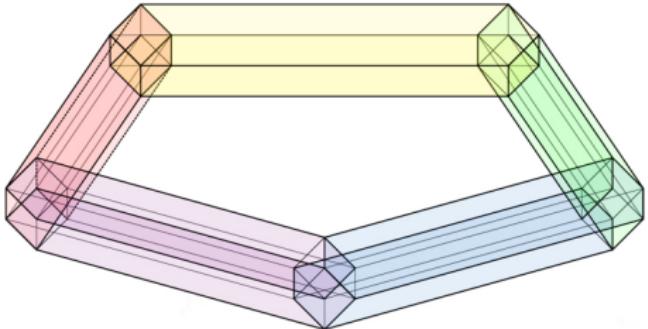




Categorified cyclic operads (in nature)

Logic and applications 2018

Jovana Obradović
Charles University Prague



Relevant examples of categorified structures

sets	categories
functions	functors
equalities	coherent isomorphisms

Relevant examples of categorified structures

sets	categories
functions	functors
equalities	coherent isomorphisms

Coherence of symmetric monoidal categories



S. Mac Lane

Categories for the Working Mathematician

Springer, 1997

$$\beta_{f,g,h} : (fg)h \rightarrow f(gh) \quad \gamma_{f,g} : fg \rightarrow gf$$

Relevant examples of categorified structures

sets	categories
functions	functors
equalities	coherent isomorphisms

Coherence of symmetric monoidal categories



S. Mac Lane

Categories for the Working Mathematician
Springer, 1997

$$\beta_{f,g,h} : (fg)h \rightarrow f(gh) \quad \gamma_{f,g} : fg \rightarrow gf$$

Coherence of categorified operads



K. Došen, Z. Petrić

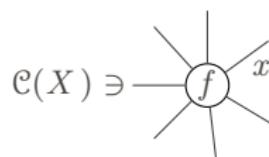
Weak Cat-operads

Logical Methods in Computer Science, 2009

$$\beta_{f,g,h}^{i,j} : (f \circ_i g) \circ_j h \rightarrow f \circ_i (g \circ_{j-i+1} h) \quad \theta_{f,g,h}^{i,j} : (f \circ_i g) \circ_j h \rightarrow (f \circ_j h) \circ_{i+n-1} g$$

Cyclic operads

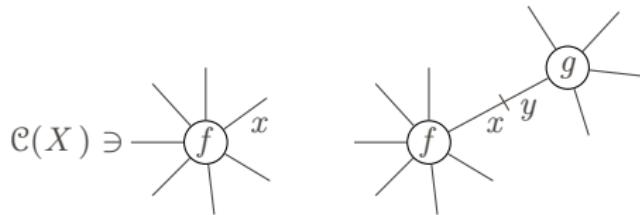
$$\mathcal{C} : \mathbf{Bij}^{op} \rightarrow \mathbf{Set}$$



Cyclic operads

$$\mathcal{C} : \mathbf{Bij}^{op} \rightarrow \mathbf{Set}$$

$$_x \circ_y : \mathcal{C}(X) \times \mathcal{C}(Y) \rightarrow \mathcal{C}(X \setminus \{x\} \cup Y \setminus \{y\})$$

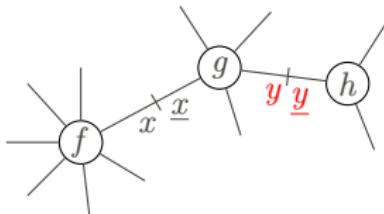
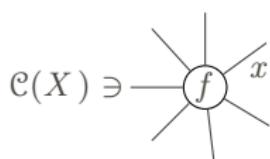


Cyclic operads

$$\mathcal{C} : \mathbf{Bij}^{op} \rightarrow \mathbf{Set}$$

$$x \circ_y : \mathcal{C}(X) \times \mathcal{C}(Y) \rightarrow \mathcal{C}(X \setminus \{x\} \cup Y \setminus \{y\})$$

$$(f \underset{x}{\circ} \underline{g}) \underset{y}{\circ} \underline{h} = f \underset{x}{\circ} \underline{g} (g \underset{y}{\circ} \underline{h}) \quad f \underset{x}{\circ} y \ g = g \underset{y}{\circ} x \ f$$

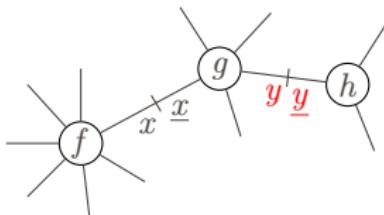
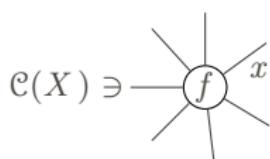


Cyclic operads

$$\mathcal{C} : \mathbf{Bij}^{op} \rightarrow \mathbf{Set}$$

$$x \circ_y : \mathcal{C}(X) \times \mathcal{C}(Y) \rightarrow \mathcal{C}(X \setminus \{x\} \cup Y \setminus \{y\})$$

$$(f \underset{x}{\circ} \underline{g}) \underset{y}{\circ} \underline{h} = f \underset{x}{\circ} \underline{g} (g \underset{y}{\circ} \underline{h}) \quad f \underset{x}{\circ} y g = g \underset{y}{\circ} x f$$



