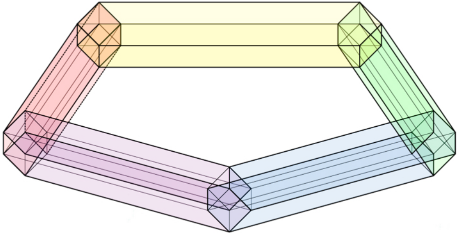
Categorified cyclic operads (in nature)

Logic and applications 2018



Jovana Obradović
Charles University Prague

Relevant examples of categorified structures

sets	categories
functions	functors
equalities	coherent isomorphisms

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Coherence of symmetric monoidal categories

S. Mac Lane
Categories for the Working Mathematician
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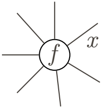
Coherence of categorified operads

K. Došen, Z. Petrić
Weak Cat-operads
Logical Methods in Computer Science, 2009

$$\beta^{i,j}_{f,g,h} : (f \circ_i g) \circ_j h \to f \circ_i (g \circ_{j-i+1} h) \qquad \theta^{i,j}_{f,g,h} : (f \circ_i g) \circ_j h \to (f \circ_j h) \circ_{i+n-1} g$$

Cyclic operads

$$\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{Set}$$

$\mathcal{C}(X) \ni$ 

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$$\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{Set}$$

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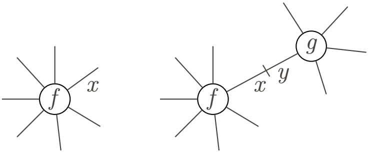
$\mathcal{C}(X) \ni$ 

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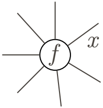
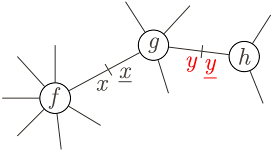
$\mathcal{C}(X) \ni$ 



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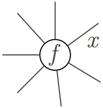
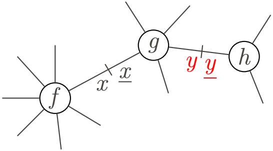
$\mathcal{C}(X) \ni$ 



$$f^{\sigma_1} \; {}_{\sigma_1^{-1}(x)}\circ_{\sigma_2^{-1}(y)} \; g^{\sigma_2} = (f \, {}_x\circ_y \, g)^{\sigma}$$

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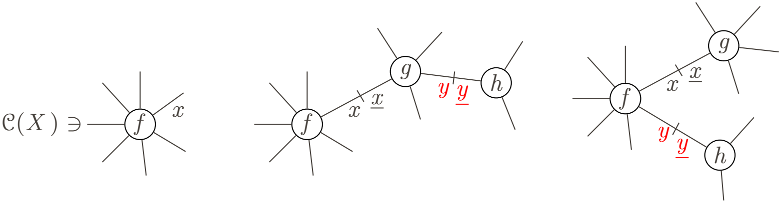

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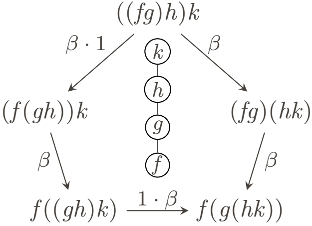
Parallel associativity

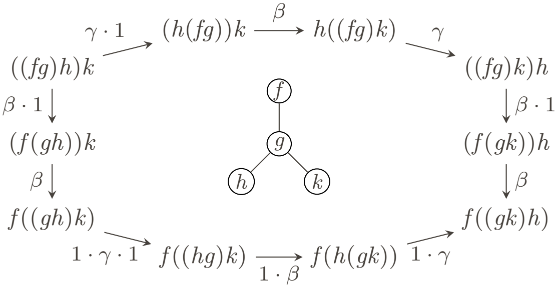
$$(f\ {}_x\circ_{\underline{x}}\ g)\ {}_y\circ_{\underline{y}}\ h = (g\ {}_{\underline{x}}\circ_x\ f)\ {}_y\circ_{\underline{y}}\ h = g\ {}_{\underline{x}}\circ_x\ (f\ {}_y\circ_{\underline{y}}\ h) = (f\ {}_y\circ_{\underline{y}}\ h)\ {}_x\circ_{\underline{x}}\ g$$

