Propositional and first order logic formalizations of social welfare functions

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Content

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Example:

Student 1	Student 2	Student 3	Student 4
Football	Basket	Chess	Basket
Basket	Chess	Basket	Football
Chess	Football	Football	Chess

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Consider this as election with four voters having to choose from three alternatives.

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Football	Basket	Chess	Basket
Basket	Chess	Basket	Football
Chess	Football	Football	Chess

Consider this as election with four voters having to choose from three alternatives.

How should n voters choose from a set of m alternatives?

Borda voting rule:

Student 1	Student 2	Student 3	Student 4
Football	Basket	Chess	Basket
Basket	Chess	Football	Football
Chess	Football	Basket	Chess

each voter gives $m-1$ points to the alternative she ranks first, $m-2$ to the alternative she ranks second,

$$\vdots$$

and the alternative with the most points wins.

Borda voting rule:

Student 1	Student 2	Student 3	Student 4
Football 2	Basket 2	Chess 2	Basket 2
Basket 1	Chess 1	Football 1	Football 1
Chess 0	Football 0	Basket 0	Chess 0

each voter gives $m-1$ points to the alternative she ranks first, $m-2$ to the alternative she ranks second,

$$\vdots$$

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Borda voting rule:

Student 1	Student 2	Student 3	Student 4
Football 2			
		Football 1	Football 1
	Football 0		

each voter gives $m-1$ points to the alternative she ranks first, $m-2$ to the alternative she ranks second,

$\vdots$

and the alternative with the most points wins.

Football gets 4 points,

Borda voting rule:

Student 1	Student 2	Student 3	Student 4
	Basket 2		Basket 2
Basket 1			
		Basket 0	

each voter gives $m-1$ points to the alternative she ranks first, $m-2$ to the alternative she ranks second,

⋮

and the alternative with the most points wins.

Football gets 4 points,
Basket gets 5 points,

Borda voting rule:

Student 1	Student 2	Student 3	Student 4
		Chess 2	
	Chess 1		
Chess 0			Chess 0

each voter gives $m-1$ points to the alternative she ranks first, $m-2$ to the alternative she ranks second,

$\vdots$

and the alternative with the most points wins.

Football gets 4 points,
Basket gets 5 points,
Chess gets 3 points.

Borda voting rule:

Student 1	Student 2	Student 3	Student 4
Football 2	Basket 2	Chess 2	Basket 2
Basket 1	Chess 1	Football 1	Football 1
Chess 0	Football 0	Basket 0	Chess 0

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$$\vdots$$

and the alternative with the most points wins.

Football gets 4 points,
Basket gets 5 points,
Chess gets 3 points.

Borda election outcome is:

Basket
Football
Chess

Notation: individuals and alternatives

{Student 1, Student 2 , Student 3, Student 4}

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{Student 1, Student 2 , Student 3, Student 4}

Individuals

$I = \{1, 2, \ldots, n\}$

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$A = \{a_1, a_2, \ldots, a_m\}$

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$I = \{1, 2, \ldots, n\}$

{Football, Basket, Chess}

Alternatives

$A = \{a_1, a_2, \ldots, a_m\}$

Individuals are expressing preferences over a set of alternatives.

Notation: Preference

P_i	preference by individual i

Notation: Preference

P_i preference by individual i

Student 1
Football
Basket
Chess

Notation: Preference

P_i	preference by individual i
aP_ib	Binary relations

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Basket
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Notation: Preference

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We want those relations to make *strict linear* orders:

Notation: Preference

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aP_ib Binary relations

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Basket
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We want those relations to make *strict linear* orders:

- irreflexive

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We want those relations to make *strict linear* orders:

- irreflexive
- transitive

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We want those relations to make *strict linear* orders:

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- complete

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Football
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We want those relations to make *strict linear* orders:

- irreflexive
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- complete

aP_ib individual i strictly prefers alternative a to alternative b

Notation: Preference profile

$\mathcal{L}(A)$ set of strict linear orders over A

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$\mathcal{L}(A)$	set of strict linear orders over A $P_1, P_2, \ldots, P_n \in \mathcal{L}(A)$

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$\mathcal{L}(A)$ — set of strict linear orders over A

$P_1, P_2, \ldots, P_n \in \mathcal{L}(A)$

$\boldsymbol{P} = (P_1, \ldots, P_n)$ — preference profile constituted from n individual preferences