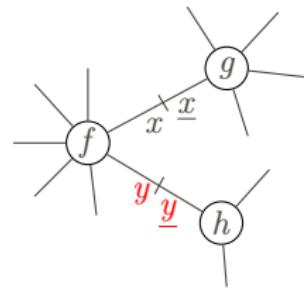
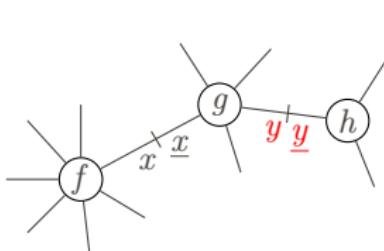
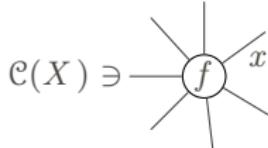
$$f^{\sigma_1} \circ_{\sigma_1^{-1}(x)} \circ_{\sigma_2^{-1}(y)} g^{\sigma_2} = (f_x \circ_y g)^{\sigma}$$

Cyclic operads

$$\mathcal{C} : \mathbf{Bij}^{op} \rightarrow \mathbf{Set}$$

$$x \circ_y : \mathcal{C}(X) \times \mathcal{C}(Y) \rightarrow \mathcal{C}(X \setminus \{x\} \cup Y \setminus \{y\})$$

$$(f \underset{x}{\circ} \underline{g}) \underset{y}{\circ} \underline{h} = f \underset{x}{\circ} \underline{g} (g \underset{y}{\circ} \underline{h}) \quad f \underset{x}{\circ} y g = g \underset{y}{\circ} x f$$



$$f^{\sigma_1} \circ_{\sigma_1^{-1}(x)} \circ_{\sigma_2^{-1}(y)} g^{\sigma_2} = (f \underset{x}{\circ} y g)^{\sigma}$$

Parallel associativity

$$(f \underset{x}{\circ} \underline{g}) \underset{y}{\circ} \underline{h} = (g \underset{x}{\circ} \underline{f}) \underset{y}{\circ} \underline{h} = g \underset{x}{\circ} \underline{f} (f \underset{y}{\circ} \underline{h}) = (f \underset{y}{\circ} \underline{h}) \underset{x}{\circ} \underline{f} g$$

Categorified cyclic operads

$$\mathcal{C} : \mathbf{Bij}^{op} \rightarrow \mathbf{Cat}$$

$$x \circ_y : \mathcal{C}(X) \times \mathcal{C}(Y) \rightarrow \mathcal{C}(X \setminus \{x\} \cup Y \setminus \{y\})$$

$$\beta : (f_{x \circ \underline{x}} g)_{y \circ \underline{y}} h \rightarrow f_{x \circ \underline{x}} (g_{y \circ \underline{y}} h) \quad \gamma : f_{x \circ y} g \rightarrow g_{y \circ x} f$$

The diagram illustrates the compatibility of the β and γ operations with the composition of morphisms. It consists of two main parts:

- Left Triangle:** Relates $((fg)h)k$ to $(f(gh))k$ and $(fg)(hk)$. The top edge is labeled $\beta \cdot 1$, the left edge $\beta \cdot 1$, and the right edge β .
- Right Triangle:** Relates $((fg)h)k$ to $h((fg)k)$ and $((fg)k)h$. The top edge is labeled $\beta \cdot 1$, the left edge $\beta \cdot 1$, and the right edge γ .
- Bottom Square:** Relates $f((gh)k)$ to $f((hg)k)$ and $f(h(gk))$. The top edge is labeled $\beta \cdot 1$, the left edge $\beta \cdot 1$, and the right edge γ .
- Bottom Square:** Relates $f((gh)k)$ to $f((hg)k)$ and $f(h(gk))$. The top edge is labeled $1 \cdot \gamma \cdot 1$, the left edge $1 \cdot \gamma \cdot 1$, and the right edge $1 \cdot \beta$.

The diagram shows the compatibility of β and γ with the composition of morphisms:

- Top Row:** $(fg)h \xrightarrow{\beta} f(gh) \xrightarrow{\gamma} (gh)f$
- Bottom Row:** $(gf)h \xrightarrow{\gamma} h(gf) \xleftarrow{\beta} (hg)f$
- Left Column:** $(fg)h \xrightarrow{\beta \cdot 1} (gf)h$
- Right Column:** $(gh)f \xrightarrow{\gamma \cdot 1} (hg)f$
- Bottom Left:** $fg \xrightarrow{\gamma} gf \xrightarrow{1} fg$
- Bottom Right:** $fg \xrightarrow{\beta} gf \xrightarrow{\gamma} fg$

