Undecidability of the Lambek Calculus with a Relevant Modality

Max Kanovich, Stepan Kuznetsov, Andre Scedrov

Basic Categorial Grammar

John loves Mary

Basic Categorial Grammar

John	loves	Mary
np	$(np \backslash s) / np$	np

Basic Categorial Grammar

John	loves	Mary	
np	$(np \backslash s) / np$	np	$\rightarrow s$

Basic Categorial Grammar

	John	loves	Mary	
$\vdash$	np	$(np \setminus s) / np$	np	$\rightarrow s$

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Non-commutativity: $\vdash np, np \backslash s \rightarrow np$ ("John runs"), but $\nvdash np \backslash s, np \rightarrow s)$ ("runs John").

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Reduction rules of BCG: $A, A \backslash B \to B$; $\qquad B / A, A \to B$

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[Ajdukiewicz 1935, Bar-Hillel *et al.* 1960]

Extending Categorial Grammar

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the girl whom John loves

Extending Categorial Grammar

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the	girl	whom	John	loves
np / n	n	$(n \backslash n) / (s / np)$	np	$(np \backslash s) / np$

Extending Categorial Grammar

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$$\underbrace{np \quad (np \backslash s) / np}_{\to s / np}$$

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Extending Categorial Grammar

$$\vdash \quad \underset{\text{John}}{np} \quad \underset{\text{loves}}{(np \backslash s) / np} \quad \underset{\text{Mary}}{np} \quad \to s$$

$$\vdash \quad \underset{\text{the}}{np / n} \quad \underset{\text{girl}}{n} \quad \underset{\text{whom}}{(n \backslash n) / (s / np)} \quad \underbrace{\underset{\text{John}}{np} \quad \underset{\text{loves}}{(np \backslash s) / np}}_{\to s / np} \quad \to np$$

Deriving principles like $np, (np \backslash s) / np \to s / np$ requires extra rules (in this particular case, associativity: $(A \backslash B) / C \leftrightarrow A \backslash (B / C)$).

Extending Categorial Grammar

$$\begin{array}{lcccl} & \text{John} & \text{loves} & \text{Mary} & \\ \vdash & np & (np \backslash s) / np & np & \rightarrow s \end{array}$$

$$\begin{array}{lccccccl} & \text{the} & \text{girl} & \text{whom} & \text{John} & \text{loves} & \\ \vdash & np / n & n & (n \backslash n) / (s / np) & \underbrace{np \quad (np \backslash s) / np}_{\rightarrow s / np} & & \rightarrow np \end{array}$$

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the boy who loves Mary

Extending Categorial Grammar

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	the	girl	whom	John	loves	
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Extending Categorial Grammar

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Extending Categorial Grammar

$$\begin{array}{lcccc} & \text{John} & \text{loves} & \text{Mary} & \\ \vdash & np & (np \backslash s) / np & np & \rightarrow s \end{array}$$

$$\begin{array}{lcccccccc} & \text{the} & \text{girl} & \text{whom}_i & \text{John} & \text{loves} & e_i & \\ \vdash & np / n & n & (n \backslash n) / (s / np) & \underbrace{np \quad (np \backslash s) / np}_{\rightarrow s / np} & & & \rightarrow np \end{array}$$

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Extending Categorial Grammar (cont.)

Another example (from Italian, see [Moot and Retoré 2012]):
"She/He watches the train passing"

Guarda	passare	il	treno
She/He watches	pass	the	train

Extending Categorial Grammar (cont.)

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$\vdash$	$s \,/\, inf$	$inf \,/\, np$	$np \,/\, n$	n	$\rightarrow s$

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Transform into a question: "What does she/he watch passing?"

Cosa	guarda	passare ?	
			$\rightarrow s$

Here we need transitivity: $A \,/\, B, B \,/\, C \rightarrow A \,/\, C$.

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Transform into a question: "What does she/he watch passing?"

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$\vdash$	$q \,/ (s \,/\, np)$	$s \,/\, inf$	$inf \,/\, np$	$\to s$

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$$\underbrace{s \,/\, inf \quad inf \,/\, np}_{\to\, s \,/\, np}$$

Here we need transitivity: $A \,/\, B, B \,/\, C \to A \,/\, C$.

