Undecidability of the Lambek Calculus Extended with Subexponential and Bracket Modalities

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Why This Talk?

Why This Talk?

A study on the borderline of **practical motivation** and **undecidability.**

Linguistic Intro: Lambek Grammar

John loves Mary

Linguistic Intro: Lambek Grammar

John	loves	Mary
N	$(N \setminus S) / N$	N

Linguistic Intro: Lambek Grammar

John	loves	Mary	
N	$(N \setminus S) / N$	N	$\rightarrow S$

Linguistic Intro: Lambek Grammar

	John	loves	Mary	
$\vdash$	N	$(N \backslash S) / N$	N	$\to S$

Linguistic Intro: Lambek Grammar

	John	loves	Mary	
$\vdash$	N	$(N \setminus S) / N$	N	$\to S$

the girl whom John loves

Linguistic Intro: Lambek Grammar

	John	loves	Mary	
$\vdash$	N	$(N \backslash S) / N$	N	$\rightarrow S$

the	girl	whom	John	loves
N / CN	CN	$(CN \backslash CN) / (S / N)$	N	$(N \backslash S) / N$

Linguistic Intro: Lambek Grammar

	John	loves	Mary	
$\vdash$	N	$(N \backslash S) / N$	N	$\rightarrow S$

the	girl	whom	John	loves
N / CN	CN	$(CN \backslash CN) / (S / N)$	N	$(N \backslash S) / N$

$$\underbrace{N \quad (N \backslash S) / N}_{\rightarrow S / N}$$

Linguistic Intro: Lambek Grammar

$$\begin{array}{ccccc} & \text{John} & \text{loves} & \text{Mary} & \\ \vdash & N & (N \backslash S) / N & N & \to S \end{array}$$

$$\begin{array}{cccccccc} & \text{the} & \text{girl} & \text{whom} & \text{John} & \text{loves} & \\ \vdash & N / CN & CN & {\color{red}(CN \backslash CN) / (S / N)} & \underbrace{N \quad (N \backslash S) / N}_{\to S / N} & & \to N \end{array}$$

Linguistic Intro: Lambek Grammar

	John	loves	Mary	
$\vdash$	N	$(N \backslash S) / N$	N	$\to S$

	the	girl	whom	John	loves	
$\vdash$	N / CN	CN	$(CN \backslash CN) / (S / N)$	N	$(N \backslash S) / N$	$\to N$

$$\underbrace{N \quad (N \backslash S) / N}_{\to S / N}$$

the	boy	who	loves	Mary

Linguistic Intro: Lambek Grammar

	John	loves	Mary	
$\vdash$	N	$(N \backslash S) / N$	N	$\rightarrow S$

	the	girl	whom	John	loves	
$\vdash$	N / CN	CN	$(CN \backslash CN) / (S / N)$	N	$(N \backslash S) / N$	$\rightarrow N$

$$\underbrace{N \quad (N \backslash S) / N}_{\rightarrow S / N}$$

the	boy	who	loves	Mary
N / CN	CN	$(CN \backslash CN) / (N \backslash S)$	$(N \backslash S) / N$	N

Linguistic Intro: Lambek Grammar

$$\vdash \underset{N}{\text{John}} \quad \underset{(N \backslash S) / N}{\text{loves}} \quad \underset{N}{\text{Mary}} \quad \to S$$

$$\vdash \underset{N / CN}{\text{the}} \quad \underset{CN}{\text{girl}} \quad \underset{\color{red}{(CN \backslash CN) / (S / N)}}{\text{whom}} \quad \underbrace{\underset{N}{\text{John}} \quad \underset{(N \backslash S) / N}{\text{loves}}}_{\to S / N} \quad \to N$$

$$\underset{N / CN}{\text{the}} \quad \underset{CN}{\text{boy}} \quad \underset{\color{red}{(CN \backslash CN) / (N \backslash S)}}{\text{who}} \quad \underbrace{\underset{(N \backslash S) / N}{\text{loves}} \quad \underset{N}{\text{Mary}}}_{\to N \backslash S}$$

