Subexponentials in Non-Commutative Linear Logic

Max Kanovich, Stepan Kuznetsov, Vivek Nigam, Andre Scedrov

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Multiplicative-Additive Lambek Calculus

$$\frac{}{A \to A}\ (\mathrm{ax})$$

$$\frac{\Gamma_1, A, B, \Gamma_2 \to C}{\Gamma_1, A \cdot B, \Gamma_2 \to C}\ (\cdot \to) \qquad \frac{\Gamma_1 \to A \quad \Gamma_2 \to B}{\Gamma_1, \Gamma_2 \to A \cdot B}\ (\to \cdot)$$

$$\frac{\Pi \to A \quad \Gamma_1, B, \Gamma_2 \to C}{\Gamma_1, \Pi, A \setminus B, \Gamma_2 \to C}\ (\setminus \to) \qquad \frac{A, \Pi \to B}{\Pi \to A \setminus B}\ (\to \setminus)$$

$$\frac{\Pi \to A \quad \Gamma_1, B, \Gamma_2 \to C}{\Gamma_1, B \,/\, A, \Pi, \Gamma_2 \to C}\ (/ \to) \qquad \frac{\Pi, A \to B}{\Pi \to B \,/\, A}\ (\to /)$$

$$\frac{\Gamma_1, \Gamma_2 \to C}{\Gamma_1, \mathbf{1}, \Gamma_2 \to C}\ (\mathbf{1} \to) \qquad \frac{}{\to \mathbf{1}}\ (\to \mathbf{1})$$

$$\frac{\Gamma_1, A_1, \Gamma_2 \to C \quad \Gamma_1, A_2, \Gamma_2 \to C}{\Gamma_1, A_1 \vee A_2, \Gamma_2 \to C}\ (\vee \to) \qquad \frac{\Gamma \to A_i}{\Gamma \to A_1 \vee A_2}\ (\to \vee),\ \text{where } i = 1 \text{ or } 2$$

$$\frac{\Gamma_1, A_i, \Gamma_2 \to C}{\Gamma_1, A_1 \wedge A_2, \Gamma_2 \to C}\ (\wedge \to),\ \text{where } i = 1 \text{ or } 2 \qquad \frac{\Gamma \to A_1 \quad \Gamma \to A_2}{\Gamma \to A_1 \wedge A_2}\ (\to \wedge)$$

Subexponentials

- Subexponential signature: $\Sigma = \langle \mathcal{I}, \preceq, \mathcal{W}, \mathcal{C}, \mathcal{E} \rangle$, where $\mathcal{I} = \{s_1, \ldots, s_n\}$ is a set of subexponential labels; $\preceq$ is a preorder; $\mathcal{W}, \mathcal{C}, \mathcal{E} \subseteq \mathcal{I}$.

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- Rules:

$$\frac{\Gamma_1, A, \Gamma_2 \to C}{\Gamma_1, !^s A, \Gamma_2 \to C}\ (! \to) \qquad \frac{!^{s_1} A_1, \ldots, !^{s_n} A_n \to B}{!^{s_1} A_1, \ldots, !^{s_n} A_n \to !^s B}\ (\to !),\ \text{where } s_j \succeq s \text{ for all } j$$

$$\frac{\Gamma_1, \Gamma_2 \to C}{\Gamma_1, !^s A, \Gamma_2 \to C}\ (\mathrm{weak}),\ \text{where } s \in \mathcal{W}$$

$$\frac{\Gamma_1, !^s A, \Delta, !^s A, \Gamma_2 \to C}{\Gamma_1, !^s A, \Delta, \Gamma_2 \to C}\ (\mathrm{ncontr}_1)\ \text{and}\ \frac{\Gamma_1, !^s A, \Delta, !^s A, \Gamma_2 \to C}{\Gamma_1, \Delta, !^s A, \Gamma_2 \to C}\ (\mathrm{ncontr}_2),\ \text{where } s \in \mathcal{C}$$

$$\frac{\Gamma_1, \Delta, !^s A, \Gamma_2 \to C}{\Gamma_1, !^s A, \Delta, \Gamma_2 \to C}\ (\mathrm{ex}_1) \quad \text{and} \quad \frac{\Gamma_1, !^s A, \Delta, \Gamma_2 \to C}{\Gamma_1, \Delta, !^s A, \Gamma_2 \to C}\ (\mathrm{ex}_2),\ \text{where } s \in \mathcal{E}$$

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- $\mathcal{W} \cap \mathcal{C} \subseteq \mathcal{E}$.

Cut

$$\frac{\Pi \to A \quad \Gamma, A, \Delta \to C}{\Gamma, \Pi, \Delta \to C}\ (\mathrm{cut})$$

Local vs. Non-local Contraction

- Non-local contraction, used in SMALC_Σ for $s \in \mathcal{C}$:

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- Counter-example:

$$r \,/\, q, !p, !(p \setminus q), q \setminus s \to r \cdot s$$

This sequent is derivable with local contraction, but only using cut. With non-local contraction, a cut-free proof exists.