Associated with these operations are several identities:

$$f^{\sigma_1} \circ_{\sigma_1^{-1}(x)} \circ_{\sigma_2^{-1}(y)} g^{\sigma_2} = (f_{x \circ y} g)^{\sigma}$$

$$\beta_{f,g,h}^{\sigma} = \beta_{f^{\sigma_1}, g^{\sigma_2}, h^{\sigma_3}}$$

$$\gamma_{f,g}^{\sigma} = \gamma_{f^{\sigma_1}, g^{\sigma_2}}$$

$$(\varphi \cdot \psi)^{\sigma} = \varphi^{\sigma_1} \cdot \psi^{\sigma_2}$$

Polytopes of categorified cyclic operads

Categorified operads: Hypergraph polytopes



K. Došen, Z. Petrić

Hypergraph polytopes

Topology and its Applications 158, pp. 1405–1444, 2011

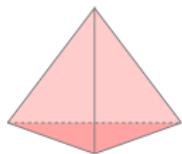


P.-L. Curien, J. Obradović, J. Ivanović

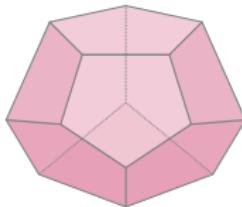
Syntactic aspects of hypergraph polytopes

Journal of Homotopy and Related Structures

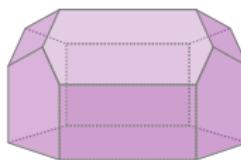
<https://doi.org/10.1007/s40062-018-0211-9>



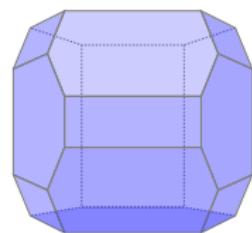
simplex



associahedron



hemiassociahedron



permutohedron



“the more hyperedges, the more truncations”

Hypergraph terminology

Hypergraph

$$\mathbf{H} = (H, \mathbf{H})$$

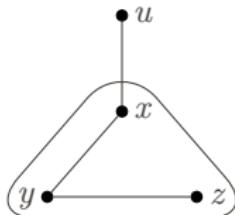
H - vertices

$$\mathbf{H} \subseteq \mathcal{P}(H) \setminus \emptyset \text{ - hyperedges}$$

$$\bigcup \mathbf{H} = H$$

$$H = \{x, y, z, u\}$$

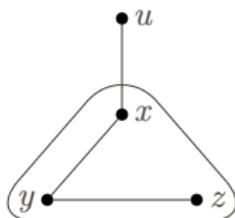
$$\mathbf{H} = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{x, u\}, \{y, z\}, \{x, y, z\}\}$$



Hypergraph terminology

Hypergraph $\mathbf{H} = (H, \mathbf{H})$
 H - vertices $\mathbf{H} \subseteq \mathcal{P}(H) \setminus \emptyset$ - hyperedges $\bigcup \mathbf{H} = H$

$H = \{x, y, z, u\}$ $\mathbf{H} = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{x, u\}, \{y, z\}, \{x, y, z\}\}$

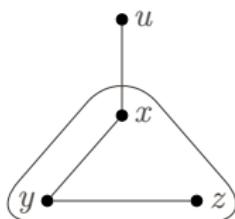


Additional properties

Hypergraph terminology

Hypergraph $\mathbf{H} = (H, \mathbf{H})$
 H - vertices $\mathbf{H} \subseteq \mathcal{P}(H) \setminus \emptyset$ - hyperedges $\bigcup \mathbf{H} = H$

$H = \{x, y, z, u\}$ $\mathbf{H} = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{x, u\}, \{y, z\}, \{x, y, z\}\}$

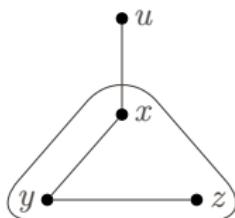


Additional properties non-empty,

Hypergraph terminology

Hypergraph $\mathbf{H} = (H, \mathbf{H})$
 H - vertices $\mathbf{H} \subseteq \mathcal{P}(H) \setminus \emptyset$ - hyperedges $\bigcup \mathbf{H} = H$

$H = \{x, y, z, u\}$ $\mathbf{H} = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{x, u\}, \{y, z\}, \{x, y, z\}\}$

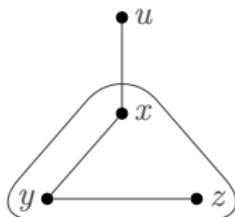


Additional properties non-empty, finite,

Hypergraph terminology

Hypergraph $\mathbf{H} = (H, \mathbf{H})$
 H - vertices $\mathbf{H} \subseteq \mathcal{P}(H) \setminus \emptyset$ - hyperedges $\bigcup \mathbf{H} = H$

$H = \{x, y, z, u\}$ $\mathbf{H} = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{x, u\}, \{y, z\}, \{x, y, z\}\}$

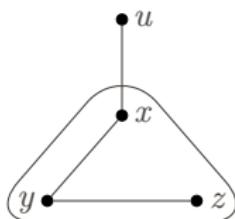


Additional properties *non-empty, finite, atomic,*

Hypergraph terminology

Hypergraph $\mathbf{H} = (H, \mathbf{H})$
 H - vertices $\mathbf{H} \subseteq \mathcal{P}(H) \setminus \emptyset$ - hyperedges $\bigcup \mathbf{H} = H$

$H = \{x, y, z, u\}$ $\mathbf{H} = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{x, u\}, \{y, z\}, \{x, y, z\}\}$