Linguistic Intro: Lambek Grammar

$$\vdash \underset{N}{\text{John}} \quad \underset{(N\backslash S)/N}{\text{loves}} \quad \underset{N}{\text{Mary}} \quad \to S$$

$$\vdash \underset{N/CN}{\text{the}} \quad \underset{CN}{\text{girl}} \quad \underset{\color{red}{(CN\backslash CN)/(S/N)}}{\text{whom}} \quad \underbrace{\underset{N}{\text{John}} \quad \underset{(N\backslash S)/N}{\text{loves}}}_{\to S/N} \quad \to N$$

$$\vdash \underset{N/CN}{\text{the}} \quad \underset{CN}{\text{boy}} \quad \underset{\color{red}{(CN\backslash CN)/(N\backslash S)}}{\text{who}} \quad \underbrace{\underset{(N\backslash S)/N}{\text{loves}} \quad \underset{N}{\text{Mary}}}_{\to N\backslash S} \quad \to N$$

Linguistic Intro: Lambek Grammar

$$
\begin{array}{lcccl}
 & \text{John} & \text{loves} & \text{Mary} & \\
\vdash & N & (N \backslash S) / N & N & \rightarrow S
\end{array}
$$

$$
\begin{array}{lcccccl}
 & \text{the} & \text{girl} & \text{whom}_i & \text{John} & \text{loves} \quad e_i & \\
\vdash & N / CN & CN & (CN \backslash CN) / (S / N) & \underbrace{N \quad (N \backslash S) / N}_{\rightarrow S / N} & & \rightarrow N
\end{array}
$$

$$
\begin{array}{lcccccl}
 & \text{the} & \text{boy} & \text{who}_i & e_i \quad \text{loves} & \text{Mary} & \\
\vdash & N / CN & CN & (CN \backslash CN) / (N \backslash S) & \underbrace{(N \backslash S) / N \quad N}_{\rightarrow N \backslash S} & & \rightarrow N
\end{array}
$$

The Lambek Calculus

[Lambek 1958, 1961]

$$\frac{}{A \to A}$$

$$\frac{\Pi \to A \quad \Delta_1, B, \Delta_2 \to C}{\Delta_1, B / A, \Pi, \Delta_2 \to C}\ (/ \to) \qquad \frac{\Pi, A \to B}{\Pi \to B / A}\ (\to /)$$

$$\frac{\Pi \to A \quad \Delta_1, B, \Delta_2 \to C}{\Delta_1, \Pi, A \backslash B, \Delta_2 \to C}\ (\backslash \to) \qquad \frac{A, \Pi \to B}{\Pi \to A \backslash B}\ (\to \backslash)$$

$$\frac{\Delta_1, A, B, \Delta_2 \to C}{\Delta_1, A \cdot B, \Delta_2 \to C}\ (\cdot \to) \qquad \frac{\Pi_1 \to A \quad \Pi_2 \to B}{\Pi_1, \Pi_2 \to A \cdot B}\ (\to \cdot)$$

$$\frac{\Pi \to A \quad \Gamma, A, \Delta \to C}{\Gamma, \Pi, \Delta \to C}\ (\mathrm{cut})$$

The Lambek Calculus

[Lambek 1958, 1961]

$$\frac{}{A \to A}$$

$$\frac{\Pi \to A \quad \Delta_1, B, \Delta_2 \to C}{\Delta_1, B \,/\, A, \Pi, \Delta_2 \to C}\ (/ \to) \qquad \frac{\Pi, A \to B}{\Pi \to B \,/\, A}\ (\to /)$$

$$\frac{\Pi \to A \quad \Delta_1, B, \Delta_2 \to C}{\Delta_1, \Pi, A \,\backslash\, B, \Delta_2 \to C}\ (\backslash \to) \qquad \frac{A, \Pi \to B}{\Pi \to A \,\backslash\, B}\ (\to \backslash)$$

$$\frac{\Delta_1, A, B, \Delta_2 \to C}{\Delta_1, A \cdot B, \Delta_2 \to C}\ (\cdot \to) \qquad \frac{\Pi_1 \to A \quad \Pi_2 \to B}{\Pi_1, \Pi_2 \to A \cdot B}\ (\to \cdot)$$

$$\frac{\Pi \to A \quad \Gamma, A, \Delta \to C}{\Gamma, \Pi, \Delta \to C}\ (\mathrm{cut})$$

- Notice: we don't impose Lambek's non-emptiness restriction on the left-hand sides of sequents.