Extending Categorial Grammar: Two Approaches

1. Add necessary principles as extra axioms to BCG
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1. Add necessary principles as extra axioms to BCG
   $\rightsquigarrow$ Combinatory Categorial Grammar (CCG) [Steedman 1996]
2. One calculus to derive them all! $\rightsquigarrow$ Lambek Grammar [Lambek 1958]

The Lambek Calculus ($\mathbf{L}^*$)

$$\overline{A \to A}$$

$$\frac{\Pi \to A \quad \Delta_1, B, \Delta_2 \to C}{\Delta_1, B \,/\, A, \Pi, \Delta_2 \to C} \; (/ \to) \qquad \frac{\Pi, A \to B}{\Pi \to B \,/\, A} \; (\to /)$$

$$\frac{\Pi \to A \quad \Delta_1, B, \Delta_2 \to C}{\Delta_1, \Pi, A \,\backslash\, B, \Delta_2 \to C} \; (\backslash \to) \qquad \frac{A, \Pi \to B}{\Pi \to A \,\backslash\, B} \; (\to \backslash)$$

[Lambek 1958, 1961, ...]

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$$\frac{\Pi \to A \quad \Delta_1, B, \Delta_2 \to C}{\Delta_1, \Pi, A \setminus B, \Delta_2 \to C}\ (\setminus \to) \qquad \frac{A, \Pi \to B}{\Pi \to A \setminus B}\ (\to \setminus)$$

[Lambek 1958, 1961, ...]

$\mathbf{L}^* \vdash (A \setminus B) \,/\, C \leftrightarrow A \setminus (B \,/\, C)$

$\mathbf{L}^* \vdash A \,/\, B, B \,/\, C \to A \,/\, C$

$\ldots$

Properties of the Lambek Calculus

- Lambek grammars generate precisely context-free languages [Pentus 1993].
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- Lambek grammars generate precisely context-free languages [Pentus 1993].
  This means that formally their expressive power is not greater than the power of BCGs.
- The Lambek calculus is NP-complete [Pentus 2006, Savateev 2008].
  (Steedman's CCGs enjoy polynomial-time parsing.)
- Polynomial-time algorithm for fragments of bounded depth [Pentus 2010].
  (Running time $O(2^d n^4)$, where n is the length of the sequent and d is the implication nesting depth.)

Medial Extraction

the girl whom John met yesterday

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the girl whom$_i$ John met e_i yesterday

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the girl whom$_i$ $\underbrace{\text{John met } e_i \text{ yesterday}}$

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“John met yesterday” has the *gap* (e_i) in the middle and therefore is neither $s \,/\, np$, nor $np \,\backslash\, s$. A substructural modality, !, equipped with permutation rules, can be used to handle such situations, placing the *np* to the correct place [Moortgat 2006, Morrill 2011].

Medial Extraction

$$\text{the girl} \qquad \text{whom}_i \qquad \underbrace{\text{John met } e_i \text{ yesterday}}_{\rightarrow s \,/\, !np}$$

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$$\frac{\Delta_1, !A, \Delta_2, \Delta_3 \rightarrow C}{\Delta_1, \Delta_2, !A, \Delta_3 \rightarrow C} \ (\mathrm{perm}_1) \qquad \frac{\Delta_1, A, \Delta_2 \rightarrow C}{\Delta_1, !A, \Delta_2 \rightarrow C} \ (! \rightarrow)$$

Medial Extraction

$$\text{the girl} \quad \underset{(n \backslash n) / (s / !np)}{\text{whom}_i} \quad \underbrace{\text{John met } e_i \text{ yesterday}}_{\to s / !np}$$

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$$\dfrac{\dfrac{\dfrac{np, (np \backslash s) / np, np, (np \backslash s) \backslash (np \backslash s) \to s}{np, (np \backslash s) / np, !np, (np \backslash s) \backslash (np \backslash s) \to s}\,(! \to)}{np, (np \backslash s) / np, (np \backslash s) \backslash (np \backslash s), !np \to s}\,(\mathrm{perm}_1)}{np, (np \backslash s) / np, (np \backslash s) \backslash (np \backslash s) \to s / !np}\,(\to /)$$

