B)$

(A12) $A \circ B \leftrightarrow \neg(A \to \neg B)$

(A13) $\Box A \to A$

(A14) $\Box(A \to B) \to (\Box A \to \Box B)$

(A15) $\Box A \to \Box\Box A$

(A16) $\Box A \wedge \Box B \to \Box(A \wedge B)$

(A17) If A is an axiom, $\Box A$

Inference Rules:

Logic NR

Relevant logic R + $S4$-style of necessity.

(A1) $A \to A$
(A2) $A \to ((A \to B) \to B)$
(A3) $(A \to B) \to ((B \to C) \to (A \to C))$
(A4) $(A \to (A \to B)) \to (A \to B)$
(A5) $A \wedge B \to A$
(A6) $A \wedge B \to B$
(A7) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
(A8) $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
(A9) $\neg\neg A \to A$
(A10) $(A \to \neg B) \to (B \to \neg A)$
(A11) $A \vee B \leftrightarrow \neg(\neg A \wedge \neg B)$
(A12) $A \circ B \leftrightarrow \neg(A \to \neg B)$

(A13) $\Box A \to A$
(A14) $\Box(A \to B) \to (\Box A \to \Box B)$
(A15) $\Box A \to \Box\Box A$
(A16) $\Box A \wedge \Box B \to \Box(A \wedge B)$
(A17) If A is an axiom, $\Box A$

Inference Rules:

From A and $A \to B$, infer B

From A and B, infer $A \wedge B$.

Semantics

NR-frame: $\langle K, 0, R, S, *\rangle$ with: $(a \leq b := \exists x(S0x \wedge Rxab))$

Semantics

NR-frame: $\langle K, 0, R, S, * \rangle$ with: $(a \leq b := \exists x(S0x \wedge Rxab))$

(P1) Saa

(P2) $Raaa$

(P3) $S^2ab \Rightarrow Sab$

(P4) $R^2abcd \Rightarrow R^2acbd$

(P5) $R|Sabc \Rightarrow \exists x \exists y(Sax \wedge Sby \wedge Rxyc)$

(P6) $a \leq a$

(P7) $a \leq b \wedge Rbcd \Rightarrow Racd$

(P8) $a \leq b \wedge Sbc \Rightarrow Sac$

(P9) $Rabc \Leftrightarrow Rac^*b^*$

(P10) $a^{**} = a$.

Semantics

NR-frame: $\langle K, 0, R, S, * \rangle$ with: $(a \leq b := \exists x(S0x \wedge Rxab))$

(P1) Saa

(P2) $Raaa$

(P3) $S^2ab \Rightarrow Sab$

(P4) $R^2abcd \Rightarrow R^2acbd$

(P5) $R|Sabc \Rightarrow \exists x \exists y(Sax \wedge Sby \wedge Rxyc)$

(P6) $a \leq a$

(P7) $a \leq b \wedge Rbcd \Rightarrow Racd$

(P8) $a \leq b \wedge Sbc \Rightarrow Sac$

(P9) $Rabc \Leftrightarrow Rac^*b^*$

(P10) $a^{**} = a.$

$a \models \Box A$ iff $b \models A$, for all $b \in K$ such that Sab

Outline

Axiom Schemes

Axiom Schemes

(A1) $A \to A$
(A2) $A \to ((A \to B) \to B)$
(A3) $(A \to B) \to ((B \to C) \to (A \to C))$
(A4) $(A \to (A \to B)) \to (A \to B)$
(A5) $A \wedge B \to A$
(A6) $A \wedge B \to B$
(A7) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
(A8) $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
(A9) $\neg\neg A \to A$
(A10) $(A \to \neg B) \to (B \to \neg A)$
(A11) $A \vee B \leftrightarrow \neg(\neg A \wedge \neg B)$
(A12) $A \circ B \leftrightarrow \neg(A \to \neg B)$

Axiom Schemes

(A1) $A \to A$
(A2) $A \to ((A \to B) \to B)$
(A3) $(A \to B) \to ((B \to C) \to (A \to C))$
(A4) $(A \to (A \to B)) \to (A \to B)$
(A5) $A \wedge B \to A$
(A6) $A \wedge B \to B$
(A7) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
(A8) $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
(A9) $\neg\neg A \to A$
(A10) $(A \to \neg B) \to (B \to \neg A)$
(A11) $A \vee B \leftrightarrow \neg(\neg A \wedge \neg B)$
(A12) $A \circ B \leftrightarrow \neg(A \to \neg B)$
(A13) $t : (A \to B) \to (s : A \to (t \cdot s) : B)$
(A14) $t : A \to !t : t : A$
(A15) $t : A \wedge s : B \to (t \tilde{\wedge} s)(A \wedge B)$
(A16) $t : A \to (t + s) : A$ and $t : A \to (s + t) : A$

Inference Rules

Inference Rules

$$\frac{F \qquad F \to G}{G} \qquad\qquad \frac{F \qquad G}{F \wedge G} \qquad\qquad \frac{(c, A) \in \mathsf{CS}}{c : A}$$

A constant specification CS is called *axiomatically appropriate* if for each axiom A there is a constant c such that $(c, A) \in \mathsf{CS}$. As usual in justification logics, we can show the following analogue of the necessitation rule.