The Lambek Calculus

[Lambek 1958, 1961]

$$\frac{}{A \to A}$$

$$\frac{\Pi \to A \quad \Delta_1, B, \Delta_2 \to C}{\Delta_1, B / A, \Pi, \Delta_2 \to C} \ (/ \to) \qquad \frac{\Pi, A \to B}{\Pi \to B / A} \ (\to /)$$

$$\frac{\Pi \to A \quad \Delta_1, B, \Delta_2 \to C}{\Delta_1, \Pi, A \backslash B, \Delta_2 \to C} \ (\backslash \to) \qquad \frac{A, \Pi \to B}{\Pi \to A \backslash B} \ (\to \backslash)$$

$$\frac{\Delta_1, A, B, \Delta_2 \to C}{\Delta_1, A \cdot B, \Delta_2 \to C} \ (\cdot \to) \qquad \frac{\Pi_1 \to A \quad \Pi_2 \to B}{\Pi_1, \Pi_2 \to A \cdot B} \ (\to \cdot)$$

$$\frac{\Pi \to A \quad \Gamma, A, \Delta \to C}{\Gamma, \Pi, \Delta \to C} \ (\mathrm{cut})$$

- Notice: we don't impose Lambek's non-emptiness restriction on the left-hand sides of sequents.
- The cut rule is eliminable [Lambek 1958].

Unwanted Derivations

book which John laughed without reading

Unwanted Derivations

book	which	John laughed without reading
CN	$(CN \backslash CN)/(S / N)$	$\underbrace{\qquad\qquad\qquad\qquad}_{S / N}$

Unwanted Derivations

$$\vdash \underset{CN}{\text{book}} \quad \underset{(CN \backslash CN)/(S/N)}{\text{which}} \quad \underbrace{\text{John laughed without reading}}_{S/N} \rightarrow CN$$

Unwanted Derivations

$$\vdash \underset{CN}{\text{book}} \quad \underset{(CN \backslash CN)/(S / N)}{\text{which}} \quad \underbrace{\text{John laughed without reading}}_{S / N} \rightarrow CN$$

Unwanted Derivations

$$\begin{array}{lllll} * & \text{book} & \text{which} & \text{John laughed without reading} & \\ \vdash & CN & (CN \setminus CN)/(S / N) & \underbrace{\hphantom{\text{John laughed without reading}}}_{S / N} & \to CN \end{array}$$

Unwanted Derivations

*	book	which	John laughed without reading	
$\vdash$	CN	$(CN \backslash CN)/(S/N)$	$\underbrace{}_{S/N}$	$\rightarrow CN$

*	girl	who	John likes Mary and Pete likes

Unwanted Derivations

$$\begin{array}{llll} * & \text{book} & \text{which} & \text{John laughed without reading} \\ \vdash & CN & (CN \backslash CN)/(S/N) & \underbrace{\hphantom{\text{John laughed without reading}}}_{S/N} \rightarrow CN \end{array}$$

$$\begin{array}{llll} * & \text{girl} & \text{who} & \text{John likes Mary and Pete likes} \\ \vdash & CN & (CN \backslash CN)/(S/N) & \underbrace{\hphantom{\text{John likes Mary and Pete likes}}}_{S/N} \rightarrow CN \end{array}$$

Unwanted Derivations

$$\begin{array}{llll} * & \text{book} & \text{which} & \text{John laughed without reading} \\ \vdash & CN & (CN \setminus CN) / (S / N) & \underbrace{\phantom{\text{John laughed without reading}}}_{S / N} \rightarrow CN \end{array}$$

$$\begin{array}{llll} * & \text{girl} & \text{who} & \text{John likes Mary and Pete likes} \\ \vdash & CN & (CN \setminus CN) / (S / N) & \underbrace{\phantom{\text{John likes Mary and Pete likes}}}_{S / N} \rightarrow CN \end{array}$$

(cf. "John likes Mary and Pete likes Kate" $\rightarrow S$; "and" is of type $S \setminus S / S$)