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- Derivation with local contraction and cut:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \to p \quad q \to q}{p, p \setminus q \to q}\,(\setminus \to)}{p, !(p \setminus q) \to q}\,(! \to)}{!p, !(p \setminus q) \to q}\,(! \to)}{!p, !(p \setminus q) \to !q}\,(\to !) \qquad \dfrac{\dfrac{\dfrac{q \to q}{!q \to q}\,(! \to) \qquad \dfrac{\dfrac{q \to q}{!q \to q}\,(! \to) \quad \dfrac{r \to r \quad s \to s}{r, s \to r \cdot s}\,(\to \cdot)}{r, !q, q \setminus s \to r \cdot s}\,(\setminus \to)}{r \,/\, q, !q, !q, q \setminus s \to r \cdot s}\,(/ \to)}{r \,/\, q, !q, q \setminus s \to r \cdot s}\,(\text{contr})}{r \,/\, q, !p, !(p \setminus q), q \setminus s \to r \cdot s}\,(\text{cut})$$

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- Not derivable without cut, if ! allows local contraction and maybe weakening, but neither exchange nor non-local contraction (shown by exhaustive proof search).

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$$r \,/\, q, !p, !(p \backslash q), q \backslash s \to r \cdot s$$

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- Not derivable without cut, if ! allows local contraction and maybe weakening, but neither exchange nor non-local contraction (shown by exhaustive proof search).
- Cut-free derivation with non-local contraction:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \to p \quad p \to p \quad \dfrac{q \to q \quad \dfrac{q \to q \quad \dfrac{r \to r \quad s \to s}{r, s \to r \cdot s}\,(\to \cdot)}{r, q, q \backslash s \to r \cdot s}\,(\backslash \to)}{r \,/\, q, q, q, q \backslash s \to r \cdot s}\,(/ \to)}{r \,/\, q, p, p \backslash q, p, p \backslash q, q \backslash s \to r \cdot s}\,(\backslash \to)\text{ twice}}{r \,/\, q, !p, !(p \backslash q), !p, !(p \backslash q), q \backslash s \to r \cdot s}\,(! \to)\text{ 4 times}}{r \,/\, q, !p, !p, !(p \backslash q), q \backslash s \to r \cdot s}\,(\mathrm{ncontr}_2)}{r \,/\, q, !p, !(p \backslash q), q \backslash s \to r \cdot s}\,(\mathrm{contr})$$

Cyclic Linear Logic with Subexponentials (SCLL_Σ)

Sequents are of the form $\vdash \Gamma$, where Γ is cyclically ordered sequence of formulae (i.e., A, B, C is the same as B, C, A and C, A, B, but not A, C, B).

$$\frac{}{\vdash A, A^\perp}\,(\mathrm{ax}) \qquad \frac{}{\vdash \mathbf{1}}\,(\mathbf{1}) \qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}\,(\bot) \qquad \frac{}{\vdash \top, \Gamma}\,(\top)$$

(no rule for $\mathbf{0}$)

$$\frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\,(\otimes) \qquad \frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\,(\parr)$$

$$\frac{\vdash A_1, \Gamma \quad \vdash A_2, \Gamma}{\vdash A_1 \,\&\, A_2, \Gamma}\,(\&) \qquad \frac{\vdash A_i, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}\,(\oplus), \text{ where } i = 1 \text{ or } 2$$

$$\frac{\vdash B, ?^{s_1} A_1, \ldots, ?^{s_n} A_n}{\vdash !^s B, ?^{s_1} A_1, \ldots, ?^{s_n} A_n}\,(!), \text{ where } s_j \succeq s \text{ for all } j$$

$$\frac{\vdash A, \Gamma}{\vdash ?^s A, \Gamma}\,(?) \qquad \frac{\vdash \Gamma}{\vdash ?^s A, \Gamma}\,(\mathrm{weak}), \text{ where } s \in \mathcal{W}$$

$$\frac{\vdash ?^s A, \Gamma, ?^s A, \Delta}{\vdash ?^s A, \Gamma, \Delta}\,(\mathrm{ncontr}), \text{ where } s \in \mathcal{C} \qquad \frac{\vdash \Gamma, ?^s A, \Delta}{\vdash ?^s A, \Gamma, \Delta}\,(\mathrm{ex}), \text{ where } s \in \mathcal{E}$$

Multiplicative-only Fragments

- For SMALC_Σ — $\mathrm{SLC}^{\mathbf{1}}_\Sigma$ (without $\vee$ and $\wedge$, but with $\mathbf{1}$ and subexponentials).
- For SCLL_Σ — SMCLL_Σ (without $\&$, $\oplus$, $\mathbf{0}$, and $\top$, but with $\mathbf{1}$, $\bot$, and subexponentials).

Cut Elimination in SCLL_Σ

- The cut rule:

$$\frac{\vdash \Gamma, A^\perp \quad \vdash A, \Delta}{\vdash \Gamma, \Delta}\ (\mathrm{cut})$$

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- A non-standard form of mix for non-local contraction:

$$\frac{\vdash \Gamma, !^s A^\perp \quad \vdash ?^s A, \Delta_1, ?^s A, \Delta_2, \ldots, ?^s A, \Delta_k}{\vdash \Gamma, \Delta_1, \Delta_2, \ldots, \Delta_k}\ (\mathrm{mix})$$

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- Eliminate cut and mix by joint nested induction.
  Outer parameter: κ, the complexity of the formula being cut.
  Inner parameter: δ, the sum of heights of cut-free derivations of the premises.