Additional properties *non-empty, finite, atomic, connected,*

Hypergraph terminology

Hypergraph

$$\mathbf{H} = (H, \mathbf{H})$$

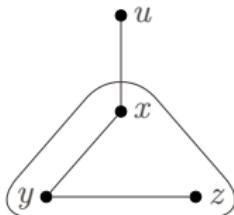
H - vertices

$$\mathbf{H} \subseteq \mathcal{P}(H) \setminus \emptyset - \text{hyperedges}$$

$$\bigcup \mathbf{H} = H$$

$$H = \{x, y, z, u\}$$

$$\mathbf{H} = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{x, u\}, \{y, z\}, \{x, y, z\}\}$$



Additional properties *non-empty, finite, atomic, connected, saturated:*

$$(\forall X, Y \in \mathbf{H}) \quad X \cap Y \neq \emptyset \Rightarrow X \cup Y \in \mathbf{H}$$

Hypergraph terminology

Hypergraph

$$\mathbf{H} = (H, \mathbf{H})$$

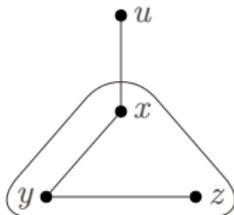
H - vertices

$$\mathbf{H} \subseteq \mathcal{P}(H) \setminus \emptyset - \text{hyperedges}$$

$$\bigcup \mathbf{H} = H$$

$$H = \{x, y, z, u\}$$

$$\mathbf{H} = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{x, u\}, \{y, z\}, \{x, y, z\}\}$$



Additional properties non-empty, finite, atomic, connected, saturated:

$$(\forall X, Y \in \mathbf{H}) \quad X \cap Y \neq \emptyset \Rightarrow X \cup Y \in \mathbf{H}$$

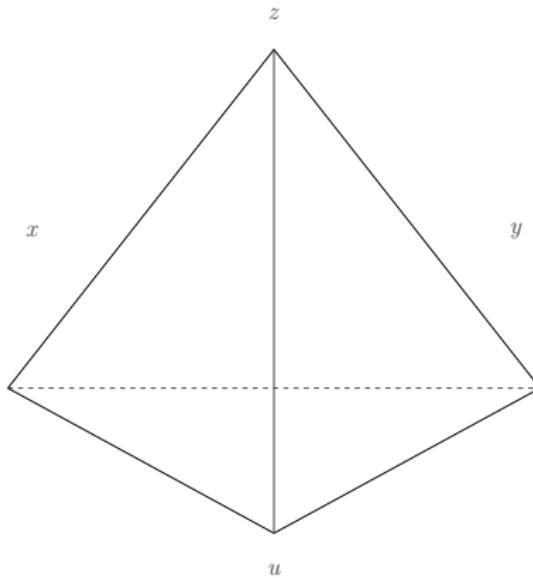
$$Sat(\mathbf{H}) = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{x, u\}, \{x, y, u\}, \{y, z\}, \{x, y, z\}, \{x, y, z, u\}\}$$

A polytope from a hypergraph

Hemiassociahedron

$$\mathbf{H} = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, x\}, \{x, u\}\}$$

$$Sat(\mathbf{H}) = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, x\}, \{x, u\}, \{x, y, z\}, \{x, y, u\}, \{x, z, u\}, \{x, y, z, u\}\}$$

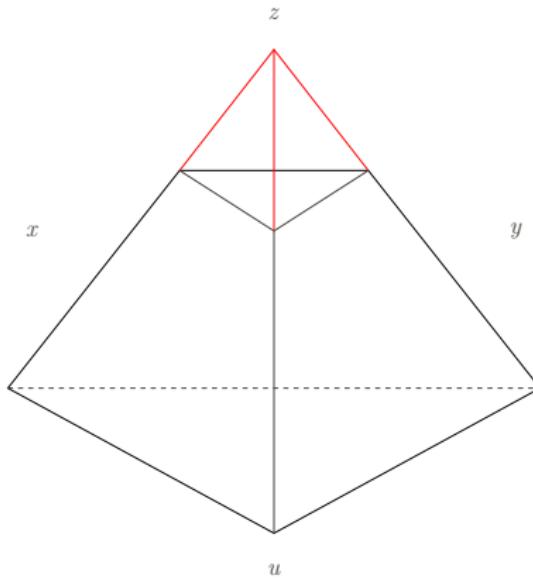


A polytope from a hypergraph

Hemiassociahedron

$$\mathbf{H} = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, x\}, \{x, u\}\}$$

$$Sat(\mathbf{H}) = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, x\}, \{x, u\}, \{x, y, z\}, \{x, y, u\}, \{x, z, u\}, \{x, y, z, u\}\}$$