P_1	P_2	P_3	P_4
F	B	C	B
B	C	B	F
C	F	F	C

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P_1	P_2	P_3	P_4
F	B	C	B
B	C	B	F
C	F	F	C

$\mathcal{L}(A)^I$ — set of profiles

$\boldsymbol{P} \in \mathcal{L}(A)^I$

Social welfare function (SWF)

$$\omega : \mathcal{L}(A)^I \longrightarrow \mathcal{L}(A)$$

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P_1	P_2	P_3	P_4
F	B	C	B
B	C	B	F
C	F	F	C

$\longrightarrow$

Society
F
B
C

Preference profile $\boldsymbol{P} \in \mathcal{L}(A)^I$

Preference $\omega(\boldsymbol{P}) \in \mathcal{L}(A)$

Social welfare function (SWF)

$$\omega : \mathcal{L}(A)^I \longrightarrow \mathcal{L}(A)$$

P_1	P_2	P_3	P_4
F	B	C	B
B	C	B	F
C	F	F	C

$\longrightarrow$

Society
F
B
C

Preference profile $\boldsymbol{P} \in \mathcal{L}(A)^I$

Preference $\omega(\boldsymbol{P}) \in \mathcal{L}(A)$

SWF associates with every preference profile $\boldsymbol{P} \in \mathcal{L}(A)^I$ a strict linear order $\omega(\boldsymbol{P})$

Unanimity

UN: A SWF ω satisfies *unanimity* if, whenever every individual strictly prefers alternative a to alternative b, so does society.

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P_1	P_2	P_3	P_4	P_5		$\omega(\boldsymbol{P})$
a	a	⋮	⋮	a		⋮
b	⋮	⋮	a	⋮		a
⋮	⋮	a	⋮	⋮	$\longrightarrow$	⋮
⋮	b	b	b	⋮		b
⋮	⋮	⋮	⋮	b		⋮

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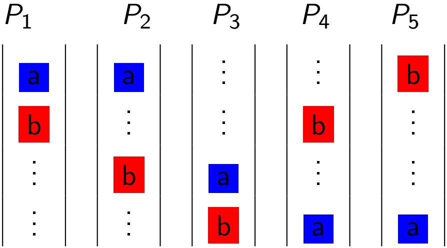
P_1	P_2	P_3	P_4	P_5		$\omega(\boldsymbol{P})$
a	a	⋮	⋮	a		⋮
b	⋮	⋮	a	⋮		a
⋮	⋮	a	⋮	⋮		⋮
⋮	b	b	b	⋮	$\longrightarrow$	b
⋮	⋮	⋮	⋮	b		⋮

Formally,
if aP_ib for every individual $i \in I$, then $a\omega(\boldsymbol{P})b$.

Independence of irrelevant alternatives

IIA: Given two preference profiles $\boldsymbol{P}$ and $\boldsymbol{P'}$, if for every individual $i \in I$ we have that $aP_i b$ if and only if $aP'_i b$, then $a\omega(\boldsymbol{P})\, b$ if and only if $a\omega(\boldsymbol{P'})\, b$.

Independence of irrelevant alternatives



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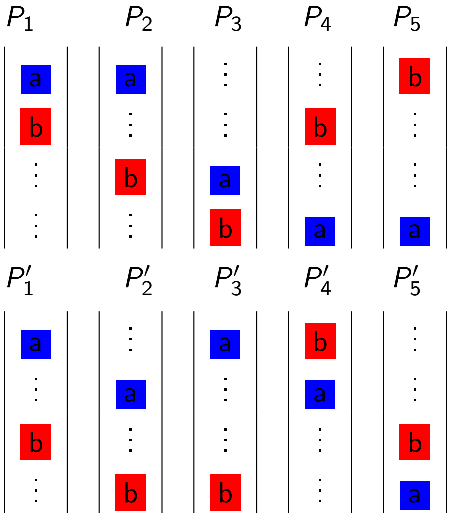

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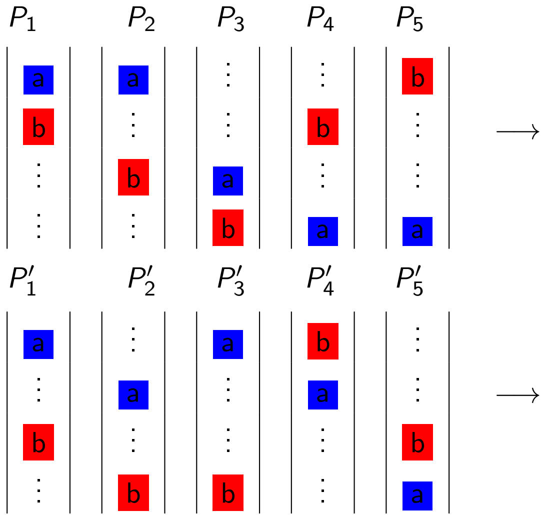

