Starlike neighbourhoods and computability

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Computable metric space

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A compact set $S \subseteq X$ is **computable** if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that

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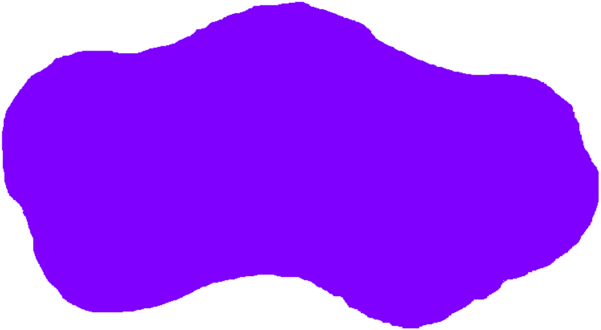

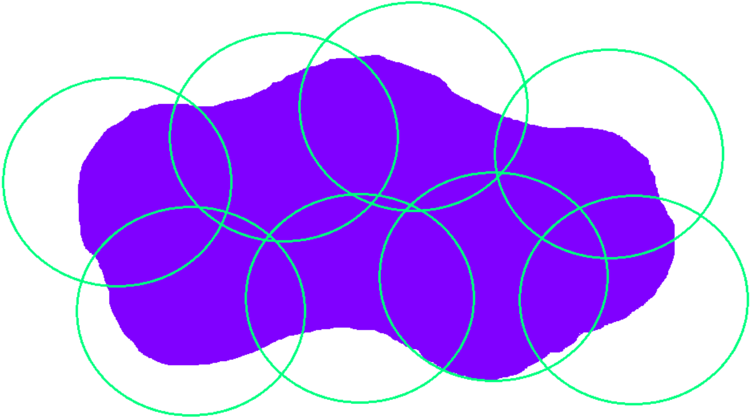

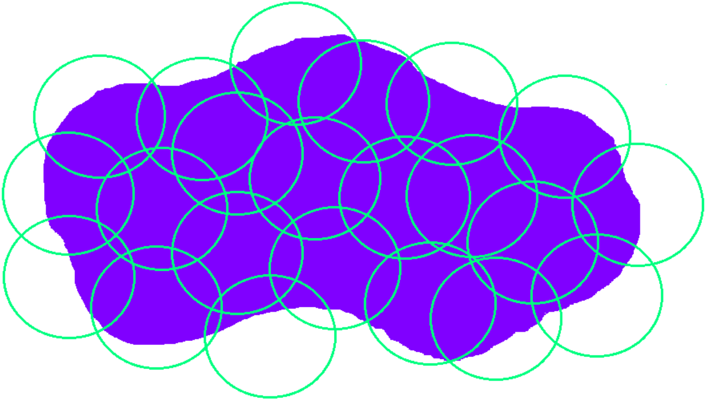

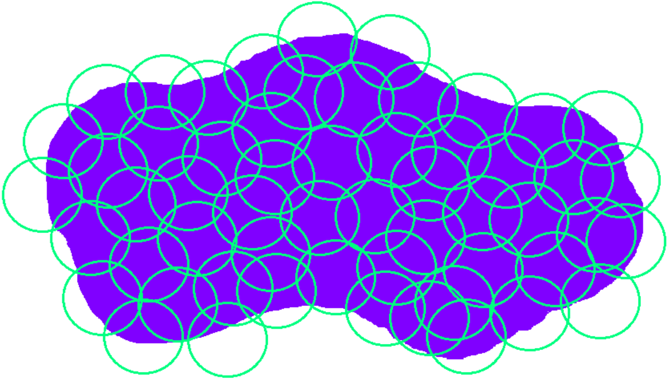

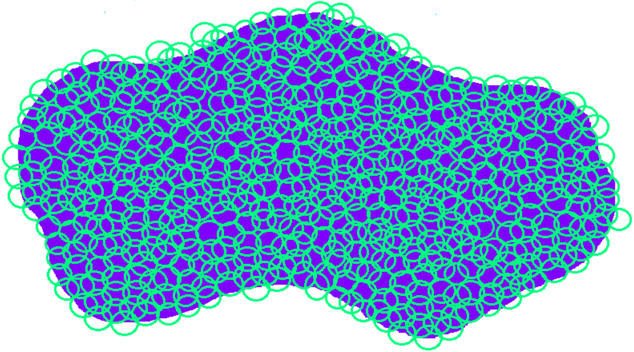

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- $(i,j) \in \mathcal{D} \ \Rightarrow \ I_i \cap I_j = \emptyset$
- $(i,j) \in \mathcal{C} \ \Rightarrow \ I_i \subseteq I_j$
- $x, y \in X, x \neq y \ \Rightarrow \ (\exists i, j \in \mathbb{N})\, x \in I_i, y \in I_j, (i,j) \in \mathcal{D}$
- $i, j \in \mathbb{N}, x \in I_i \cap I_j \ \Rightarrow \ (\exists k \in \mathbb{N})\, x \in I_k, (k,i) \in \mathcal{C}, (k,j) \in \mathcal{C}$