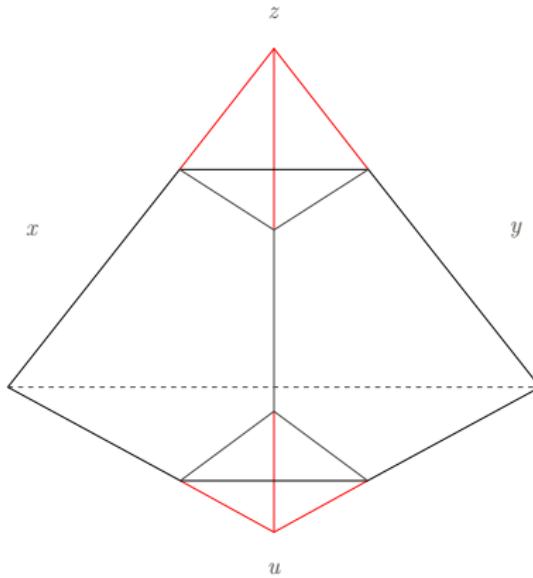


A polytope from a hypergraph

Hemiassociahedron

$$\mathbf{H} = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, x\}, \{x, u\}\}$$

$$Sat(\mathbf{H}) = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, x\}, \{x, u\}, \{x, y, z\}, \{x, y, u\}, \{x, z, u\}, \{x, y, z, u\}\}$$

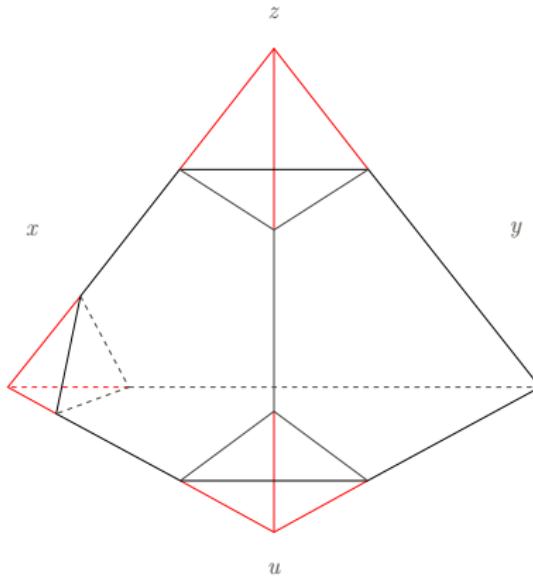


A polytope from a hypergraph

Hemiassociahedron

$$\mathbf{H} = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, x\}, \{x, u\}\}$$

$$Sat(\mathbf{H}) = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, x\}, \{x, u\}, \{x, y, z\}, \{x, y, u\}, \{x, z, u\}, \{x, y, z, u\}\}$$

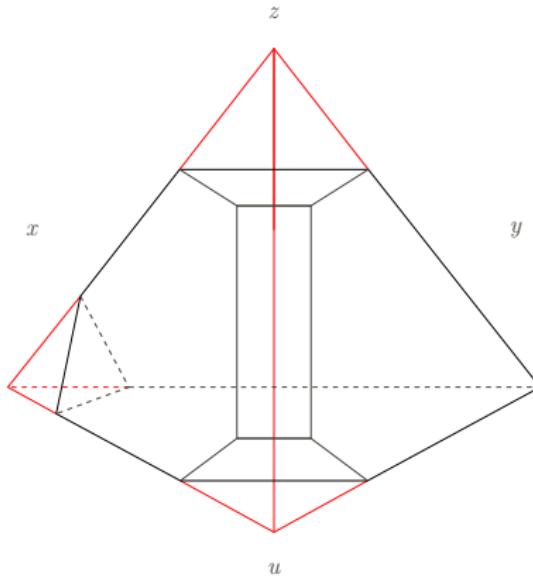


A polytope from a hypergraph

Hemiassociahedron

$$\mathbf{H} = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, x\}, \{x, u\}\}$$

$$Sat(\mathbf{H}) = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, x\}, \{x, u\}, \{x, y, z\}, \{x, y, u\}, \{x, z, u\}, \{x, y, z, u\}\}$$

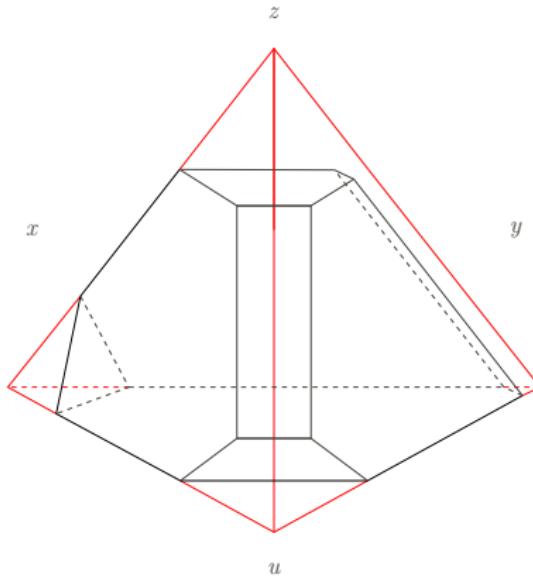


A polytope from a hypergraph

Hemiassociahedron

$$\mathbf{H} = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, x\}, \{x, u\}\}$$

$$Sat(\mathbf{H}) = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, x\}, \{x, u\}, \{x, y, z\}, \{x, y, u\}, \{x, z, u\}, \{x, y, z, u\}\}$$