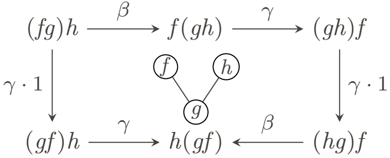
$$\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{Cat}$$

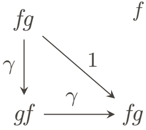
$$_x\circ_y : \mathcal{C}(X) \times \mathcal{C}(Y) \to \mathcal{C}(X\backslash\{x\} \cup Y\backslash\{y\})$$

$$\beta : (f\ _x\circ_{\underline{x}}\ g)\ _y\circ_{\underline{y}}\ h \to f\ _x\circ_{\underline{x}}\ (g\ _y\circ_{\underline{y}}\ h) \qquad \gamma : f\ _x\circ_y\ g \to g\ _y\circ_x\ f$$









$$f^{\sigma_1}\ _{\sigma_1^{-1}(x)}\circ_{\sigma_2^{-1}(y)}\ g^{\sigma_2} = (f\,_x\circ_y\, g)^\sigma$$

$$\beta^\sigma_{f,g,h} = \beta_{f^{\sigma_1}, g^{\sigma_2}, h^{\sigma_3}}$$

$$\gamma^\sigma_{f,g} = \gamma_{f^{\sigma_1}, g^{\sigma_2}}$$

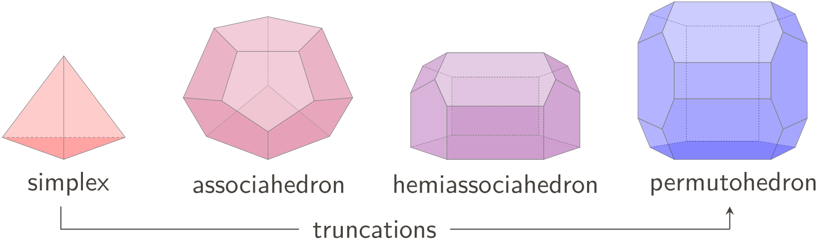
$$(\varphi \cdot \psi)^\sigma = \varphi^{\sigma_1} \cdot \psi^{\sigma_2}$$

Polytopes of categorified cyclic operads

Categorified operads: Hypergraph polytopes

K. Došen, Z. Petrić
Hypergraph polytopes
Topology and its Applications 158, pp. 1405–1444, 2011

P.-L. Curien, J. Obradović, J. Ivanović
Syntactic aspects of hypergraph polytopes
Journal of Homotopy and Related Structures
https://doi.org/10.1007/s40062-018-0211-9



"the more hyperedges, the more truncations"

Hypergraph terminology

Hypergraph $\mathbf{H} = (H, \mathbf{H})$

H - vertices $\mathbf{H} \subseteq \mathcal{P}(H) \backslash \emptyset$ - hyperedges $\bigcup \mathbf{H} = H$

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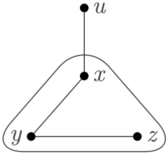

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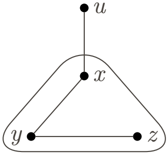

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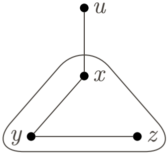

Additional properties *non-empty*,

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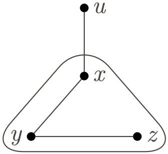

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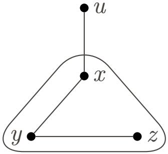

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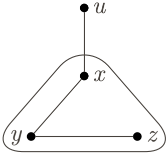