The Lambek Calculus with Brackets

[Morrill 1992, Moortgat 1995]

$$\frac{}{A \to A}$$

$$\frac{\Pi \to A \quad \Delta(B) \to C}{\Delta(\Pi, A \setminus B) \to C} \qquad \frac{A, \Pi \to B}{\Pi \to A \setminus B} \qquad \frac{\Gamma(A, B) \to C}{\Gamma(A \cdot B) \to C}$$

$$\frac{\Pi \to A \quad \Delta(B) \to C}{\Delta(B \,/\, A, \Pi) \to C} \qquad \frac{\Pi, A \to B}{\Pi \to B \,/\, A} \qquad \frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A \cdot B}$$

$$\frac{\Delta([A]) \to C}{\Delta(\langle\rangle A) \to C} \qquad \frac{\Pi \to A}{[\Pi] \to \langle\rangle A} \qquad \frac{\Delta(A) \to C}{\Delta([[]^{-1} A]) \to C} \qquad \frac{[\Pi] \to A}{\Pi \to []^{-1} A}$$

The Lambek Calculus with Brackets

[Morrill 1992, Moortgat 1995]

$$\frac{}{A \to A}$$

$$\frac{\Pi \to A \quad \Delta(B) \to C}{\Delta(\Pi, A \backslash B) \to C} \qquad \frac{A, \Pi \to B}{\Pi \to A \backslash B} \qquad \frac{\Gamma(A, B) \to C}{\Gamma(A \cdot B) \to C}$$

$$\frac{\Pi \to A \quad \Delta(B) \to C}{\Delta(B / A, \Pi) \to C} \qquad \frac{\Pi, A \to B}{\Pi \to B / A} \qquad \frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A \cdot B}$$

$$\frac{\Delta([A]) \to C}{\Delta(\langle\rangle A) \to C} \qquad \frac{\Pi \to A}{[\Pi] \to \langle\rangle A} \qquad \frac{\Delta(A) \to C}{\Delta([[]^{-1} A]) \to C} \qquad \frac{[\Pi] \to A}{\Pi \to []^{-1} A}$$

- Brackets introduce *controlled non-associativity.*

The Lambek Calculus with Brackets

[Morrill 1992, Moortgat 1995]

$$\frac{}{A \to A}$$

$$\frac{\Pi \to A \quad \Delta(B) \to C}{\Delta(\Pi, A \setminus B) \to C} \qquad \frac{A, \Pi \to B}{\Pi \to A \setminus B} \qquad \frac{\Gamma(A, B) \to C}{\Gamma(A \cdot B) \to C}$$

$$\frac{\Pi \to A \quad \Delta(B) \to C}{\Delta(B \,/\, A, \Pi) \to C} \qquad \frac{\Pi, A \to B}{\Pi \to B \,/\, A} \qquad \frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A \cdot B}$$

$$\frac{\Delta([A]) \to C}{\Delta(\langle\rangle A) \to C} \qquad \frac{\Pi \to A}{[\Pi] \to \langle\rangle A} \qquad \frac{\Delta(A) \to C}{\Delta([[]^{-1} A]) \to C} \qquad \frac{[\Pi] \to A}{\Pi \to []^{-1} A}$$

- Brackets introduce *controlled non-associativity.*
- Cut elimination proved by Moortgat [1996].

Islands: Blocking Unwanted Derivations Using Brackets

Islands: Blocking Unwanted Derivations Using Brackets

- book which John laughed without reading

Islands: Blocking Unwanted Derivations Using Brackets

- book which John laughed [without reading]

Islands: Blocking Unwanted Derivations Using Brackets

- book which John laughed [without reading]

$$CN, (CN \backslash CN) / (S / CN), N, N \backslash S, [[]^{-1}((N \backslash S) \backslash (N \backslash S)) / (N \backslash S), (N \backslash S) / N] \rightarrow CN$$

Islands: Blocking Unwanted Derivations Using Brackets

- book which John laughed [without reading]

$$CN, (CN \backslash CN) / (S / CN), N, N \backslash S, [[]^{-1}((N \backslash S) \backslash (N \backslash S)) / (N \backslash S), (N \backslash S) / N] \rightarrow CN$$

This sequent is not derivable.

Islands: Blocking Unwanted Derivations Using Brackets

- book which John laughed [without reading]

$CN, (CN \backslash CN) / (S / CN), N, N \backslash S, [[]^{-1}((N \backslash S) \backslash (N \backslash S)) / (N \backslash S), (N \backslash S) / N] \to CN$

  This sequent is not derivable.