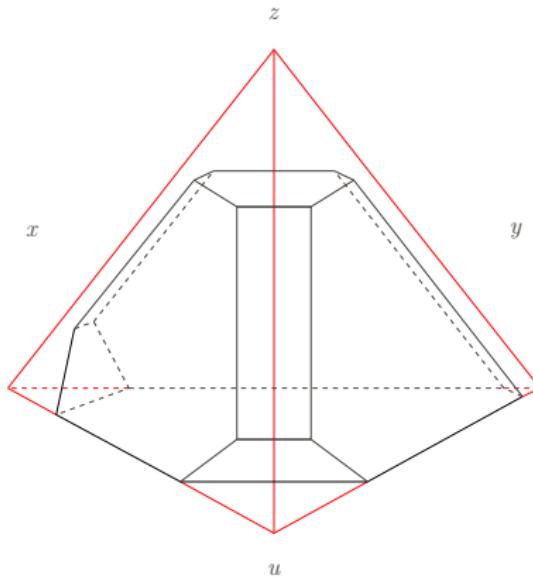


A polytope from a hypergraph

Hemiassociahedron

$$\mathbf{H} = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, x\}, \{x, u\}\}$$

$$Sat(\mathbf{H}) = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, x\}, \{x, u\}, \{x, y, z\}, \{x, y, u\}, \{x, z, u\}, \{x, y, z, u\}\}$$

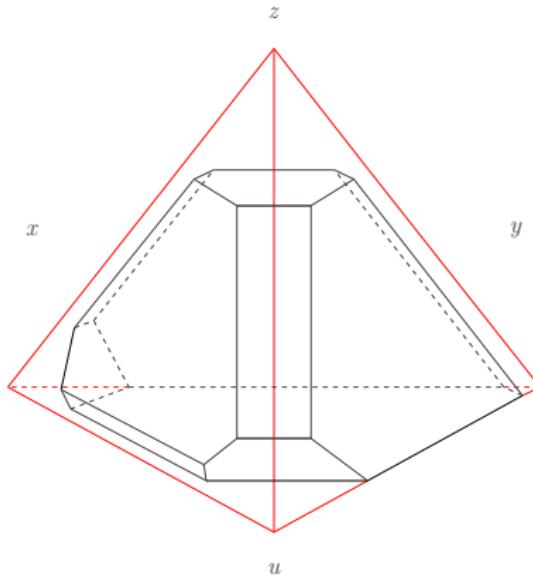


A polytope from a hypergraph

Hemiassociahedron

$$\mathbf{H} = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, x\}, \{x, u\}\}$$

$$Sat(\mathbf{H}) = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, x\}, \{x, u\}, \{x, y, z\}, \{x, y, u\}, \{x, z, u\}, \{x, y, z, u\}\}$$

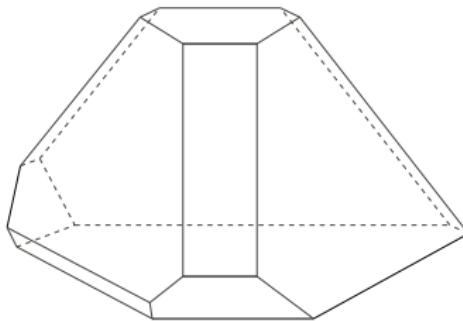


A polytope from a hypergraph

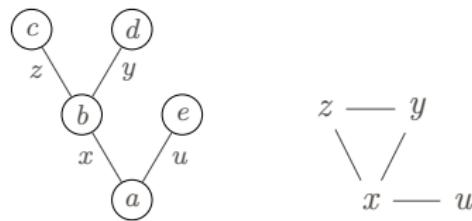
Hemiassociahedron

$$\mathbf{H} = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, x\}, \{x, u\}\}$$

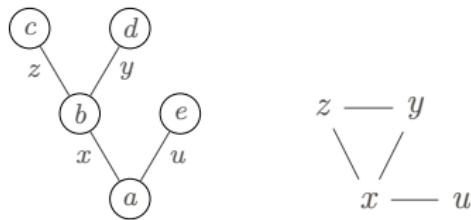
$$Sat(\mathbf{H}) = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, x\}, \{x, u\}, \{x, y, z\}, \{x, y, u\}, \{x, z, u\}, \{x, y, z, u\}\}$$



Polytopes of categorified operads

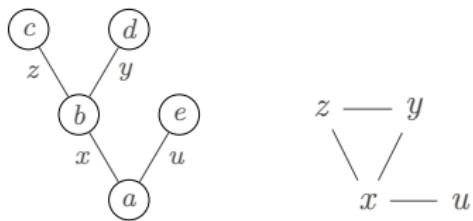


Polytopes of categorified operads

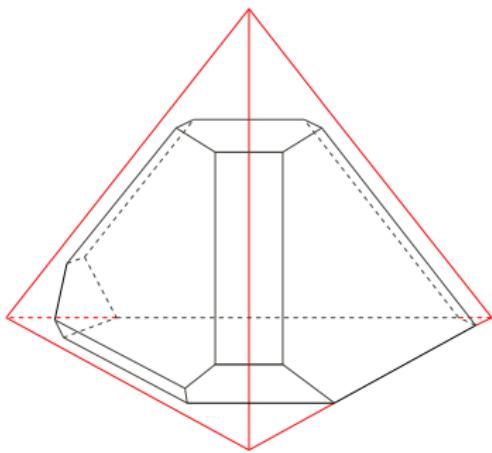


$$\mathbf{H} = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, x\}, \{x, u\}\}$$