Embedding SMALC_Σ into SCLL_Σ

- The Lambek calculus is "intuitionistic" (one formula in the succedent in the Gentzen-style calculus). Cyclic linear logic is "classical." Thus, SMALC_Σ should not be a conservative fragment of SCLL_Σ.

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  - *tertium non datur*, $A \mathbin{⅋} A^\perp$: no ⅋ and no linear negation in SMALC_Σ;
  - Peirce's law (with subexponentials for enabling necessary structural rules), $(x \setminus ?^w y) \setminus x \to ?^c x$, $c \in \mathcal{C}$, $w \in \mathcal{W}$: no ? in SMALC_Σ, only !;

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  - Schellinx' (1991) example with the zero constant (a non-commutative modification), $(r \,/\, (\mathbf{0} \setminus q)) \,/\, p, (s \,/\, p) \setminus \mathbf{0} \to r$: no $\mathbf{0}$ in SMALC_Σ.

A Non-Commutative Version of Schellinx' Example

- $(r \,/\, (\mathbf{0} \setminus q)) \,/\, p, (s \,/\, p) \setminus \mathbf{0} \rightarrow r$

A Non-Commutative Version of Schellinx' Example

- $(r \,/ (\mathbf{0} \setminus q)) \,/\, p, (s \,/\, p) \setminus \mathbf{0} \to r$
- Translation into cyclic linear logic:
  $\vdash \top \otimes (s \parr \bar{p}), p \otimes (\top \parr q) \otimes \bar{r}, r$

A Non-Commutative Version of Schellinx' Example

- $(r \,/ (\mathbf{0} \setminus q)) \,/\, p, (s \,/\, p) \setminus \mathbf{0} \to r$
- Translation into cyclic linear logic:
  $\vdash \top \otimes (s \parr \bar{p}), p \otimes (\top \parr q) \otimes \bar{r}, r$
- Derivation in SCLL_{Σ}:

$$\dfrac{\dfrac{}{\vdash \top}\,(\top) \qquad \dfrac{\dfrac{\dfrac{}{\vdash \bar{p}, p}\,(\mathrm{ax}) \qquad \dfrac{\dfrac{\dfrac{}{\vdash s, \top, q}\,(\top)}{\vdash s, \top \parr q}\,(\parr) \qquad \dfrac{}{\vdash \bar{r}, r}\,(\mathrm{ax})}{\vdash s, (\top \parr q) \otimes \bar{r}, r}\,(\otimes)}{\vdash s, \bar{p}, p \otimes ((\top \parr q) \otimes \bar{r}), r}\,(\otimes)}{\vdash s \parr \bar{p}, p \otimes ((\top \parr q) \otimes \bar{r}), r}\,(\parr)}{\vdash \top \otimes (s \parr \bar{p}), p \otimes ((\top \parr q) \otimes \bar{r}), r}\,(\otimes)$$

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  Rule for $\mathbf{0}$:

$$
\dfrac{}{\Gamma, \mathbf{0}, \Delta \to C}\,(\mathbf{0} \to)
$$

  (and no right rule).

Embedding SMALC_Σ into SCLL_Σ

- Translation of formulae (with negations):

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\begin{array}{lll}
\widehat{p_i} = p_i & \widehat{p_i}^\perp = p_i^\perp & \widehat{\Pi}^\perp = \widehat{A}_n^\perp, \ldots, \widehat{A}_1^\perp \\
\widehat{A \cdot B} = \widehat{A} \otimes \widehat{B} & (\widehat{A \cdot B})^\perp = \widehat{B}^\perp \parr \widehat{A}^\perp & \text{for } \Pi = A_1, \ldots, A_n \\
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\widehat{\mathbf{1}} = \mathbf{1} & \widehat{\mathbf{1}}^\perp = \perp & \\
\widehat{A \wedge B} = \widehat{A} \,\&\, \widehat{B} & (\widehat{A \wedge B})^\perp = \widehat{A}^\perp \oplus \widehat{B}^\perp & \\
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$$\widehat{\Pi}^\perp = \widehat{A}_n^\perp, \ldots, \widehat{A}_1^\perp$$
for $\Pi = A_1, \ldots, A_n$

- **Theorem.** The following are equivalent:
  1. the sequent $\Pi \to B$ is derivable in SMALC_Σ;
  2. the sequent $\Pi \to B$ is derivable in $\mathrm{SMALC}_\Sigma + (\mathrm{cut})$;
  3. the sequent $\vdash \widehat{\Pi}^\perp, \widehat{B}$ is derivable in $\mathrm{SCLL}_\Sigma + (\mathrm{cut})$;
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- Yields both embedding and cut elimination for SMALC_Σ.
- The interesting step is $4 \Rightarrow 1$ ($3 \Rightarrow 4$ discussed earlier, others are straightforward).

Embedding SMALC_Σ into SCLL_Σ: the $\natural$ Counter

Proving 4 $\Rightarrow$ 1, extending ideas of Schellinx (1991) and Pentus (1998).