Lemma (Constructive necessitation)

Let CS be an axiomatically appropriate constant specification. For each formula A,

$$\mathsf{RJ}_{\mathsf{CS}} \vdash A \quad \textit{implies} \quad \mathsf{RJ}_{\mathsf{CS}} \vdash t : A \textit{ for some term } t.$$

Outline

The semantics for RJ

The semantics for RJ

An $\mathsf{RJ}_{\mathsf{CS}}$-model is a tuple of the form $(K, 0, R, *, \spadesuit, \nu)$ where

1. K is a set;
2. $0 \in K$;
3. R is a ternary relation on K;
4. $*$ is a function $* : K \to K$;
5. $\spadesuit$ is a function $\spadesuit : \mathsf{Tm} \times K \to \mathcal{P}(\mathsf{For})$;
6. ν is a function $\nu : K \to \mathcal{P}(\mathsf{Prop})$.

The semantics for RJ

An RJ_{CS}-model $(K, 0, R, *, \spadesuit, \nu)$ must satisfy the following conditions:

$Raaa$	$R^2abcd \Rightarrow R^2acbd$	$Rabc \Rightarrow t_a^\spadesuit \cdot s_b^\spadesuit \subseteq (t \cdot s)_c^\spadesuit$
$a \leq a$	$a \leq b \wedge Rbcd \Rightarrow Racd$	$a \leq b \Rightarrow t_a^\spadesuit \subseteq t_b^\spadesuit$
$Rabc \Leftrightarrow Rac^*b^*$	$a^{**} = a$	$s_a^\spadesuit \cdot t_a^\spadesuit \subseteq (s \cdot t)_a^\spadesuit$
$s_a^\spadesuit \cup t_a^\spadesuit \subseteq (s + t)_a^\spadesuit$	$A \in t_0^\spadesuit$ if $(t, A) \in$ CS	$t : (t_a^\spadesuit) \subseteq (!t)_a^\spadesuit$
$s_a^\spadesuit \wedge t_a^\spadesuit \subseteq (s \tilde{\wedge} t)_a^\spadesuit$	$a \leq b \Rightarrow \nu(a) \subseteq \nu(b)$	

Satisfiability Relation

Satisfiability Relation

Given a model $\mathcal{M} = (K, 0, R, *, \spadesuit, \nu)$ and $a \in K$ we define:

$$
\begin{array}{rcl}
\mathcal{M}, a \models p & \text{iff} & p \in \nu(a), \text{for } p \in Prop \\
\mathcal{M}, a \models A \wedge B & \text{iff} & \mathcal{M}, a \models A \text{ and } \mathcal{M}, a \models B \\
\mathcal{M}, a \models A \vee B & \text{iff} & \mathcal{M}, a \models A \text{ or } \mathcal{M}, a \models B \\
\mathcal{M}, a \models A \rightarrow B & \text{iff} & Raxy \text{ and } \mathcal{M}, x \models A \text{ imply } \mathcal{M}, y \models B, \text{for all } x, y \in K \\
\mathcal{M}, a \models \neg A & \text{iff} & \mathcal{M}, a* \not\models A \\
\mathcal{M}, a \models t : A & \text{iff} & A \in t_a^{\spadesuit}
\end{array}
$$

Outline

Goal 1

Goal 1

Conjecture 1. [Soundness and Completeness] Let CS be any constant specification. For each formula A we have

$$\mathsf{RJ}_{\mathsf{CS}} \vdash A \quad \text{iff} \quad A \text{ is CS-valid.}$$

Goal 2

Goal 2

Let RLP be the system RJ plus the axiom $t : A \to A$ based on the total constant specification, i.e., every constant justifies every axiom (including $t : A \to A$). A *realization* is a mapping from modal formulas to formulas of justification logic that replaces each $\Box$ with some expression t: (different occurrences of $\Box$ may be replaced with different terms).

Goal 2

Let RLP be the system RJ plus the axiom $t : A \to A$ based on the total constant specification, i.e., every constant justifies every axiom (including $t : A \to A$). A *realization* is a mapping from modal formulas to formulas of justification logic that replaces each $\Box$ with some expression $t:$ (different occurrences of $\Box$ may be replaced with different terms).

Conjecture 2. [Realization] There is a realization r such that for each modal formula A

$$\mathsf{NR} \vdash A \quad \text{implies} \quad \mathsf{RLP} \vdash r(A).$$

Bibliography

Artemov, S.: Explicit provabilty and constructive semantics. Bulletin of Symbolic Logic 7(1), 1–36 (2001)

Meyer, R.K.: Entailment and Relevant Implication. Logique et Analyse no. 44, 472–479 (1968)

Routley, R., Meyer, R.K.: The Semantics of Entailment II. Journal of Philosophical Logic 1, 53–73 (1972)