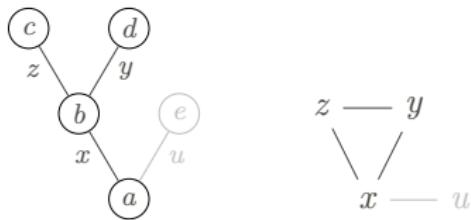
Polytopes of categorified operads



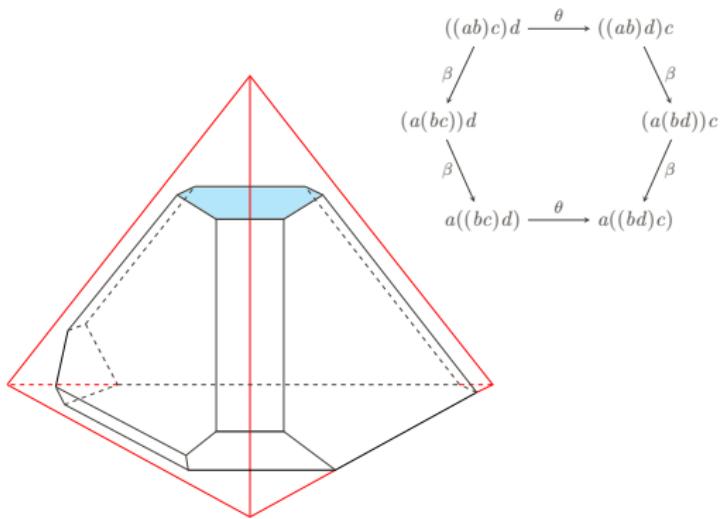
$$\mathbf{H} = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, x\}, \{x, u\}\}$$



Polytopes of categorified operads



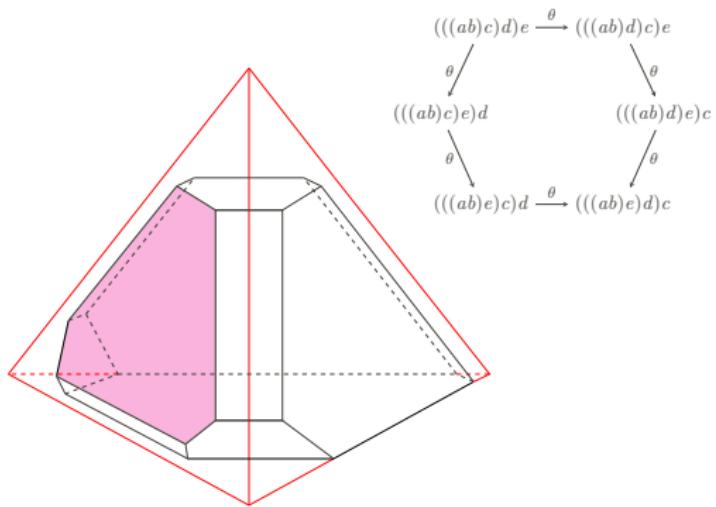
$$\mathbf{H} = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, x\}, \{x, u\}\}$$