- girl who John likes Mary and Pete likes

Islands: Blocking Unwanted Derivations Using Brackets

- book which John laughed [without reading]

$CN, (CN \setminus CN) / (S / CN), N, N \setminus S, [[]^{-1}((N \setminus S) \setminus (N \setminus S)) / (N \setminus S), (N \setminus S) / N] \rightarrow CN$

  This sequent is not derivable.

- girl who [John likes Mary and Pete likes]

Islands: Blocking Unwanted Derivations Using Brackets

- book which John laughed [without reading]

$CN, (CN \backslash CN) / (S / CN), N, N \backslash S, [[]^{-1}((N \backslash S) \backslash (N \backslash S)) / (N \backslash S), (N \backslash S) / N] \to CN$

This sequent is not derivable.

- girl who [John likes Mary and Pete likes]

$CN, (CN \backslash CN) / (S / CN), [N, (N \backslash S) / N, N, (S \backslash []^{-1}S) / S, N, (N \backslash S) / N] \to CN$

Islands: Blocking Unwanted Derivations Using Brackets

- book which John laughed [without reading]

$CN, (CN \backslash CN) / (S / CN), N, N \backslash S, [[]^{-1}((N \backslash S) \backslash (N \backslash S)) / (N \backslash S), (N \backslash S) / N] \to CN$

  This sequent is not derivable.

- girl who [John likes Mary and Pete likes]

$CN, (CN \backslash CN) / (S / CN), [N, (N \backslash S) / N, N, (S \backslash []^{-1} S) / S, N, (N \backslash S) / N] \to CN$

  Neither is this one.

Subexponential: Medial Extraction

the girl whom John met yesterday

Subexponential: Medial Extraction

the girl whom$_i$ John met e_i yesterday

Subexponential: Medial Extraction

the girl whom$_i$ John met e_i yesterday

Subexponential: Medial Extraction

the girl whom$_i$ $\underbrace{\text{John met } e_i \text{ yesterday}}_{\to S \,/\, !N}$

$$\frac{\Delta(!A, \Gamma) \to C}{\Delta(\Gamma, !A) \to C}\ (\mathrm{perm}_1) \qquad \frac{\Delta(A) \to C}{\Delta(!A) \to C}\ (! \to)$$

Subexponential: Medial Extraction

the girl whom$_i$ $\underbrace{\text{John met } e_i \text{ yesterday}}_{\to\, S \,/\, !N}$

$$\frac{\Delta(!A, \Gamma) \to C}{\Delta(\Gamma, !A) \to C}\ (\mathrm{perm}_1) \qquad \frac{\Delta(A) \to C}{\Delta(!A) \to C}\ (! \to)$$

$$\cfrac{\cfrac{\cfrac{N, (N \setminus S) / N, N, (N \setminus S) \setminus (N \setminus S) \to S}{N, (N \setminus S) / N, !N, (N \setminus S) \setminus (N \setminus S) \to S}\ (! \to)}{N, (N \setminus S) / N, (N \setminus S) \setminus (N \setminus S), !N \to S}\ (\mathrm{perm}_1)}{N, (N \setminus S) / N, (N \setminus S) \setminus (N \setminus S) \to S \,/\, !N}\ (\to /)$$

Subexponential: Medial Extraction

the girl — whom$_i$ $(CN \setminus CN) / (S / {!N})$ — $\underbrace{\text{John met } e_i \text{ yesterday}}_{\to S / {!N}}$

$$\frac{\Delta(!A, \Gamma) \to C}{\Delta(\Gamma, !A) \to C}\ (\mathrm{perm}_1) \qquad \frac{\Delta(A) \to C}{\Delta(!A) \to C}\ (! \to)$$

$$\cfrac{\cfrac{\cfrac{N, (N \setminus S) / N, N, (N \setminus S) \setminus (N \setminus S) \to S}{N, (N \setminus S) / N, !N, (N \setminus S) \setminus (N \setminus S) \to S}\ (! \to)}{N, (N \setminus S) / N, (N \setminus S) \setminus (N \setminus S), !N \to S}\ (\mathrm{perm}_1)}{N, (N \setminus S) / N, (N \setminus S) \setminus (N \setminus S) \to S / !N}\ (\to /)$$