- The main issue: maintain the fact that in a cut-free SCLL_Σ-derivation of $\vdash \widehat{\Pi}^\perp, \widehat{B}$ all sequents are again of the form $\vdash \widehat{\Phi}^\perp, \widehat{C}$. (Then we can just map it onto a SMALC_Σ-derivation.)

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- In other words, each sequent should contain exactly one formula of the form $\widehat{C}$, and other formulae should be of the form $\widehat{B}_i^\perp$. The only possible violation is the $\otimes$ rule, where both formulae of the form $\widehat{C}$ could go into one branch.

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- The $\natural$ counter:

$$\natural(p_i) = 0 \qquad \natural(A \parr B) = \natural(A) + \natural(B) - 1$$

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- For a derivable sequent $\vdash E_1, \ldots, E_n$ we have $\natural(E_1) + \ldots + \natural(E_n) = n - 1$. This maintains the necessary invariant.

Undecidability and Decidability

Theorem

If $\mathcal{C} \neq \varnothing$ (i.e., at least one subexponential allows non-local contraction), then the derivability problem in $\mathrm{SLC}^{\mathbf{1}}_{\Sigma}$ is undecidable.

Proof.

Encoding semi-Thue systems.

For each rewriting rule $u_1 \dots u_k \Rightarrow v_1 \dots v_m$ let $B_i = (u_1 \cdot \ldots \cdot u_k) / (v_1 \cdot \ldots \cdot v_m)$ and add $\mathbf{1} \,/\, !^s B_i, !^s B_i$ (where $s \in \mathcal{C}$) to the antecedent Φ. Then $\Phi, b_1, \dots, b_k \to a_1 \cdot \ldots \cdot a_m$ is derivable in $\mathrm{SLC}^{\mathbf{1}}_{\Sigma}$ iff $a_1 \dots a_m$ yields $b_1 \dots b_k$ in the semi-Thue system. □

Theorem

If $\mathcal{C} = \varnothing$, then the derivability problem in SCLL_{Σ} is decidable and belongs to PSPACE and the derivability problem in SMCLL_{Σ} (without additives) belongs to NP.

Proof.

By cut-free proof search, exactly as in the case without subexponentials. □

Related Work

- Lincoln et al. (1992): undecidability and cut elimination for propositional linear logic with one exponential, including the non-commutative (cyclic) case.
- Ordered Logical Frameworks [Polakow 2000; Simmons and Pfenning 2011]
- Categorial grammar parsers / theorem-provers:
  - CatLog [Morrill 2012], based on the Lambek calculus with brackets (introduce controlled non-associativity);
  - Grail [Moot 2017], based on non-commutative multi-modal Lambek calculus (modalities can restore associativity).

Focusing

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- We propose a system based on non-commutative linear logic, with both commutative and non-commutative subexponentials.
- Ongoing work, paper in IJCAR 2018 ("A Logical Framework with Commutative and Non-Commutative Subexponentials"), which we discuss next.

Logical Frameworks

Logical Specifications allow for the specification of deductive systems, logics, and operational semantics.

- **Linear Logical Frameworks**: Specify state conscious systems;

$$\Theta; \Gamma \longrightarrow G$$

Unbounded Formulas: Interpreted as a **set** of formulas.

Linear Formulas: Interpreted as a **multiset** of formulas.

Logical Frameworks

Two extensions of **Linear Logical Frameworks**:

Subexponentials

[Nigam,Olarte,Pimentel, Reis]

$$\Theta_1; \ldots; \Theta_n; \Gamma_1; \ldots; \Gamma_m \longrightarrow G$$

Allows for **many unbounded and linear contexts**.

- **Extended expressiveness:** specificaiton of systems with several contexts: logics, concurrent programming, etc.

Ordered Logics

[Pfenning,Simmons,Polakow]

$$\Theta; \Gamma; L \longrightarrow G$$

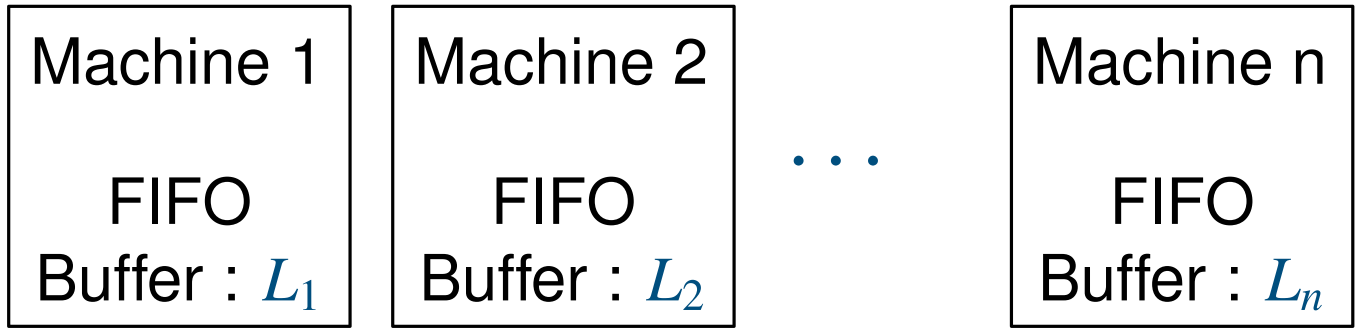
L - **Ordered Formulas:** Interpreted as a **list** of formulas.