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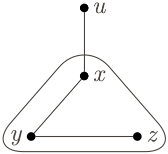

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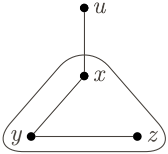

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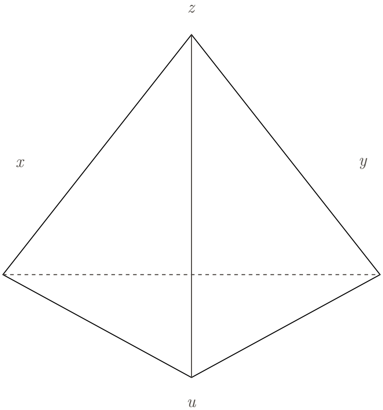

A polytope from a hypergraph

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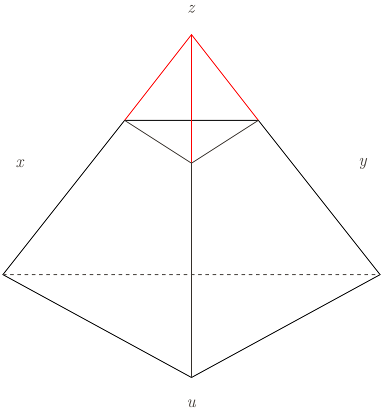

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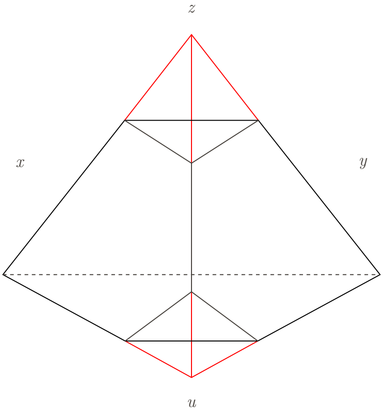

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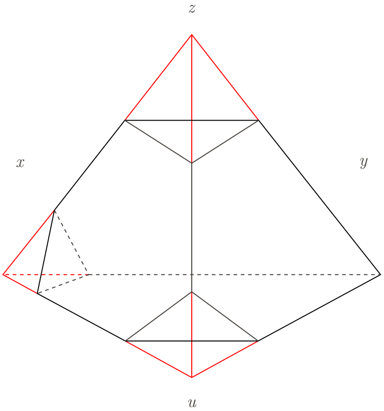

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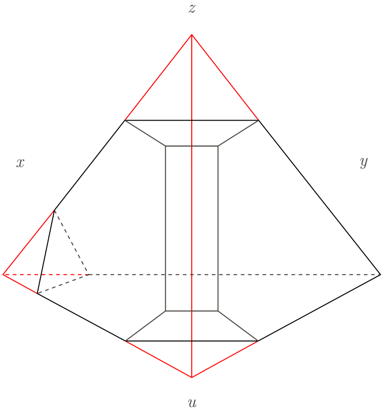

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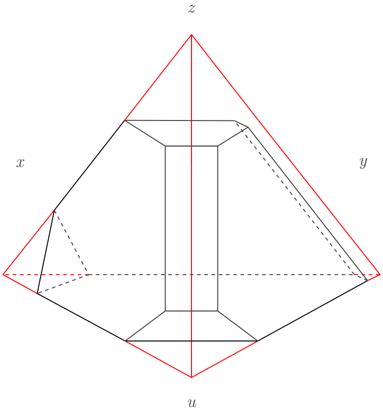

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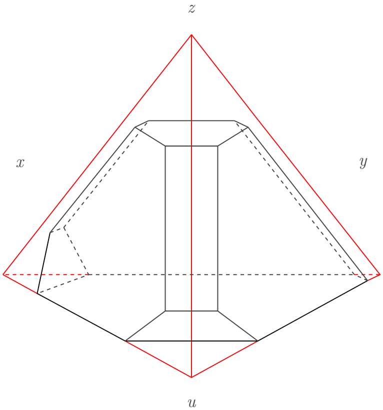