Subexponential: Medial Extraction

the girl	whom$_i$	John met e_i yesterday	
$\ldots$	$(CN \setminus CN) / (S / {!N})$	$\underbrace{\hphantom{John met e yesterday}}_{\to S / {!N}}$	$\to N$

$$\frac{\Delta(!A, \Gamma) \to C}{\Delta(\Gamma, !A) \to C}\ (\mathrm{perm}_1) \qquad \frac{\Delta(A) \to C}{\Delta(!A) \to C}\ (! \to)$$

$$\cfrac{\cfrac{\cfrac{N, (N \setminus S) / N, N, (N \setminus S) \setminus (N \setminus S) \to S}{N, (N \setminus S) / N, !N, (N \setminus S) \setminus (N \setminus S) \to S}\ (! \to)}{N, (N \setminus S) / N, (N \setminus S) \setminus (N \setminus S), !N \to S}\ (\mathrm{perm}_1)}{N, (N \setminus S) / N, (N \setminus S) \setminus (N \setminus S) \to S / !N}\ (\to /)$$

Subexponential: Parasitic Extraction

the paper   that   John signed   without reading

Subexponential: Parasitic Extraction

the paper that$_i$ John signed e_i without reading e_i

Subexponential: Parasitic Extraction

the paper that$_i$ $\underbrace{\text{John signed } e_i \text{ without reading } e_i}_{\rightarrow S \,/\, !N}$

Subexponential: Parasitic Extraction

$$\text{the paper} \quad \text{that}_i \quad \underbrace{\text{John signed } e_i \text{ without reading } e_i}_{\to S \,/\, !N}$$

$$\frac{\Delta(!A, \Gamma) \to C}{\Delta(\Gamma, !A) \to C}\ (\mathrm{perm}_1) \qquad \frac{\Delta(A) \to C}{\Delta(!A) \to C}\ (! \to)$$

$$\frac{\Delta(!A, !A) \to C}{\Delta(!A) \to C}\ (\mathrm{contr})$$

Subexponential: Parasitic Extraction

the paper that$_i$ $\underbrace{\text{John signed } e_i \text{ [without reading } e_i]}_{\to S \,/\, !N}$

$$\frac{\Delta(!A, \Gamma) \to C}{\Delta(\Gamma, !A) \to C} \ (\text{perm}_1) \qquad \frac{\Delta(A) \to C}{\Delta(!A) \to C} \ (! \to)$$

$$\frac{\Delta(!A, !A) \to C}{\Delta(!A) \to C} \ (\text{contr})$$

Subexponential: Parasitic Extraction

the paper that$_i$ $\underbrace{\text{John signed } e_i \text{ [without reading } e_i]}_{\to S \,/\, !N}$

$$\frac{\Delta(!A, \Gamma) \to C}{\Delta(\Gamma, !A) \to C}\ (\mathrm{perm}_1) \qquad \frac{\Delta(A) \to C}{\Delta(!A) \to C}\ (! \to)$$

$$\frac{\Delta(!A_1, \ldots, !A_n, [!A_1, \ldots, !A_n, \Gamma]) \to B}{\Delta(!A_1, \ldots, !A_n, \Gamma) \to B}\ (\mathrm{contr}_{\mathbf{b}})$$

Subexponential: Parasitic Extraction

the paper that$_i$ $\underbrace{\text{John signed } e_i \text{ [without reading } e_i]}_{\to S \,/\, !N}$

$$\frac{\Delta(!A, \Gamma) \to C}{\Delta(\Gamma, !A) \to C}\ (\mathrm{perm}_1) \qquad \frac{\Delta(A) \to C}{\Delta(!A) \to C}\ (! \to)$$

$$\frac{\Delta(!A_1, \ldots, !A_n, [!A_1, \ldots, !A_n, \Gamma]) \to B}{\Delta(!A_1, \ldots, !A_n, \Gamma) \to B}\ (\mathrm{contr}_{\mathbf{b}})$$

causes undecidability

The Lambek Calculus with Subexponential and Bracket Modalities ($\mathbf{!_b L^1}$)

$$\frac{}{A \to A} \qquad \frac{}{\Lambda \to \mathbf{1}}$$

$$\frac{\Gamma \to B \quad \Delta(C) \to D}{\Delta(C \,/\, B, \Gamma) \to D}\ (/ \to) \qquad \frac{\Gamma, B \to C}{\Gamma \to C \,/\, B}\ (\to /) \qquad \frac{\Delta(A, B) \to D}{\Delta(A \cdot B) \to D}\ (\cdot \to)$$