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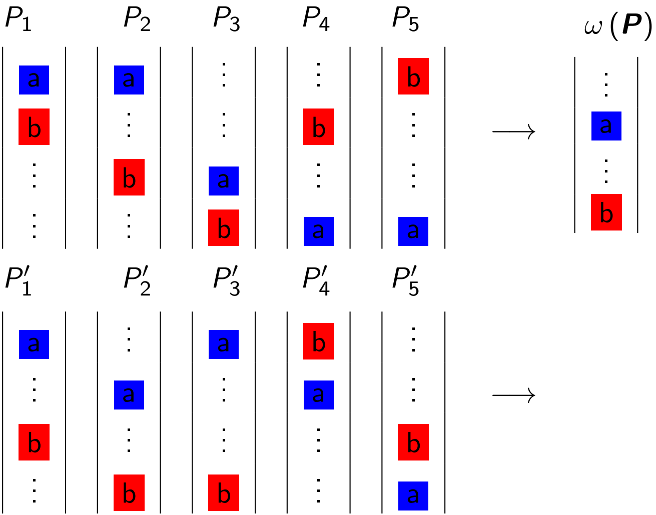

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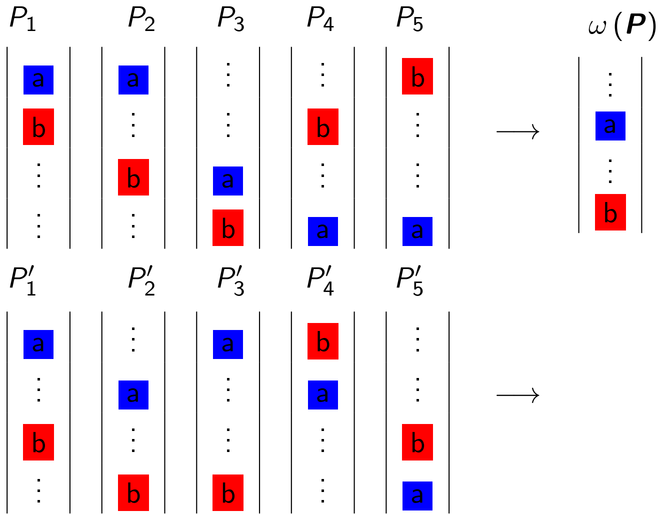

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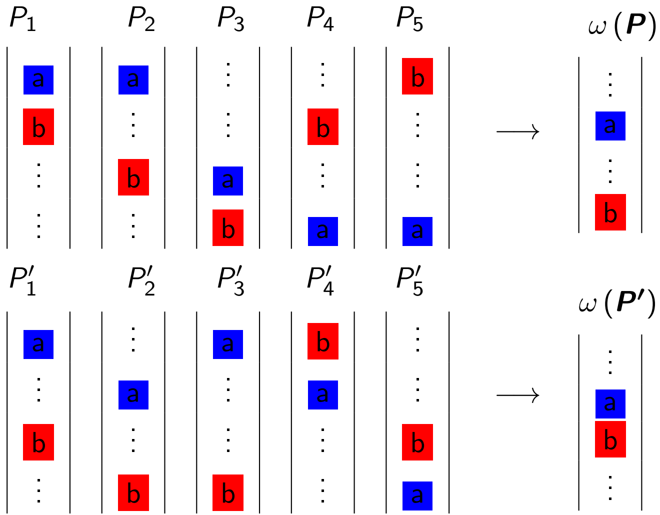