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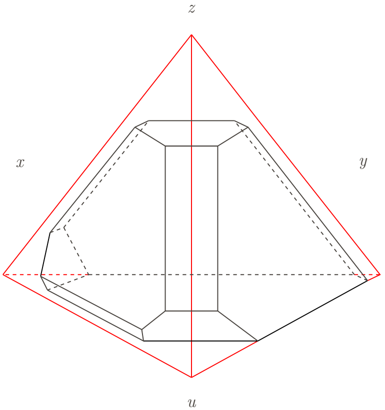

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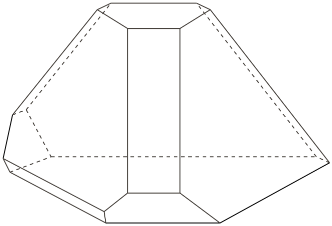
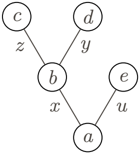
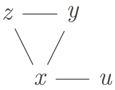


Polytopes of categorified operads





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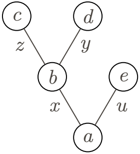

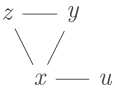

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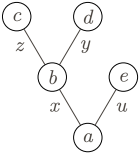

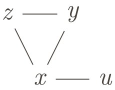

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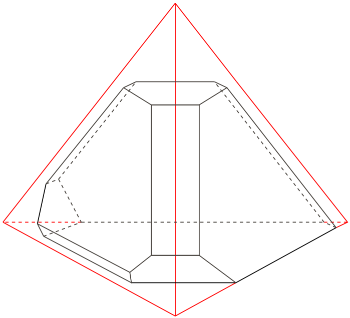
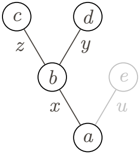
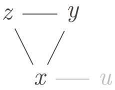
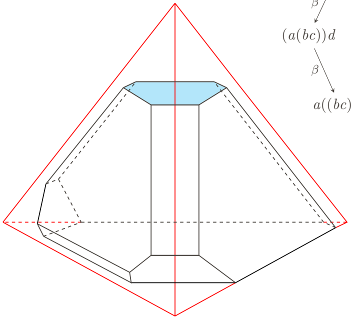


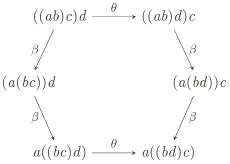
Polytopes of categorified operads



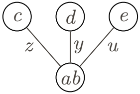


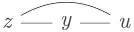
$$\mathbf{H} = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, x\}, \{x, u\}\}$$



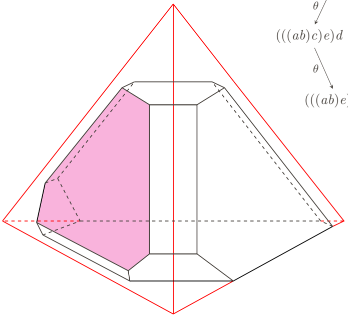


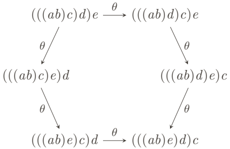
Polytopes of categorified operads





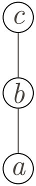
$$\mathbf{H} = \{\{y\}, \{z\}, \{u\}, \{y, z\}, \{y, u\}, \{z, u\}\}$$