$$\frac{\Gamma \to A \quad \Delta(C) \to D}{\Delta(\Gamma, A \setminus C) \to D}\ (\setminus \to) \qquad \frac{A, \Gamma \to C}{\Gamma \to A \setminus C}\ (\to \setminus) \qquad \frac{\Gamma_1 \to A \quad \Gamma_2 \to B}{\Gamma_1, \Gamma_2 \to A \cdot B}\ (\to \cdot)$$

$$\frac{\Delta(\Lambda) \to A}{\Delta(\mathbf{1}) \to A}\ (\mathbf{1} \to) \qquad \frac{\Delta([A]) \to C}{\Delta(\langle\rangle A) \to C}\ (\langle\rangle \to) \qquad \frac{\Pi \to A}{[\Pi] \to \langle\rangle A}\ (\to \langle\rangle)$$

$$\frac{\Gamma(A) \to B}{\Gamma(!A) \to B}\ (! \to) \qquad \frac{\Delta(A) \to C}{\Delta([[]^{-1}A]) \to C}\ ([]^{-1} \to) \qquad \frac{[\Pi] \to A}{\Pi \to []^{-1}A}\ (\to []^{-1})$$

$$\frac{!A_1, \ldots, !A_n \to A}{!A_1, \ldots, !A_n \to !A}\ (\to !) \qquad \frac{\Delta(!A_1, \ldots, !A_n, [!A_1, \ldots, !A_n, \Gamma]) \to B}{\Delta(!A_1, \ldots, !A_n, \Gamma) \to B}\ (\mathrm{contr}_{\mathbf{b}})$$

$$\frac{\Delta(!A, \Gamma) \to B}{\Delta(\Gamma, !A) \to B}\ (\mathrm{perm}_1) \qquad \frac{\Delta(\Gamma, !A) \to B}{\Delta(!A, \Gamma) \to B}\ (\mathrm{perm}_2) \qquad \frac{\Pi \to A \quad \Delta(A) \to C}{\Delta(\Pi) \to C}\ (\mathrm{cut})$$

The Lambek Calculus with Subexponential and Bracket Modalities ($!_{\mathbf{b}}\mathbf{L}^{\mathbf{1}}$)

$$\frac{}{A \to A} \qquad \frac{}{\Lambda \to \mathbf{1}}$$

$$\frac{\Gamma \to B \quad \Delta(C) \to D}{\Delta(C \,/\, B, \Gamma) \to D}\ (/\ \to) \qquad \frac{\Gamma, B \to C}{\Gamma \to C \,/\, B}\ (\to\ /) \qquad \frac{\Delta(A, B) \to D}{\Delta(A \cdot B) \to D}\ (\cdot \to)$$

$$\frac{\Gamma \to A \quad \Delta(C) \to D}{\Delta(\Gamma, A \setminus C) \to D}\ (\setminus\ \to) \qquad \frac{A, \Gamma \to C}{\Gamma \to A \setminus C}\ (\to\ \setminus) \qquad \frac{\Gamma_1 \to A \quad \Gamma_2 \to B}{\Gamma_1, \Gamma_2 \to A \cdot B}\ (\to \cdot)$$

$$\frac{\Delta(\Lambda) \to A}{\Delta(\mathbf{1}) \to A}\ (\mathbf{1} \to) \qquad \frac{\Delta([A]) \to C}{\Delta(\langle\rangle A) \to C}\ (\langle\rangle \to) \qquad \frac{\Pi \to A}{[\Pi] \to \langle\rangle A}\ (\to \langle\rangle)$$

$$\frac{\Gamma(A) \to B}{\Gamma(!A) \to B}\ (! \to) \qquad \frac{\Delta(A) \to C}{\Delta([[]^{-1}A]) \to C}\ ([]^{-1} \to) \qquad \frac{[\Pi] \to A}{\Pi \to []^{-1}A}\ (\to []^{-1})$$

$$\frac{!A_1, \ldots, !A_n \to A}{!A_1, \ldots, !A_n \to !A}\ (\to !) \qquad \frac{\Delta(!A_1, \ldots, !A_n, [!A_1, \ldots, !A_n, \Gamma]) \to B}{\Delta(!A_1, \ldots, !A_n, \Gamma) \to B}\ (\mathrm{contr}_{\mathbf{b}})$$

$$\frac{\Delta(!A, \Gamma) \to B}{\Delta(\Gamma, !A) \to B}\ (\mathrm{perm}_1) \qquad \frac{\Delta(\Gamma, !A) \to B}{\Delta(!A, \Gamma) \to B}\ (\mathrm{perm}_2) \qquad \frac{\Pi \to A \quad \Delta(A) \to C}{\Delta(\Pi) \to C}\ (\mathrm{cut})$$

- A fragment of $\mathbf{Db!_b}$ by Morrill and Valentín, 2015.