- **Extended expressiveness:** specificaiton of systems with some order (PL evaluation strategies, systems with lists, etc.)

Contribution 1: A logical framework with commutative and non-commutative subexponentials.

Application

Example: Distributed System Semantics



List of formulas

$$\Theta; [\mathsf{start}, \Gamma_1, \mathsf{end}]_{\mathsf{m}1}; [\mathsf{start}, \Gamma_2, \mathsf{end}]_{\mathsf{m}2}; \cdots ; [\mathsf{start}, \Gamma_n, \mathsf{end}]_{\mathsf{m}n} \longrightarrow G$$

Specification of the behavior of the system.

Lambek Proof System

$$\frac{}{F \to F}\ I \qquad \frac{\Gamma_1, \Gamma_2 \to C}{\Gamma_1, \mathbf{1}, \Gamma_2 \to C}\ \mathbf{1}_L \qquad \frac{}{\to \mathbf{1}}\ \mathbf{1}_R$$

Initial and Unit

$$\frac{\Pi \to G \quad \Gamma_1, F, \Gamma_2 \to C}{\Gamma_1, F / G, \Pi, \Gamma_2 \to C}\ /_L \qquad \frac{\Pi, F \to G}{\Pi \to G / F}\ /_R$$

Right Division

$$\frac{\Pi \to F \quad \Gamma_1, G, \Gamma_2 \to C}{\Gamma_1, \Pi, F \backslash G, \Gamma_2 \to C}\ \backslash_L \qquad \frac{F, \Pi \to G}{\Pi \to F \backslash G}\ \backslash_R$$

Left Division

$$\frac{\Gamma_1, F, G, \Gamma_2 \to C}{\Gamma_1, F \cdot G, \Gamma_2 \to C}\ \cdot_L \qquad \frac{\Gamma_1 \to F \quad \Gamma_2 \to G}{\Gamma_1, \Gamma_2 \to F \cdot G}\ \cdot_R$$

Product

$$\frac{\Pi \to F\{e/x\}}{\Pi \to \forall x.F}\ \forall_R \qquad \frac{\Gamma_1, F\{t/x\}, \Gamma_2 \to C}{\Gamma_1, \forall x.F, \Gamma_2 \to C}\ \forall_L$$

Quantifier

The order of formulas is important.

Proof System with Subexponentials

Subexponential Signature

$$\Sigma = \langle \mathcal{I}, \preceq, \mathcal{W}, \mathcal{C}, \mathcal{E} \rangle$$

$SNILL_\Sigma$ proof system.

- $\mathcal{I}$ is a set of lables, $\mathcal{W}, \mathcal{C}, \mathcal{E} \subseteq \mathcal{I}$
- $\preceq$ is a pre-order relation over $\mathcal{I}$ upwardly closed w.r.t. $\mathcal{W}, \mathcal{C}, \mathcal{E}$.

For each $s \in \mathcal{I}$:

$$\frac{\Gamma_1, F, \Gamma_2 \to G}{\Gamma_1, !^{s} F, \Gamma_2 \to G}\ Der \qquad \frac{!^{s_1} F_1, \ldots, !^{s_n} F_n \longrightarrow F}{!^{s_1} F_1, \ldots, !^{s_n} F_n \longrightarrow !^{s} F}\ !^{s}{}_R, \text{provided, } s \preceq s_i, 1 \leq i \leq n$$

For each $w \in \mathcal{W}$ and $c \in \mathcal{C}$:

$$\frac{\Gamma, \Delta \longrightarrow G}{\Gamma, !^{w} F, \Delta \longrightarrow G}\ W \qquad \frac{\Gamma_1, !^{c} F, \Delta, !^{c} F, \Gamma_2 \to G}{\Gamma_1, !^{c} F, \Delta, \Gamma_2 \to G}\ C_1 \qquad \frac{\Gamma_1, !^{c} F, \Delta, !^{c} F, \Gamma_2 \to G}{\Gamma_1, \Delta, !^{c} F, \Gamma_2 \to G}\ C_2$$

For each $e \in \mathcal{E}$:

$$\frac{\Gamma_1, \Delta, !^{e} F, \Gamma_2 \to C}{\Gamma_1, !^{e} F, \Delta, \Gamma_2 \to C}\ E_1 \qquad \frac{\Gamma_1, !^{e} F, \Delta, \Gamma_2 \to C}{\Gamma_1, \Delta, !^{e} F, \Gamma_2 \to C}\ E_2$$

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- $\preceq$ is a pre-order relation over $\mathcal{I}$ upwardly closed w.r.t. $\mathcal{W}, \mathcal{C}, \mathcal{E}$.

- **Theorem** For any well formed Σ, SNILL_Σ admits cut-elimination.

Proof Extends our previous results [Dale-Fest, MSCS 18] with quantifiers.