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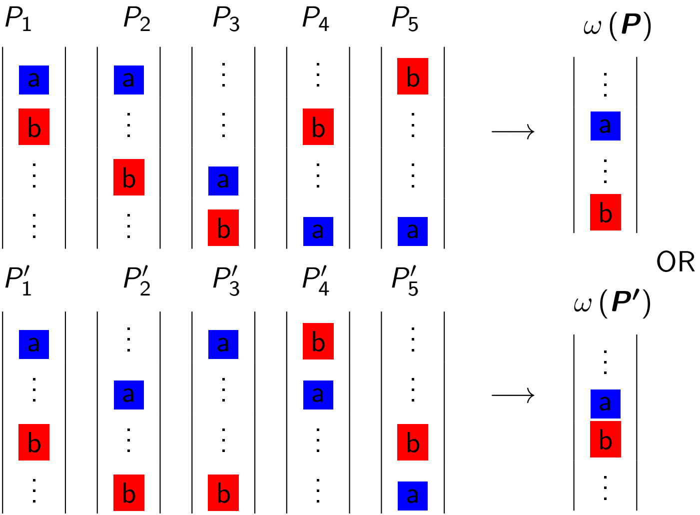

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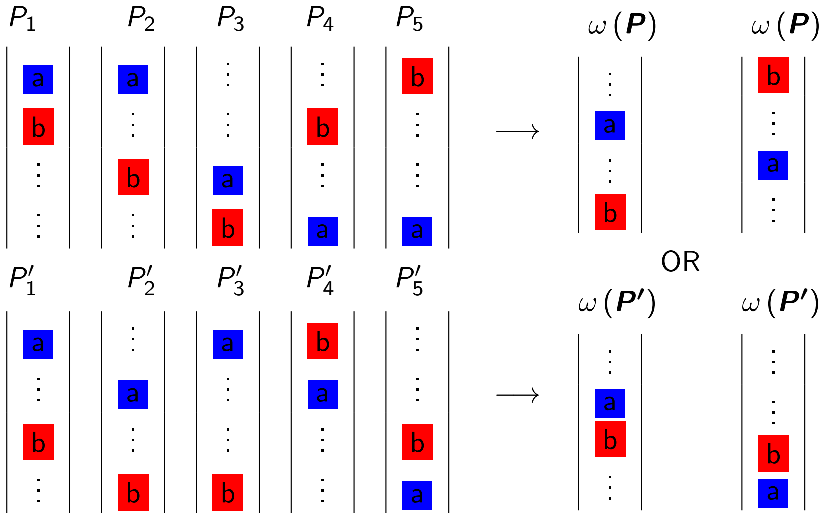

Non-dictatorship

ND: A SWF ω is *non-dictatorial* if there is no individual $i \in I$ such that for every profile $\boldsymbol{P}$ the social preference order $\omega(\boldsymbol{P})$ is equal to P_i.

Arrow's Theorem

If A and I are finite and non-empty, and if $|A| \geq 3$, then there exists no SWF for A and I that satisfies **UN, IIA** and **ND**.

Formalization in FOL

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We have to quantify over individuals, alternatives and profiles.

Individual preference: $P_i \subseteq A^2$, $i \in I$

Profile: $\boldsymbol{P} = (P_1, P_2, \ldots, P_n)$

Set of situations S

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Let $\boldsymbol{P}^u$ be the preference profile associated with situation u.

We would like to have a situation for every profile.

- The finiteness of the domain.
- The fact that two strict linear orders can be generated from each other using a sequence of swaps.

Relational first-order signature

relational signature

$\mathcal{L}_{SWF} = \{A^{(1)}, I^{(1)}, S^{(1)},$

Three unary predicates to mark alternatives(A), individuals (I) and situations(S)

relational first-order signature

relational signature

$\mathcal{L}_{SWF} = \{A^{(1)}, I^{(1)}, S^{(1)}, p^{(4)},$

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A predicate p of arity 4.

relational first-order signature

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A predicate p of arity 4.

$p(z, x, y, u)$ indicates that individual z prefers x over y in situation u.

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$p\,(z, x, y, u)$ indicates that individual z prefers x over y in situation u.

If we choose z and u then we get binary relation P_z^u.

relational first-order signature

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$$\mathcal{L}_{SWF} = \{A^{(1)}, I^{(1)}, S^{(1)}, p^{(4)}, \omega^{(3)}\}$$

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A ternary relation ω that stands for the SWF.

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$\omega(x, y, u)$ translates as x is preffered over y in the collective order associated with situation u.

relational first-order signature

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$$\mathcal{L}_{SWF} = \{A^{(1)}, I^{(1)}, S^{(1)}, p^{(4)}, \omega^{(3)}\}$$

A ternary relation ω that stands for the SWF.

$\omega(x, y, u)$ translates as x is preffered over y in the collective order associated with situation u.