Medial Extraction

the girl	whom$_i$	John met e_i yesterday	
...	$(n \backslash n) / (s / !np)$	$\underbrace{}_{\to s / !np}$	$\to np$

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$$\cfrac{\cfrac{\cfrac{np, (np \backslash s) / np, np, (np \backslash s) \backslash (np \backslash s) \to s}{np, (np \backslash s) / np, !np, (np \backslash s) \backslash (np \backslash s) \to s}\ (! \to)}{np, (np \backslash s) / np, (np \backslash s) \backslash (np \backslash s), !np \to s}\ (\mathrm{perm}_1)}{np, (np \backslash s) / np, (np \backslash s) \backslash (np \backslash s) \to s / !np}\ (\to /)$$

“Parasitic” extraction

the paper that John signed without reading

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the paper that$_i$ John signed e_i without reading e_i

“Parasitic” extraction

$$\text{the paper} \quad \text{that}_i \quad \underbrace{\text{John signed } e_i \text{ without reading } e_i}_{\rightarrow\, s \,/\, !np}$$

In this example we have 2 gaps and use the contraction rule for ! [Morrill 2011, Morrill and Valentín 2015].

“Parasitic” extraction

$$\text{the paper} \quad \text{that}_i \quad \underbrace{\text{John signed } e_i \text{ without reading } e_i}_{\to\ s\,/\,!np}$$

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$$\frac{\Delta_1, !A, !A, \Delta_2 \to C}{\Delta_1, !A, \Delta_2 \to C}\ (\mathrm{contr})$$

"Parasitic" extraction

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$$\frac{\Delta_1, !A, \Delta_2, \Delta_3 \to C}{\Delta_1, \Delta_2, !A, \Delta_3 \to C}\ (\mathrm{perm}_1) \qquad \frac{\Delta_1, A, \Delta_2 \to C}{\Delta_1, !A, \Delta_2 \to C}\ (! \to)$$

$$\frac{\Delta_1, !A, !A, \Delta_2 \to C}{\Delta_1, !A, \Delta_2 \to C}\ (\mathrm{contr})$$

yields undecidability

The Lambek Calculus with Product and the Unit ($\mathbf{L}^{\mathbf{1}}$)

$$\frac{}{A \to A}$$

$$\frac{\Pi \to A \quad \Delta_1, B, \Delta_2 \to C}{\Delta_1, B \,/\, A, \Pi, \Delta_2 \to C}\ (/ \to) \qquad \frac{\Pi, A \to B}{\Pi \to B \,/\, A}\ (\to /)$$

$$\frac{\Pi \to A \quad \Delta_1, B, \Delta_2 \to C}{\Delta_1, \Pi, A \setminus B, \Delta_2 \to C}\ (\setminus \to) \qquad \frac{A, \Pi \to B}{\Pi \to A \setminus B}\ (\to \setminus)$$

$$\frac{\Delta_1, A, B, \Delta_2 \to C}{\Delta_1, A \cdot B, \Delta_2 \to C}\ (\cdot \to) \qquad \frac{\Pi_1 \to A \quad \Pi_2 \to B}{\Pi_1, \Pi_2 \to A \cdot B}\ (\to \cdot)$$

$$\frac{\Delta_1, \Delta_2 \to C}{\Delta_1, \mathbf{1}, \Delta_2 \to C}\ (\mathbf{1} \to) \qquad \frac{}{\to \mathbf{1}}\ (\to \mathbf{1})$$

[Lambek 1969]

$!\mathbf{L}^1$ (The Lambek Calculus with a Relevant Modality)

$$\overline{A \to A}$$

$(\backslash \to)$ $(\to \backslash)$ $(/ \to)$ $(\to /)$ $(\cdot \to)$ $(\to \cdot)$ $(\mathbf{1} \to)$ $(\to \mathbf{1})$

$$\frac{\Delta_1, A, \Delta_2 \to C}{\Delta_1, !A, \Delta_2 \to C}\ (! \to) \qquad \frac{!A_1, \ldots, !A_n \to C}{!A_1, \ldots, !A_n \to !C}\ (\to !)$$

$$\frac{\Delta_1, !A, \Delta_2, \Delta_3 \to C}{\Delta_1, \Delta_2, !A, \Delta_3 \to C}\ (\mathrm{perm}_1) \qquad \frac{\Delta_1, \Delta_2, !A, \Delta_3 \to C}{\Delta_1, !A, \Delta_2, \Delta_3 \to C}\ (\mathrm{perm}_2)$$

$$\frac{\Delta_1, !A, !A, \Delta_2 \to C}{\Delta_1, !A, \Delta_2 \to C}\ (\mathrm{contr})$$