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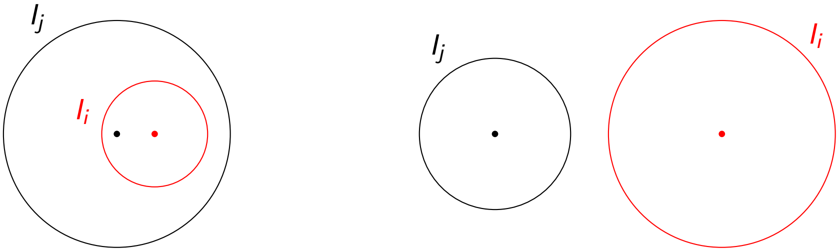

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$x \in X$ is a **computable point** if $\{x\}$ is a computable set.

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Need not hold if:

- S is a line segment
- S is a cell

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For a compact manifold with boundary S:

$$T = \partial S$$

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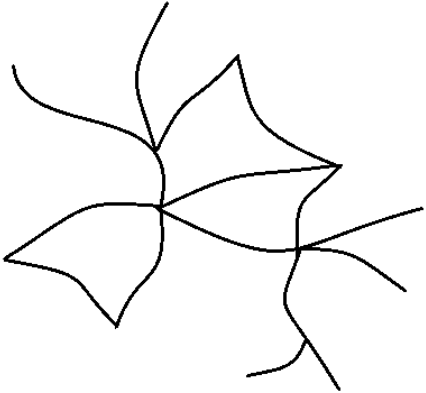
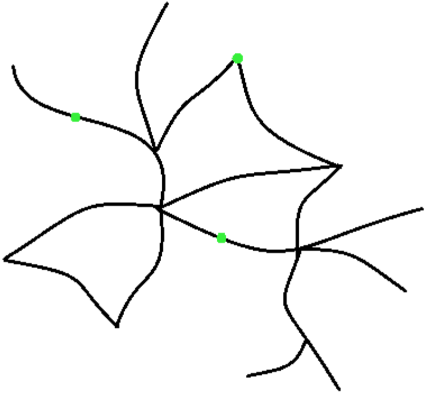
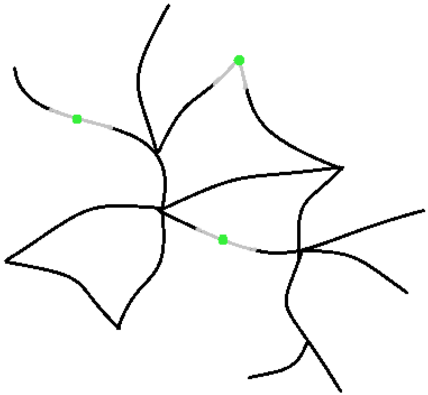
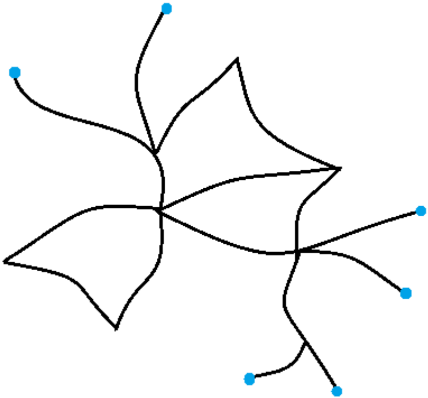