For every situation u we have a binary relation $\omega(\boldsymbol{P}^u)$.

LINp

Axioms of strict linear order for $p\left(z,\cdot,\cdot,u\right)$.

- $I\left(z\right)\wedge S\left(u\right)\wedge A\left(x\right)\wedge A\left(y\right)\rightarrow \left(p\left(z,x,y,u\right)\vee p\left(z,y,x,u\right)\vee x=y\right)$ complete

LINp

Axioms of strict linear order for $p(z, \cdot, \cdot, u)$.

- $I(z) \wedge S(u) \wedge A(x) \wedge A(y) \rightarrow (p(z, x, y, u) \vee p(z, y, x, u) \vee x = y)$ — complete
- $I(z) \wedge S(u) \wedge A(x) \rightarrow \neg p(z, x, x, u)$ — irreflexive

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- $I(z) \wedge S(u) \wedge A(x) \rightarrow \neg p(z,x,x,u)$ — irreflexive
- $I(z) \wedge S(u) \wedge A(x_1) \wedge A(x_2) \wedge A(x_3) \wedge p(z,x_1,x_2,u) \wedge p(z,x_2,x_3,u) \rightarrow p(z,x_1,x_3,u)$ — transitive

LINω

Axioms of strict linear order for $\omega(\cdot, \cdot, u)$.

- $S(u) \wedge A(x) \wedge A(y) \rightarrow (\omega(x, y, u) \vee \omega(y, x, u) \vee x = y)$

LINω

Axioms of strict linear order for $\omega(\cdot, \cdot, u)$.

- $S(u) \wedge A(x) \wedge A(y) \rightarrow (\omega(x, y, u) \vee \omega(y, x, u) \vee x = y)$
- $S(u) \wedge A(x) \rightarrow \neg\omega(x, x, u)$

LINω

Axioms of strict linear order for $\omega(\cdot,\cdot,u)$.

- $S(u) \wedge A(x) \wedge A(y) \rightarrow (\omega(x,y,u) \vee \omega(y,x,u) \vee x = y)$
- $S(u) \wedge A(x) \rightarrow \neg\omega(x,x,u)$
- $S(u) \wedge A(x_1) \wedge A(x_2) \wedge A(x_3) \wedge \omega(x_1,x_2,u) \wedge \omega(x_2,x_3,u) \rightarrow \omega(x_1,x_3,u)$

MIN

There are at least 3 different alternatives, and I and S are non-empty.

- $\exists x_1.\exists x_2.\exists x_3.A(x_1) \wedge A(x_2) \wedge A(x_3) \wedge ((x_1 \neq x_2) \wedge (x_1 \neq x_3) \wedge (x_2 \neq x_3))$

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- $\exists z.I(z)$

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- $\exists u.S(u)$

PART

I,A and S form a partition of the universe.

- $A(x) \rightarrow (\neg I(x) \wedge \neg S(x))$

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- $I(x) \to (\neg A(x) \wedge \neg S(x))$
- $S(x) \to (\neg I(x) \wedge \neg A(x))$
- $A(x) \vee I(x) \vee S(x)$

INJ

The encoding of situations into preference profiles must be injective.

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INJ:

- $S(u) \wedge S(v) \wedge u \neq v \rightarrow \exists z.\exists x.\exists y.\left[I(z) \wedge A(x) \wedge A(x) \wedge p(z,x,y,u) \wedge p(z,y,x,v)\right]$

The fact about swaps

Two linear orders can be generated from each other using a sequence of swaps.

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Two linear orders can be generated from each other using a sequence of swaps.

b	c
a	b
c	d
d	a

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a		b
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The same fact stands for profiles, but we need more steps because we have to repeat the same procedure for each individual preference.

PERM

PERM: $p(z, x, y, u) \rightarrow \exists v.\{S(v) \wedge p(z, y, x, v) \wedge$

PERM

PERM: $p(z,x,y,u) \to \exists v.\{S(v) \land p(z,y,x,v) \land$
$\forall x_1.[p(z,x,x_1,u) \land p(z,x_1,y,u) \to p(z,x_1,x,v) \land p(z,y,x_1,v)] \land$
$\forall x_1.[(p(z,x_1,x,u) \to p(z,x_1,y,v)) \land (p(z,y,x_1,u) \to p(z,x,x_1,v))] \land$
$\forall x_1.\forall y_1.[x_1 \neq x \land x_1 \neq y \land y_1 \neq y \land y_1 \neq x \to (p(z,x_1,y_1,u) \leftrightarrow p(z,x_1$
$\forall z_1.\forall x_1.\forall y_1.[z_1 \neq z \to (p(z_1,x_1,y_1,u) \leftrightarrow p(z_1,x_1,y_1,v))]\}$

T_{SWF}

Call T_{SWF} the theory composed of all the axioms from previous slides.