Kinds of Formulas

Assumption:

- $\mathcal{W} \subseteq \mathcal{E}$
- $\mathcal{C} \subseteq \mathcal{E}$

These assumptions are enough for our examples and facilitate proof search (focused proof system for SNILL).

A formula of the form $!^{\mathsf{s}} F$ is

- **Linear Formulas** if $\mathsf{s} \notin \mathcal{W} \cup \mathcal{C}$. They can be **non-commutative** if $\mathsf{s} \notin \mathcal{E}$ and **commutative** otherwise if $\mathsf{s} \in \mathcal{E}$;
- **Unbounded Formulas** if $\mathsf{s} \in \mathcal{W} \cap \mathcal{C}$;
- **Affine Formulas** if $\mathsf{s} \in \mathcal{W}$ and $\mathsf{s} \notin \mathcal{C}$;
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Logical frameworks have been proposed with unbounded, linear and affine formulas, but without relevant formulas.

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Logical frameworks have been proposed with unbounded, linear and affine formulas, but without relevant formulas.

Safe to contract unbounded formulas as one does not lose provability.

$$\dfrac{\dfrac{!^{\mathsf{u}}F, !^{\mathsf{r}}H, \Gamma \longrightarrow G_1 \qquad !^{\mathsf{u}}F, \Delta \longrightarrow G_2}{!^{\mathsf{u}}F, !^{\mathsf{r}}H, \Gamma, !^{\mathsf{u}}F, \Delta \longrightarrow G_1 \cdot G_2}\ \otimes_R}{!^{\mathsf{u}}F, !^{\mathsf{r}}H, \Gamma, \Delta \longrightarrow G_1 \cdot G_2}\ C$$

Not always safe to contract relevant formulas as one may lose provability.

$$\dfrac{\dfrac{!^{\mathsf{u}}F, !^{\mathsf{r}}H, \Gamma \longrightarrow G_1 \qquad !^{\mathsf{u}}F, !^{\mathsf{r}}H, \Delta \longrightarrow G_2}{!^{\mathsf{u}}F, !^{\mathsf{r}}H, \Gamma, !^{\mathsf{u}}F, !^{\mathsf{r}}H, \Delta \longrightarrow G_1 \cdot G_2}\ \otimes_R}{!^{\mathsf{u}}F, !^{\mathsf{r}}H, \Gamma, \Delta \longrightarrow G_1 \cdot G_2}\ 2 \times C$$

Contribution 2: Logical framework with relevant formulas.

Application: Type-Logical Grammar

Assign logical formulas (or types) to sentences.

$$\overset{N}{\text{John}}\ \overset{N \backslash S / N}{\text{loves}}\ \overset{N}{\text{Mary.}}$$

“John loves Mary.”

$$\dfrac{N \rightarrow N \qquad \dfrac{N \rightarrow N \quad S \rightarrow S}{N, N \backslash S \rightarrow S}}{N, N \backslash S / N, N \rightarrow S}$$

The proof of formulas for sentences **may have contraction**: parasitic extraction.

“John signed the paper without reading it”

“The paper that John signed without reading.”

“It” has been omitted twice.

Application: Type-Logical Grammar

"The paper that John signed without reading."

$$\dfrac{\dfrac{\dfrac{\dfrac{N, N \backslash S \,/\, N, N, (N \backslash S) \backslash (N \backslash S) \,/\, GC, GC \,/\, N, N \to S}{N, N \backslash S \,/\, N, !^{s}N, (N \backslash S) \backslash (N \backslash S) \,/\, GC, GC \,/\, N, !^{s}N \to S}\ Der}{N, N \backslash S \,/\, N, (N \backslash S) \backslash (N \backslash S) \,/\, GC, GC \,/\, N, !^{s}N \to S}\ C_L}{N, N \backslash S \,/\, N, (N \backslash S) \backslash (N \backslash S) \,/\, GC, GC \,/\, N \to S \,/\, !^{s}N} \qquad N \,/\, CN, CN, CN \backslash CN \to N}{N \,/\, CN, CN, (CN \backslash CN) \,/\, (S \,/\, !^{s}N), N, N \backslash S \,/\, N, (N \backslash S) \backslash (N \backslash S) \,/\, GC, GC \,/\, N \to N}$$

Contraction to fill the gap.

Application: Type-Logical Grammar

On the other hand, weakening should be avoided:

"The girl whom John loves Mary."

Is a mal-formed sentence which can be typed if weakening is allowed:

$$\dfrac{\dfrac{\dfrac{N, N \backslash S / N, N \rightarrow S}{N, N \backslash S / N, N, !^{s} N \rightarrow S}\, W_L}{N, N \backslash S / N, N \rightarrow S / !^{s} N} \qquad N / CN, CN, CN \backslash CN \rightarrow N}{N / CN, CN, (CN \backslash CN) / (S / !^{s} N), N, N \backslash S / N, N \rightarrow N}$$

Relevant formulas are useful for Type-Logical Grammars.

Relevant Formulas

Contribution 2: Logical framework with relevant formulas.

Lemma 1: Contraction rules permute over all rules except rules $\cdot_R, \backslash_L, /_L$ and Der.