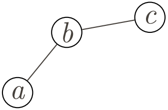


Polytopes of categorified cyclic operads

“Rooted associativity” vs. “Unrooted associativity”



$$(ab)c \xrightarrow{\beta} a(bc)$$



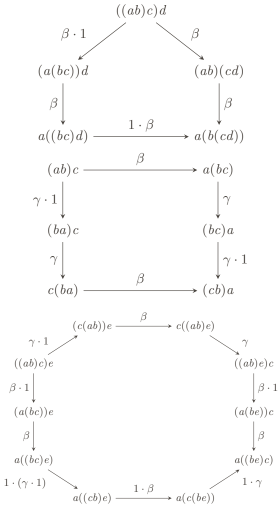
$$(ab)c \xrightarrow{\beta} a(bc)$$

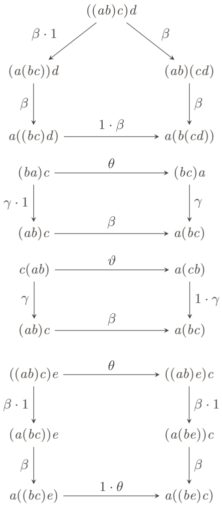
$$(ba)c \dashrightarrow (bc)a$$

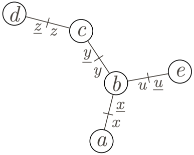
$$c(ba) \xrightarrow{\beta^{-1}} (cb)a$$

$$c(ab) \dashrightarrow a(cb)$$

$\beta,\, \gamma$	$\beta,\, \theta,\, \vartheta,\, \gamma$
$\beta^{x,\underline{x};y,\underline{y}}_{a,b,c} : (a \,{}_x{\circ}_{\underline{x}}\, b) \,{}_y{\circ}_{\underline{y}}\, c \to a \,{}_x{\circ}_{\underline{x}}\, (b \,{}_y{\circ}_{\underline{y}}\, c)$	$\beta^{x,\underline{x};y,\underline{y}}_{a,b,c} : (a \,{}_x{\circ}_{\underline{x}}\, b) \,{}_y{\circ}_{\underline{y}}\, c \to a \,{}_x{\circ}_{\underline{x}}\, (b \,{}_y{\circ}_{\underline{y}}\, c)$
$\gamma^{x,\underline{x}}_{a,b} : a \,{}_x{\circ}_{\underline{x}}\, b \to b \,{}_{\underline{x}}{\circ}_x\, a$	$\theta^{\underline{x},x;y,\underline{y}}_{b,a,c} : (b \,{}_{\underline{x}}{\circ}_x\, a) \,{}_y{\circ}_{\underline{y}}\, c \to (b \,{}_y{\circ}_{\underline{y}}\, c) \,{}_{\underline{x}}{\circ}_x\, a$
	$\vartheta^{\underline{y},y;x,\underline{x}}_{c,a,b} : c \,{}_{\underline{y}}{\circ}_y\, (a \,{}_x{\circ}_{\underline{x}}\, b) \to a \,{}_x{\circ}_{\underline{x}}\, (c \,{}_{\underline{y}}{\circ}_y\, b)$
	$\gamma^{x,\underline{x}}_{a,b} : a \,{}_x{\circ}_{\underline{x}}\, b \to b \,{}_{\underline{x}}{\circ}_x\, a$



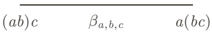


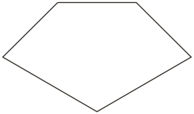


Polytopes of categorified cyclic operads

Categorified operads

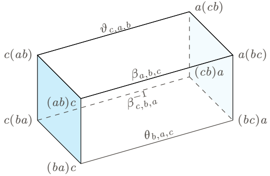
$$\mathcal{A}(\mathbf{H}) = \begin{cases} \text{associahedron} \\ \text{hemiassociahedron} \\ \text{permutohedron} \end{cases}$$

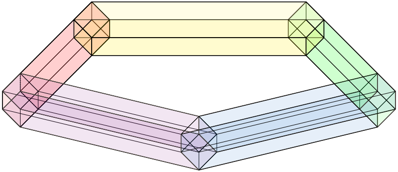




Categorified cyclic operads

$$\mathcal{A}_{ha}(\mathbf{H}) \simeq \mathcal{A}(\mathbf{H}) \times \mathbf{Q}_{|H|}$$





Thank you!