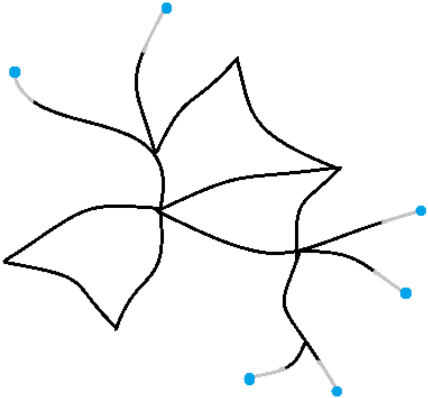
Euclidean points



neighbourhood $\cong \langle 0, 1 \rangle$



boundary points



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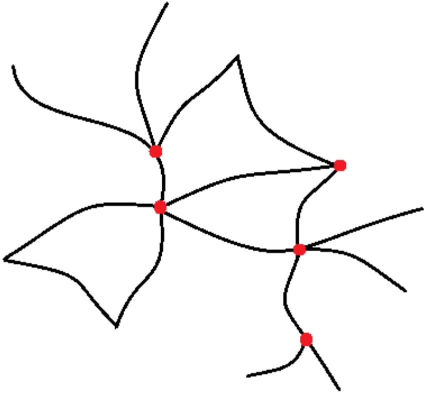
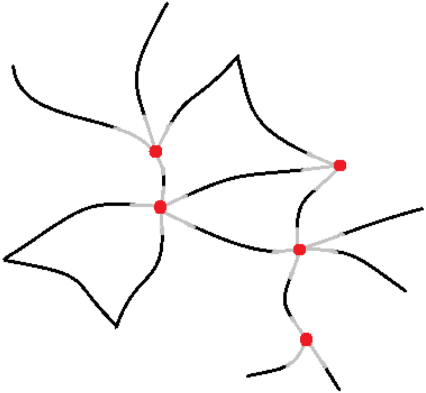


starlike points



starlike neighbourhood

$n \in \mathbb{N} \setminus \{0\}$, $i \in \{1, \dots, n\}$

$$I_i^n = \{(t_1, t_2, \dots, t_n) \in \mathbb{R}^n \mid t_k = 0 \text{ for } k \neq i,\ t_i \in [0, 1]\}$$

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$(X, \mathcal{T})$ t.s., $S \subseteq X$
$x \in S$ is a **starlike point** in S if there exist $n \in \mathbb{N}, n \geq 3$ and a continuous injective map $f : T_n \to S$ such that $f(0) = x$ and $f(\mathring{T}^n)$ is an open set in S.

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$x \in \partial S$ if there exists a neighbourhood N of x in S and a homeomorphism $f : [0, 1\rangle \to N$ such that $f(0) = x$.

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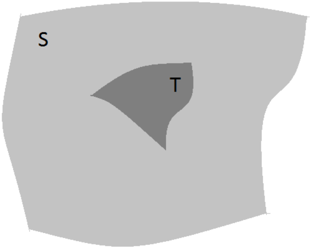

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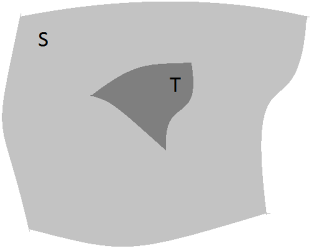

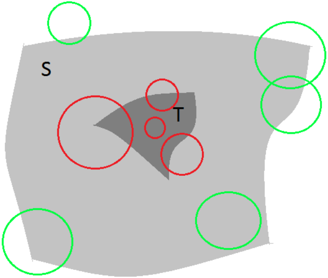

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S a compact set

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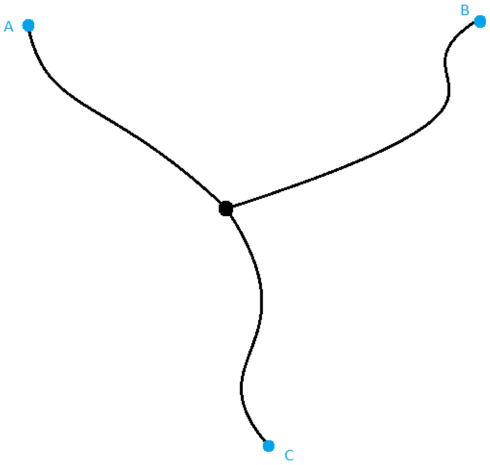
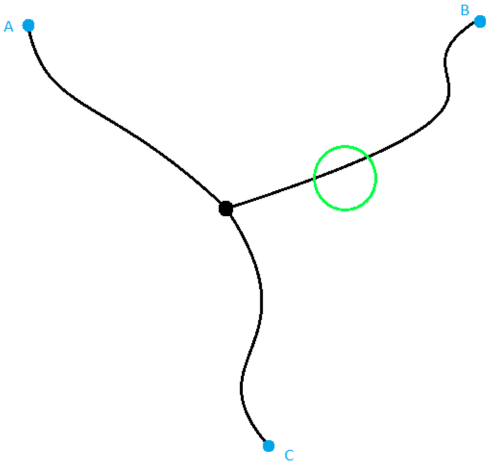
Main theorem

Let $(X, \mathcal{T}, (I_i))$ be a computable topological space and S a semicomputable set in this space. If x is a starlike point in S, then S is computably enumerable at x.