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$$\frac{\Delta_1, A, \Delta_2 \to C}{\Delta_1, !A, \Delta_2 \to C}\ (! \to) \qquad \frac{!A_1, \ldots, !A_n \to C}{!A_1, \ldots, !A_n \to !C}\ (\to !)$$

$$\frac{\Delta_1, !A, \Delta_2, \Delta_3 \to C}{\Delta_1, \Delta_2, !A, \Delta_3 \to C}\ (\mathrm{perm}_1) \qquad \frac{\Delta_1, \Delta_2, !A, \Delta_3 \to C}{\Delta_1, !A, \Delta_2, \Delta_3 \to C}\ (\mathrm{perm}_2)$$

$$\frac{\Delta_1, !A, !A, \Delta_2 \to C}{\Delta_1, !A, \Delta_2 \to C}\ (\mathrm{contr})$$

A conservative fragment of **Db!** by Morrill and Valentín (2015).

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$$\frac{\Delta_1, !A, !A, \Delta_2 \to C}{\Delta_1, !A, \Delta_2 \to C}\ (\mathrm{contr})$$

Theorem

The derivability problem for $!\mathbf{L}^{\mathbf{1}}$ *is undecidable.*

[Kanovich, Kuznetsov, Scedrov 2016] (Solves an open problem by Morrill and Valentín.)

Proof Idea

Encoding type-0 grammars (semi-Thue systems), cf. Lincoln et al. (1992).
Type-0 (unrestricted) grammars can generate arbitrary r.e. languages.

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Rewriting rule application: $s \Rightarrow \ldots \Rightarrow \eta\, u_1 \ldots u_k\, \theta \Rightarrow \eta\, v_1 \ldots v_m\, \theta$

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In categorial grammar:

$$\dfrac{\dfrac{\eta, u_1, \ldots, u_k, \theta \to s}{\ldots}}{\eta, v_1, \ldots, v_m, \theta \to s}$$

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In categorial grammar:

$$\dfrac{\dfrac{v_1, \ldots, v_m \to v_1 \cdot \ldots \cdot v_m \quad \eta, u_1, \ldots, u_k, \theta \to s}{\eta, (u_1 \cdot \ldots u_k) / (v_1 \cdot \ldots \cdot v_m), v_1, \ldots, v_m, \theta \to s}\,(/ \to), (\cdot \to)^*}{\eta, v_1, \ldots, v_m, \theta \to s}$$

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In categorial grammar:

$$\dfrac{\dfrac{v_1, \ldots, v_m \to v_1 \cdot \ldots \cdot v_m \quad !\Gamma', \eta, u_1, \ldots, u_k, \theta \to s}{!\Gamma', \eta, (u_1 \cdot \ldots u_k) / (v_1 \cdot \ldots \cdot v_m), v_1, \ldots, v_m, \theta \to s} (/ \to), (\cdot \to)^*}{!\Gamma, \eta, v_1, \ldots, v_m, \theta \to s} (\mathrm{contr}), (\mathrm{perm})$$

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In categorial grammar:

$$\dfrac{\dfrac{v_1, \ldots, v_m \to v_1 \cdot \ldots \cdot v_m \qquad !\Gamma', \eta, u_1, \ldots, u_k, \theta \to s}{!\Gamma', \eta, (u_1 \cdot \ldots u_k) / (v_1 \cdot \ldots \cdot v_m), v_1, \ldots, v_m, \theta \to s}\ (/\to), (\cdot\to)^*}{!\Gamma, \eta, v_1, \ldots, v_m, \theta \to s}\ (\mathrm{contr}), (\mathrm{perm})$$

In $!\Gamma$, we keep only the formulae **relevant** to the derivation above (i.e., actually used in it). Thus, we end up with an empty $!\Gamma$ at the axiom level.