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T_{SWF} characterizes the class of SWFs.

Every SWF gives us a model for T_{SWF}.

If a model for $\mathcal{L}_{SWF}$ represents a SWF then it satisfies the theory T_{SWF}.

Arrow's axioms

UN: $S(u) \wedge A(x) \wedge A(y) \rightarrow$
$[(\forall z.(I(z) \rightarrow p(z, x, y, u))) \rightarrow \omega(x, y, u)]$

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IIA: $S(u_1) \wedge S(u_2) \wedge A(x) \wedge A(y) \rightarrow$
$[\forall z.(I(z) \rightarrow (p(z,x,y,u_1) \leftrightarrow p(z,x,y,u_2))) \rightarrow (\omega(x,y,u_1) \leftrightarrow \omega(x$

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$[(\forall z.(I(z) \rightarrow p(z,x,y,u))) \rightarrow \omega(x,y,u)]$

IIA: $S(u_1) \wedge S(u_2) \wedge A(x) \wedge A(y) \rightarrow$
$[\forall z.(I(z) \rightarrow (p(z,x,y,u_1) \leftrightarrow p(z,x,y,u_2))) \rightarrow (\omega(x,y,u_1) \leftrightarrow \omega(x$

ND: $I(z) \rightarrow$
$\exists x.\exists y.\exists u.[S(u) \wedge A(x) \wedge A(y) \wedge p(z,x,y,u) \wedge \omega(y,x,u)]$

Arrow's axioms

UN: $S(u) \wedge A(x) \wedge A(y) \rightarrow$
$[(\forall z.(I(z) \rightarrow p(z,x,y,u))) \rightarrow \omega(x,y,u)]$

IIA: $S(u_1) \wedge S(u_2) \wedge A(x) \wedge A(y) \rightarrow$
$[\forall z.(I(z) \rightarrow (p(z,x,y,u_1) \leftrightarrow p(z,x,y,u_2))) \rightarrow (\omega(x,y,u_1) \leftrightarrow \omega(x$

ND: $I(z) \rightarrow$
$\exists x.\exists y.\exists u.[S(u) \wedge A(x) \wedge A(y) \wedge p(z,x,y,u) \wedge \omega(y,x,u)]$

Adding to T_{SWF} those three axioms we obtain a theory that we shall call T_{ARROW}.

Why propositional logic?

Arrow's original proof contained an error.

Automatically derived proof can give additional assurences for the correctness of a result.

T_{ARROW} has no finite models.

We can't express the Arrow's theorem in a sentence of T_{ARROW}.

Propositional Logic

For the special case of $n = 2$ and $m = 3$(or indeed any fixed sizes) we can rewrite the FOL representation in propositional logic:

- predicates $p\,(z, x, y, u)$ becomes atomic propositions $p_{z,x,y,u}$
- predicates $\omega\,(x, y, u)$ become atomic propositions $\omega_{x,y,u}$
- universal quantifications become conjuctions and existential quantifications become disjunctions.

That is, we need $2 \cdot 3^2 \cdot (3!)^2 + 3^2 \cdot (3!)^2 = 972$ propositional variables.

Example of sentence in prop logic

The following formula express the unanimity:

$$\bigwedge_{\substack{i,j\in\{1,2,3\}\\ k\in\{1,\ldots,36\}}} \left(p_{z_1,x_i,y_j,u_k} \wedge p_{z_2,x_i,y_j,u_k} \rightarrow \omega_{x_i,y_j,u_k}\right)$$

Inductive lemmas

Tang and Lin (2009) proved two inductive lemmas:

- If there exists an Arrovian SWF for n individuals and $m+1$ alternatives, then there exists one for n and m (if $n \geq 2$, $m \geq 3$).
- If there exists an Arrovian SWF for $n+1$ individuals and m alternatives, then there exists one for n and m (if $n \geq 2$, $m \geq 3$).

That is, Arrow's Theorem holds iff its "base case" for 2 individuals and 3 alternatives is ture - which we've modelled in propositional logic.