The Lambek Calculus with Subexponential and Bracket Modalities ($\mathbf{!_b L^1}$)

$$\frac{}{A \to A} \qquad \frac{}{\Lambda \to \mathbf{1}}$$

$$\frac{\Gamma \to B \quad \Delta(C) \to D}{\Delta(C \,/\, B, \Gamma) \to D}\ (/\to) \qquad \frac{\Gamma, B \to C}{\Gamma \to C \,/\, B}\ (\to /) \qquad \frac{\Delta(A, B) \to D}{\Delta(A \cdot B) \to D}\ (\cdot \to)$$

$$\frac{\Gamma \to A \quad \Delta(C) \to D}{\Delta(\Gamma, A \setminus C) \to D}\ (\setminus \to) \qquad \frac{A, \Gamma \to C}{\Gamma \to A \setminus C}\ (\to \setminus) \qquad \frac{\Gamma_1 \to A \quad \Gamma_2 \to B}{\Gamma_1, \Gamma_2 \to A \cdot B}\ (\to \cdot)$$

$$\frac{\Delta(\Lambda) \to A}{\Delta(\mathbf{1}) \to A}\ (\mathbf{1} \to) \qquad \frac{\Delta([A]) \to C}{\Delta(\langle\rangle A) \to C}\ (\langle\rangle \to) \qquad \frac{\Pi \to A}{[\Pi] \to \langle\rangle A}\ (\to \langle\rangle)$$

$$\frac{\Gamma(A) \to B}{\Gamma(!A) \to B}\ (! \to) \qquad \frac{\Delta(A) \to C}{\Delta([[]^{-1} A]) \to C}\ ([]^{-1} \to) \qquad \frac{[\Pi] \to A}{\Pi \to []^{-1} A}\ (\to []^{-1})$$

$$\frac{!A_1, \ldots, !A_n \to A}{!A_1, \ldots, !A_n \to !A}\ (\to !) \qquad \frac{\Delta(!A_1, \ldots, !A_n, [!A_1, \ldots, !A_n, \Gamma]) \to B}{\Delta(!A_1, \ldots, !A_n, \Gamma) \to B}\ (\mathrm{contr}_{\mathbf{b}})$$

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- Our analysis of syntactic phenomena is due to Morrill, 2011–2017.

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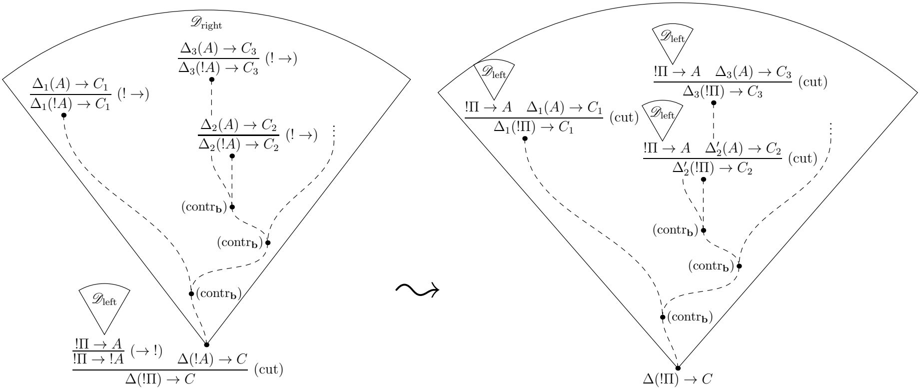

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$$!\Gamma = !B_1, \ldots, !B_n,$$

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$$!\Phi = !(\mathbf{1}\,/(!B_1)), \ldots, !(\mathbf{1}\,/(!B_n)), \text{ and}$$

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4. $s \Rightarrow^* a_1 \ldots a_n$ *in the type-0 grammar.*

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Thank you !