This means that it is safe to not contract formulas for rules other than $\cdot_R, \backslash_L, /_L$ and Der, but not safe otherwise.

$$\dfrac{\dfrac{\Pi_1, !^{\mathsf{r}} F, \Pi_2 \longrightarrow F_1 \qquad \Gamma_1, !^{\mathsf{r}} F, \Gamma_2, F_2, \Gamma_3 \longrightarrow G}{\Gamma_1, !^{\mathsf{r}} F, \Gamma_2, \Pi_1, !^{\mathsf{r}} F, \Pi_2, F_1 \backslash F_2, \Gamma_3 \longrightarrow G}\ \backslash_L}{\Gamma_1, \Gamma_2, \Pi_1, !^{\mathsf{r}} F, \Pi_2, F_1 \backslash F_2, \Gamma_3 \longrightarrow G}\ C_L$$

Let us take a closer look at the rules $\cdot_R, \backslash_L, /_L$ and Der.

Relevant Formulas

How about if this other branch requires a copy of $!^{\mathsf{r}}H$ to be proved?

This formula has to be necessarily be used in this branch.

$$\frac{\Gamma_1 \to F \quad \Gamma_2, !^{\mathsf{r}}H, \Gamma_3 \to G}{\Gamma_1, \Gamma_2, !^{\mathsf{r}}H, \Gamma_3 \to F \cdot G} \cdot_R$$

$$\cfrac{\cfrac{\Gamma_1' \to F \quad \Gamma_2, !^{\mathsf{r}}H, \Gamma_3 \to G}{\Gamma_1', \Gamma_2, !^{\mathsf{r}}H, \Gamma_3 \to F \cdot G} \cdot_R}{\Gamma_1, \Gamma_2, !^{\mathsf{r}}H, \Gamma_3 \to F \cdot G} \, n \times C_L$$

We could make as many copies as needed and move them to this branch. **This decision can be done in a lazy fashion.**

Key Observation 1: *During proof search, any relevant formula moved to one premise of $\cdot_R, \backslash_L, /_L$ can be considered unbounded in the other premise.*

Relevant Formulas

If this branch needs more copies of $!^{\mathsf{r}}H$ to be proved?

Copies can be made before dereliction. Moreover **this decision can be made in a lazy fashion.**

$$\frac{\Gamma_1, H, \Gamma_2 \longrightarrow G}{\Gamma_1, !^{\mathsf{r}}H, \Gamma_2 \longrightarrow G}\ Der$$

$$\cfrac{\cfrac{\Gamma_1, H, !^{\mathsf{r}}H, \ldots, !^{\mathsf{r}}H, \Gamma_2 \longrightarrow G}{\Gamma_1, !^{\mathsf{r}}H, !^{\mathsf{r}}H, \ldots, !^{\mathsf{r}}H, \Gamma_2 \longrightarrow G}\ Der}{\Gamma_1, !^{\mathsf{r}}H, \Gamma_2 \longrightarrow G}\ n \times C_R$$

Key Observation 2: *During proof search, any relevant formula derelicted by* Der *can be considered unbounded in its premise.*

Relevant Formulas

Sound to weaken this formula.

$$
\dfrac{\dfrac{}{!^{\mathsf{r}}A \longrightarrow A}\,Der, I
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{!^{\mathsf{r}}A \longrightarrow A}\,Der, I \quad \dfrac{\dfrac{}{A' \longrightarrow A'}\,I}{!^{\mathsf{r}}A, A' \longrightarrow A'}\,W_L \quad \dfrac{}{!^{\mathsf{r}}A \longrightarrow A}\,Der, I}{!^{\mathsf{r}}A, A' \longrightarrow A \cdot A' \cdot A}\,2 \times \cdot_R}{!^{\mathsf{r}}A, A' \longrightarrow A \cdot A' \cdot A}}{}
}{!^{\mathsf{r}}A, A \setminus A' \longrightarrow A \cdot A' \cdot A}\,\setminus_L
$$

Considered as an Unbounded Formula

Corresponds to the proof

$$
\dfrac{\dfrac{\dfrac{}{!^{\mathsf{r}}A \longrightarrow A}\,Der, I \qquad !^{\mathsf{r}}A, A', !^{\mathsf{r}}A \longrightarrow A \cdot A' \cdot A}{!^{\mathsf{r}}A, !^{\mathsf{r}}A, A \setminus A', !^{\mathsf{r}}A \longrightarrow A \cdot A' \cdot A}\,\setminus_L}{!^{\mathsf{r}}A, A \setminus A' \longrightarrow A \cdot A' \cdot A}\,2 \times C_L
$$

Relevant Formulas

How about non-commutative relevant formulas? Assume $\mathsf{s} \in \mathcal{C}$ and $\mathsf{s} \notin \mathcal{E} \cup \mathcal{W}$.