Proof Idea

Encoding type-0 grammars (semi-Thue systems), cf. Lincoln et al. (1992). Type-0 (unrestricted) grammars can generate arbitrary r.e. languages.

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In categorial grammar:

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In $!\Gamma$, we keep only the formulae **relevant** to the derivation above (i.e., actually used in it). Thus, we end up with an empty $!\Gamma$ at the axiom level.

Buszkowski (1982): encode type-0 grammars with only one operation, $/$.

Decidable Fragment of $\mathbf{!L}^1$

If ! is applied only to variables (e.g., $!np$), the calculus is decidable.

Decidable Fragment of $\mathbf{!L^1}$

If ! is applied only to variables (e.g., $!np$), the calculus is NP-complete.

Lambek's Non-Emptiness Restriction

In the original Lambek calculus, all antecedents are forced to be non-empty.

very interesting book

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book	$\triangleright$	n	(noun)
interesting	$\triangleright$	n / n	(adjective = left noun modifier)
very	$\triangleright$	$(n / n) / (n / n)$	(adverb = left adjective modifier)

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$$\underbrace{(n \,/\, n) \,/ (n \,/\, n)}_{\text{very}}, \quad \underbrace{n \,/\, n}_{\text{interesting}}, \quad \underbrace{n}_{\text{book}}$$

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$$\underset{\text{very}}{(n / n) / (n / n),} \quad \underset{\text{interesting}}{n / n,} \quad \underset{\text{book}}{n} \quad \to n$$

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$$\mathbf{L} \vdash \underbrace{(n / n) / (n / n)}_{\text{very}}, \quad \underbrace{n / n}_{\text{interesting}}, \quad \underbrace{n}_{\text{book}} \quad \to n$$

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$$\mathbf{L} \vdash \underset{\text{very}}{(n / n) / (n / n)}, \quad \underset{\text{interesting}}{n / n}, \quad \underset{\text{book}}{n} \quad \to n$$

$$\mathbf{L}^* \vdash (n / n) / (n / n), \quad n \quad \to n$$

Lambek's Non-Emptiness Restriction

In the original Lambek calculus, all antecedents are forced to be non-empty.

book	$\triangleright$	n	(noun)
interesting	$\triangleright$	$n \,/\, n$	(adjective $=$ left noun modifier)
very	$\triangleright$	$(n \,/\, n) \,/ (n \,/\, n)$	(adverb $=$ left adjective modifier)

$$\mathbf{L} \vdash \underbrace{(n \,/\, n) \,/ (n \,/\, n)}_{\text{very}}, \quad \underbrace{n \,/\, n}_{\text{interesting}}, \quad \underbrace{n}_{\text{book}} \quad \to n$$

$$\mathbf{L}^{*} \vdash \underbrace{(n \,/\, n) \,/ (n \,/\, n)}_{\text{very}}, \quad \underbrace{n}_{\text{book}} \quad \to n$$

Issues with Lambek's Restriction with !

[Kanovich, Kuznetsov, Scedrov arXiv 1608.02254]
Let **EL** be an arbitrary extension of the Lambek calculus (**L**) with Lambek's restriction that satisfies the following natural conditions:

1. If $\mathbf{L} \vdash \Pi \to A$, then $\mathbf{EL} \vdash \Pi \to A$.
2. Cut, substitution and monotonicity rules are admissible in **EL**.
3. The rules $(/\to)$, $(\backslash\to)$, $(\cdot\to)$, (contr), and $(\mathrm{perm}_{1,2})$ are admissible in **EL**.
4. If Π contains a formula without occurrences of ! and B does not contain occurrences of !, than the rules $(\to /)$ and $(\to \backslash)$ are admissible in **EL**.
5. If Δ_1 or Δ_2 contains a formula without occurrences of !, then the $(! \to)$ rule is admissible.

Then, for a sequent $\Pi \to B$ with only one variable p, if it is deriable in $\mathbf{L}^*$, the sequent $!(p\,/\,p), \Pi \to B$ is derivable in **EL**. Thus, Lambek's restriction is violated by adding only one formula with !.

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!(Thank you)