Polytopes of categorified operads

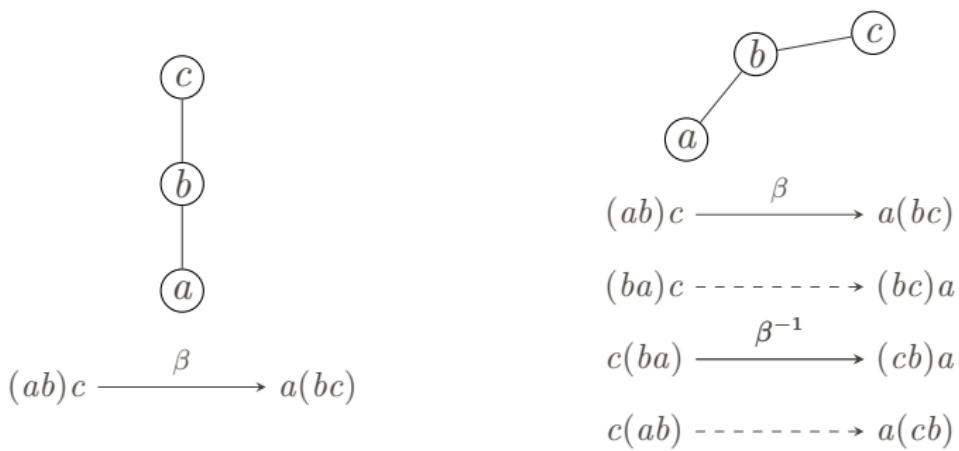


$$\mathbf{H} = \{\{y\}, \{z\}, \{u\}, \{y, z\}, \{y, u\}, \{z, u\}\}$$

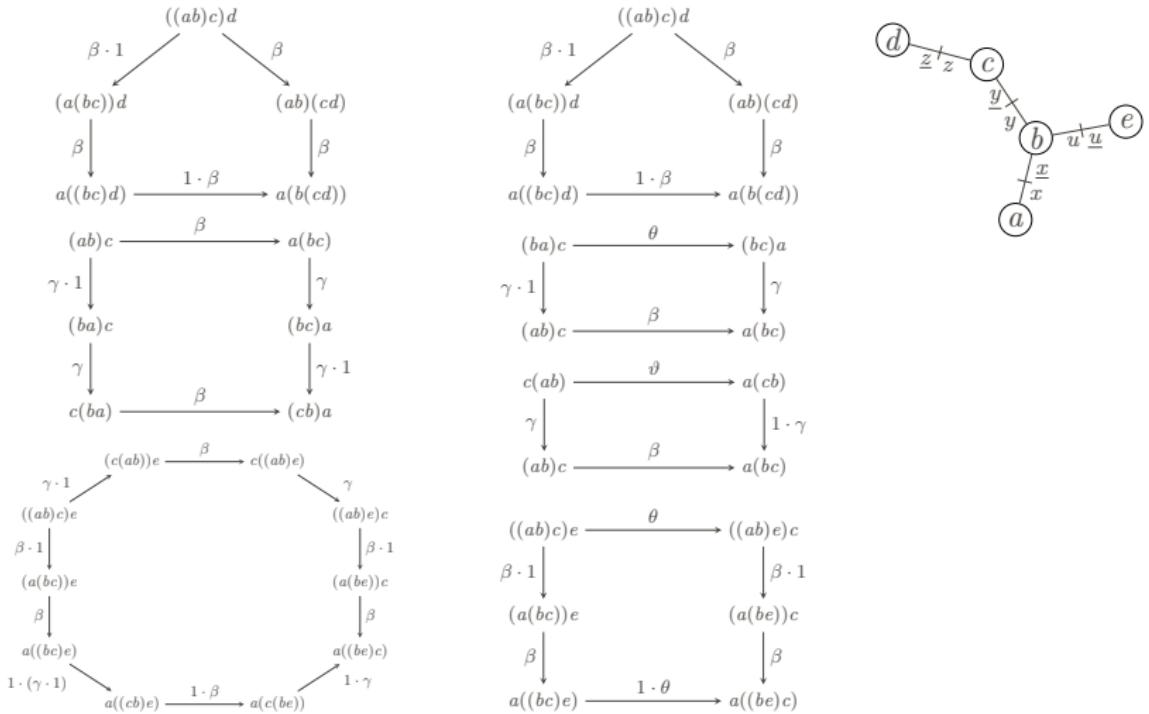


Polytopes of categorified cyclic operads

“Rooted associativity” vs. “Unrooted associativity”



β, γ	$\beta, \theta, \vartheta, \gamma$
$\beta_{a,b,c}^{x,\underline{x};y,\underline{y}} : (a \circ_{\underline{x}} b) \circ_{\underline{y}} c \rightarrow a \circ_{\underline{x}} (b \circ_{\underline{y}} c)$	$\beta_{a,b,c}^{x,\underline{x};y,\underline{y}} : (a \circ_{\underline{x}} b) \circ_{\underline{y}} c \rightarrow a \circ_{\underline{x}} (b \circ_{\underline{y}} c)$
$\gamma_{a,b}^{x,\underline{x}} : a \circ_{\underline{x}} b \rightarrow b \circ_{\underline{x}} a$	$\theta_{b,a,c}^{\underline{x},x;y,\underline{y}} : (b \circ_{\underline{x}} a) \circ_{\underline{y}} c \rightarrow (b \circ_{\underline{y}} c) \circ_{\underline{x}} a$
	$\vartheta_{c,a,b}^{y,\underline{y};x,\underline{x}} : c \circ_{\underline{y}} (a \circ_{\underline{x}} b) \rightarrow a \circ_{\underline{x}} (c \circ_{\underline{y}} b)$
	$\gamma_{a,b}^{x,\underline{x}} : a \circ_{\underline{x}} b \rightarrow b \circ_{\underline{x}} a$

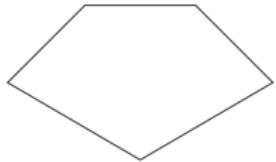


Polytopes of categorified cyclic operads

Categorified operads

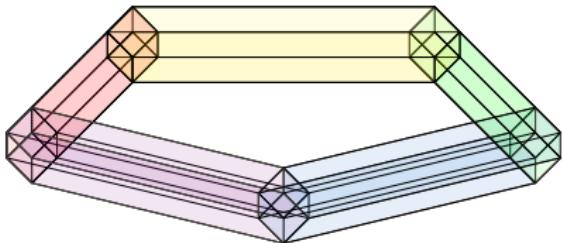
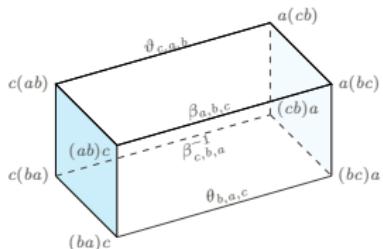
$$\mathcal{A}(\mathbf{H}) = \begin{cases} \text{associahedron} \\ \text{hemiassociahedron} \\ \text{permutohedron} \end{cases}$$

$$(ab) \quad \overbrace{\hspace{1cm}} \quad \beta_{a,b,c} \quad a(bc)$$



Categorified cyclic operads

$$\mathcal{A}_{ha}(\mathbf{H}) \simeq \mathcal{A}(\mathbf{H}) \times \mathbf{Q}_{|H|}$$



Thank you!