Starlike neighbourhoods and computability

Zvonko Iljazović, Lucija Validžić

University of Zagreb
Faculty of Science
Department of Mathematics

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Computable metric space

\( (X, d, \alpha) \)

\( \alpha \) a sequence dense in \((X, d)\)

\((i, j) \mapsto d(\alpha_i, \alpha_j)\) a computable function

\( x \in X \) is a computable point if there exists a computable function \( f : \mathbb{N} \to \mathbb{N} \) such that

\[ d(x, \alpha f(k)) < 2^{-k}, \quad k \in \mathbb{N}. \]

\( (\Lambda_i)_{i \in \mathbb{N}} \)

An effective enumeration of finite subsets of \( \text{Im} \alpha \)

A compact set \( S \subseteq X \) is computable if there exists a computable function \( f : \mathbb{N} \to \mathbb{N} \) such that

\[ d_H(S, \Lambda f(k)) < 2^{-k}, \quad k \in \mathbb{N}. \]
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\(\{I_i \mid i \in \mathbb{N}\}\) a basis for \(\mathcal{T}\), there are c.e. subsets \(C, D\) of \(\mathbb{N}^2\) such that:

1. \((i, j) \in D \Rightarrow I_i \cap I_j = \emptyset\)
2. \((i, j) \in C \Rightarrow I_i \subseteq I_j\)
3. \(x, y \in X, x \neq y \Rightarrow (\exists i, j \in \mathbb{N}) x \in I_i, y \in I_j, (i, j) \in D\)
4. \(i, j \in \mathbb{N}, x \in I_i \cap I_j \Rightarrow (\exists k \in \mathbb{N}) x \in I_k, (k, i) \in C, (k, j) \in C\)
Computable topological space

\((X, \mathcal{T}, (I_i)_{i \in \mathbb{N}})\)

\(\{I_i \mid i \in \mathbb{N}\}\) a basis for \(\mathcal{T}\), there are c.e. subsets \(\mathcal{C}, \mathcal{D}\) of \(\mathbb{N}^2\) such that:

- \((i, j) \in \mathcal{D} \Rightarrow I_i \cap I_j = \emptyset\)
- \((i, j) \in \mathcal{C} \Rightarrow I_i \subseteq I_j\)
- \(x, y \in X, x \neq y \Rightarrow (\exists i, j \in \mathbb{N}) x \in I_i, y \in I_j, (i, j) \in \mathcal{D}\)
- \(i, j \in \mathbb{N}, x \in I_i \cap I_j \Rightarrow (\exists k \in \mathbb{N}) x \in I_k, (k, i) \in \mathcal{C}, (k, j) \in \mathcal{C}\)
In a computable metric space:
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\[ I_i = B(\lambda_i, \varrho_i) \]
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\[ C = \{(i, j) \in \mathbb{N}^2 \mid d(\lambda_i, \lambda_j) + \varrho_i < \varrho_j \} \]

\[ D = \{(i, j) \in \mathbb{N}^2 \mid d(\lambda_i, \lambda_j) > \varrho_i + \varrho_j \} \]
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A compact set \(S \subseteq X\) is:

- **semicomputable** if \( \{j \in \mathbb{N} \mid S \subseteq J_j\} \) is c.e.
$(J_j)_{j \in \mathbb{N}}$ effective enumeration of finite unions of $I_i$s

A compact set $S \subseteq X$ is:

- **semicomputable** if $\{j \in \mathbb{N} \mid S \subseteq J_j\}$ is c.e.
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- **computable** if \(S\) is semicomputable and computably enumerable

\(x \in X\) is a **computable point** if \(\{x\}\) is a computable set.
$S$ semicomputable $\Rightarrow S$ computable

Holds if:
- $S$ is a compact manifold
- $S$ is a circularly chainable continuum

Need not hold if:
- $S$ is a line segment
- $S$ is a cell
\( S \text{ semicomputable} \Rightarrow S \text{ computable} \)

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If $S$ is semicomputable, can we find $T \subseteq S, T \neq S$ such that $T$ computable $\implies S$ computable?
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\[ T \text{ computable} \implies S \text{ computable?} \]

For a compact manifold with boundary $S$:

\[ T = \partial S \]
Topological 1-polyhedra

$(\mathcal{X}, \mathcal{T})$ t.s.
Topological 1-polyhedra

$(X, T)$ t.s.
If $S \subseteq X$ is homeomorphic to a finite union of line segments in $\mathbb{R}^n$, we say that $S$ is a **topological 1-polyhedron**.
Topological 1-polyhedra