Not possible to finish the proof as s does not allow exchange.

$$\dfrac{\dfrac{}{!^{\mathsf{s}}A \longrightarrow A} \qquad \dfrac{\dfrac{\underline{!^{\mathsf{s}}A}, A_1, A_2 \longrightarrow A_1 \cdot A \cdot A_2}{\underline{!^{\mathsf{s}}A}, A_1 \cdot A_2 \longrightarrow A_1 \cdot A \cdot A_2}}{\underline{!^{\mathsf{s}}A} \longrightarrow (A_1 \cdot A_2 \,/\, A_1 \cdot A \cdot A_2)}}{!^{\mathsf{s}}A \longrightarrow A \cdot (A_1 \cdot A_2 \,/\, A_1 \cdot A \cdot A_2)}$$

Key observation 1 does not work. It should be possible to **refine it by remembering the positions where non-commututative relevant formulas can be contracted to.** **Not needed for our applications and left for future work.**

Logical Framework

We propose a logical framework with commutative and non-commutative subexponentials which incorporates the two key observations.

- Focused proof system for SNILL;
- Prove to be sound and complete with respect to SNILL;
- Details of the system can be found in the paper.

Application

Example: Distributed System Semantics

Machine 1	Machine 2	$\cdots$	Machine n
FIFO Buffer : L_1	FIFO Buffer : L_2		FIFO Buffer : L_n

$$\Theta; [\mathsf{start}, \Gamma_1, \mathsf{end}]_{\mathsf{m}1}; [\mathsf{start}, \Gamma_2, \mathsf{end}]_{\mathsf{m}2}; \cdots ; [\mathsf{start}, \Gamma_n, \mathsf{end}]_{\mathsf{m}n} \longrightarrow G$$

Dequeues a syn message and sends an ack to the network.

$$\mathsf{Deq(i,j)} = !^{\mathsf{mi}}\mathsf{syn}_{\mathsf{mj}} \cdot !^{\mathsf{mi}}\mathsf{end} \setminus !^{\mathsf{mi}}\mathsf{end} \cdot !^{\mathsf{N}}\ \mathsf{ack}_{\mathsf{mj}}$$

$$\mathsf{Enq(i,j)} = !^{\mathsf{mj}}\mathsf{start} \cdot !^{\mathsf{mj}}\mathsf{ack}_{\mathsf{mj}} \ / \ !^{\mathsf{N}}\mathsf{ack}_{\mathsf{mj}} \cdot !^{\mathsf{mj}}\mathsf{start}$$

Receives an ack from the network and enqueues it.

Application

$$\Theta, [\mathsf{start}, \Gamma_1, \mathsf{end}]_{\mathsf{m}1}\ [\mathsf{start}, \Gamma_2, \mathsf{end}]_{\mathsf{m}2}\ \cdots [\mathsf{start}, \Gamma_n, \mathsf{end}]_{\mathsf{m}n} \longrightarrow G$$

Dequeues a syn message and sends an ack to the network.

$$\mathsf{Deq}(\mathsf{i},\mathsf{j}) = !^{\mathsf{mi}}\mathsf{syn}_{\mathsf{mj}} \cdot !^{\mathsf{mi}}\mathsf{end} \setminus !^{\mathsf{mi}}\mathsf{end} \cdot !^{\mathsf{N}}\ \mathsf{ack}_{\mathsf{mj}}$$
$$\mathsf{Enq}(\mathsf{i},\mathsf{j}) = !^{\mathsf{mj}}\mathsf{start} \cdot !^{\mathsf{mj}}\mathsf{ack}_{\mathsf{mj}} \,/\, !^{\mathsf{N}}\mathsf{ack}_{\mathsf{mj}} \cdot !^{\mathsf{mj}}\mathsf{start}$$

Receives an ack from the network and enqueues it.

- Our logical framework **reduces considerably proof search.**
- **Adequacy on the level of derivation:** A focused derivation corresponds exactly to a step of enqueueing or dequeueing.

Application: Type-Logical Grammar

"The paper that John signed without reading."

$$
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\dfrac{N, N \backslash S / N, N, (N \backslash S) \backslash (N \backslash S) / GC, GC / N, N \to S}
{N, N \backslash S / N, !^{\mathsf{s}}N, (N \backslash S) \backslash (N \backslash S) / GC, GC / N, !^{\mathsf{s}}N \to S}\ Der
}
{N, N \backslash S / N, (N \backslash S) \backslash (N \backslash S) / GC, GC / N, !^{\mathsf{s}}N \to S}\ C_L
}
{N, N \backslash S / N, (N \backslash S) \backslash (N \backslash S) / GC, GC / N \to S / !^{\mathsf{s}}N}
\qquad
N / CN, CN, CN \backslash CN \to N
}
{N / CN, CN, (CN \backslash CN) / (S / !^{\mathsf{s}}N), N, N \backslash S / N, (N \backslash S) \backslash (N \backslash S) / GC, GC / N \to N}
$$

- Our logical framework **reduces considerably proof search.**
- Proof search **naturally follows a backward search strategy;**
- **No need to reason** when a relevant formula should be contracted or not.

Conclusions and Future Work

- We proposed a **sound and complete logical framework** with both commutative and non-commutative subexponentials;
- Proposed general techniques to **reduce non-determinism for commutative relevant formulas;**
- Demonstrated its use in **two applications**: distributed systems and tpe-logical grammars;

- We are investigating the impact of our logical framework for categorial parsers;
- Classical logic versions of our logical framework;
- Reduce the non-determinism of non-commutative relevant formulas;
- Semantic interpretations for subexponentials.

Related Work

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