A
B
C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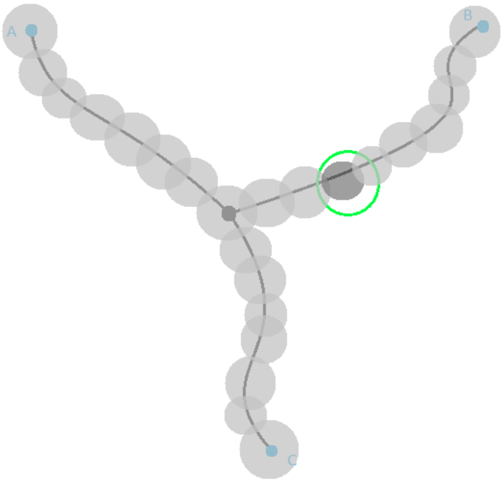
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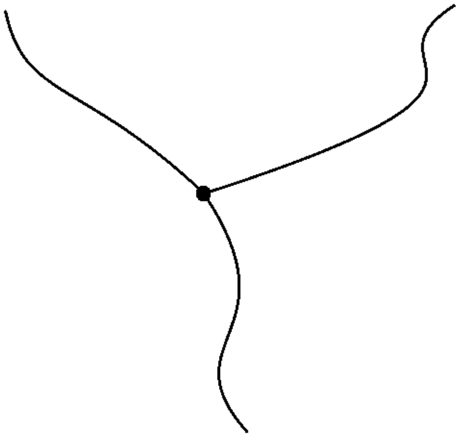

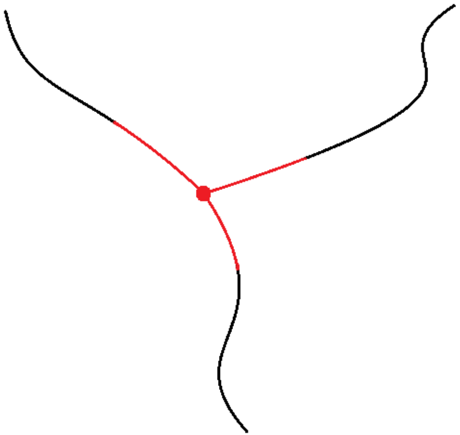

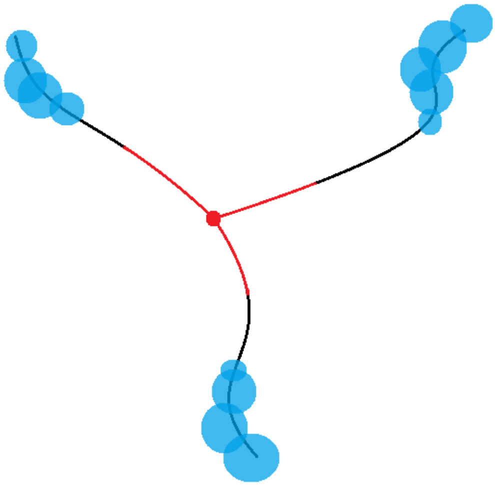

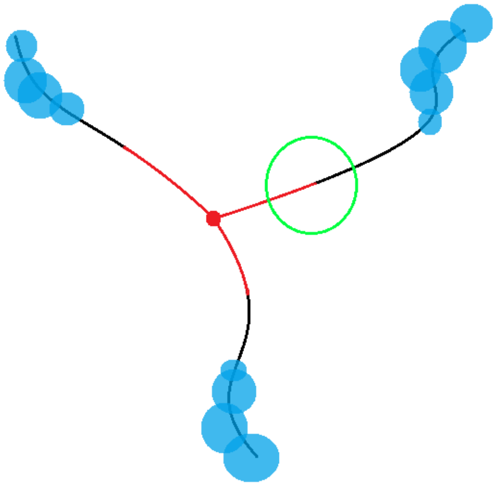

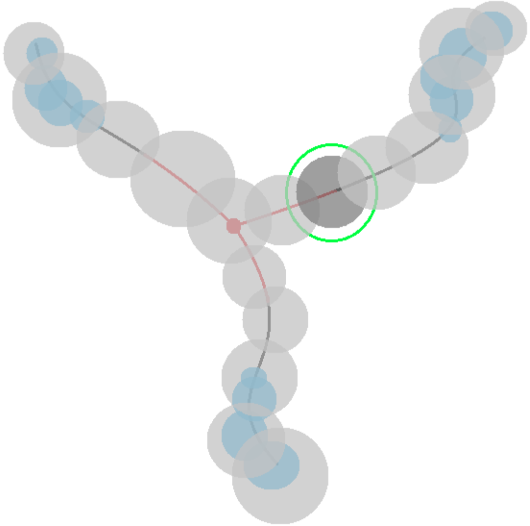

Main result

Let $(X, \mathcal{T}, (I_i))$ be a computable topological space and $S \subseteq X$ a topological 1*-polyhedron. If S and ∂S are semicomputable, then S is computable.*