$(X, T)$ t.s.
If $S \subseteq X$ is homeomorphic to a finite union of line segments in $\mathbb{R}^n$, we say that $S$ is a topological 1-polyhedron.
Euclidean points
neighbourhood $\simeq \langle 0, 1 \rangle$
boundary points
neighbourhood $\cong [0, 1]$
starlike points
starlike neighbourhood
\[ n \in \mathbb{N} \setminus \{0\}, \; i \in \{1, \ldots, n\} \]

\[ I^n_i = \{(t_1, t_2, \ldots, t_n) \in \mathbb{R}^n \mid t_k = 0 \text{ for } k \neq i, \; t_i \in [0, 1]\} \]

\[ T^n = I^n_1 \cup I^n_2 \cup \cdots \cup I^n_n \]

\[ \hat{I}^n_i = \{(t_1, t_2, \ldots, t_n) \in \mathbb{R}^n \mid t_k = 0 \text{ for } k \neq i, \; t_i \in [0, 1]\} \]

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\((X, \mathcal{T})\) t.s., \(S \subseteq X\)

\(x \in S\) is a **starlike point** in \(S\) if there exist \(n \in \mathbb{N}, \; n \geq 3\) and a continuous injective map \(f : T^n \rightarrow S\) such that \(f(0) = x\) and \(f(\hat{T}^n)\) is an open set in \(S\).
Main goal

S a topological 1-polyhedron S semicomputable, ∂S computable ⇒ S computable

x ∈ ∂S if there exists a neighbourhood N of x in S and a homeomorphism f: [0, 1) → N such that f(0) = x.
Main goal

A topological 1-polyhedron $S$ is semicomputable if its boundary $\partial S$ is computable. Therefore, if $S$ is computable, it follows that $S$ is semicomputable and $\partial S$ is computable.
Main goal

$S$ a topological 1-polyhedron

$S$ semicomputable, $\partial S$ computable $\Rightarrow S$ computable

$x \in \partial S$ if there exists a neighbourhood $N$ of $x$ in $S$ and a homeomorphism $f : [0, 1) \to N$ such that $f(0) = x$. 
Local computable enumerability

Let $S, T \subseteq X$, $T \subseteq S$. $T$ is computably enumerable up to $S$ if there exists c.e. $\Omega \subseteq \mathbb{N}$ such that for each $i \in \mathbb{N}$:

$I_i \cap T \neq \emptyset \Rightarrow i \in \Omega$

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Local computable enumerability

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\[ i \in \Omega \Rightarrow I_i \cap S \neq \emptyset \]
Local computable enumerability

$(X, T, (l_i))$ c.t.s.

Let $S, T \subseteq X$, $T \subseteq S$. $T$ is computably enumerable up to $S$ if there exists c.e. $\Omega \subseteq \mathbb{N}$ such that for each $i \in \mathbb{N}$:

$$l_i \cap T \neq \emptyset \Rightarrow i \in \Omega$$

$$i \in \Omega \Rightarrow l_i \cap S \neq \emptyset$$

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Local computable enumerability

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I_i \cap T \neq \emptyset \Rightarrow i \in \Omega
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Local computable enumerability

Properties:

▶ $S$ is c.e. ⇔ $S$ is c.e. up to $S$

▶ $A$ and $B$ c.e. up to $S$ ⇒ $A \cup B$ is c.e. up to $S$

$S$ a compact set

$S$ is c.e. ⇔ $S$ is c.e. at each $x \in S$
Local computable enumerability

\[ S \text{ is computably enumerable at } x \text{ if there exists a neighbourhood } N \text{ of } x \text{ in } S \text{ such that } N \text{ is c.e. up to } S. \]
Local computable enumerability

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$S$ is c.e. $\iff$ $S$ is c.e. at each $x \in S$
Some topological properties of a semicomputable set imply that it is c.e. at a certain point!
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**Theorem**

Let \((X, \mathcal{T}, (I_i))\) be a computable topological space, \(S\) a semicomputable set in this space and \(x \in S\). If one of the following holds:

1. \(x\) has a neighbourhood in \(S\) homeomorphic to some \(\mathbb{R}^n\)
2. there is a semicomputable set \(T \subseteq S\), a neighbourhood \(N\) of \(x\) in \(S\) and a homeomorphism \(f: \mathbb{H}^n \to N\) such that \(x \in f(Bd \mathbb{H}^n)\), \(f(Bd \mathbb{H}^n) = N \cap T\)

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\(S\) is computably enumerable at \(x\).
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- $S$ is c.e. at its Euclidean points
\( S \) a semicomputable topological 1-poyhedron

- \( S \) is c.e. at its Euclidean points
- if \( \partial S \) is (semi)computable, \( S \) is c.e. at its boundary points
$S$ a semicomputable topological 1-poyhedron

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**Main theorem**

Let $(X, \mathcal{T}, (I_i))$ be a computable topological space and $S$ a semicomputable set in this space. If $x$ is a starlike point in $S$, then $S$ is computably enumerable at $x$. 

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Main result

Let \((X, \mathcal{T}, (I_i))\) be a computable topological space and \(S \subseteq X\) a topological 1-polyhedron. If \(S\) and \(\partial S\) are semicomputable, then \(S\) is computable.