Sveučilište u Zagrebu

On geometric aspects of multiple conclusion deductions

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   On multiple conclusion deductions (MCDs) in classical logic (CPL, CFL)
2. “Geometry” of MCDs
   Some visual notions, related to the underlying formula graph

Introduction

Multiple conclusion deductions

Based on inferences with multiple conclusions

(by W. Kneale)

(I∧) $\dfrac{A \qquad B}{A \wedge B}$

(E∧) $\dfrac{A \wedge B}{A} \qquad \dfrac{A \wedge B}{B}$

(I∨) $\dfrac{A}{A \vee B} \qquad \dfrac{B}{A \vee B}$

(E∨) $\dfrac{A \vee B}{A \qquad B}$

(I→) $\dfrac{B}{A \to B} \qquad \dfrac{}{A \qquad A \to B}$

(E→) $\dfrac{A \qquad A \to B}{B}$

(I¬) $\dfrac{}{A \qquad \neg A}$

(E¬) $\dfrac{A \qquad \neg A}{}$

Local inference rules (unlike hypothetical in NK).

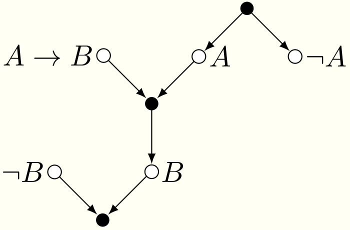
MCD example

Proof of *Modus tolens*

$$\dfrac{\neg B \qquad \dfrac{A \to B \qquad \overline{A \qquad\qquad \neg A}}{B}}{}$$

Proofs branch (both) upward and downward.

What is underneath?



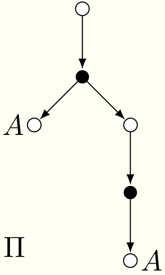
MCD is a formula graph (bipartite DAG)

Bipartition of formula nodes and “stroke” nodes

Premisses (conclusions) are minimal (maximal) formula nodes in induced p.o.

Contraction

Contraction of duplicate premisses (conclusions)



Contraction is Kneale’s missing piece

for completeness (of MCD for CPL)

Styled in a “traditional” fashion

Contraction of conclusion A

$$\dfrac{A \vee (A \wedge B)}{\cancel{A}_{(1)} \qquad \dfrac{A \wedge B}{A_{(1)}}}$$

Discharged, indexed

Graph-related notions

DAG transpose

Reversing arrows = turning Π upside down? Symmetry?

What to do with formulas?

1. Dual proof $\Pi \mapsto \Pi^d$

   Replace formulas with duals

$$\frac{A \qquad B}{A \wedge B} \xrightarrow{\text{transpose}} \frac{(A \wedge B)^d}{A^d \qquad B^d} = \frac{A \vee B}{A \qquad B}$$

2. Negation

   N.B. downward reading (the usual) is “truth preserving”

   Upward reading (from conclusions to premisses) is “falsity preserving”.

   Useful for semantic analysis

Going against the flow

- $$\frac{A \qquad B}{A \wedge B}$$ read as $$\frac{A\bot \qquad B\bot}{(A \wedge B)\bot}$$

  $(A \wedge B)\top$ implies $A\bot$ or $B\bot$

- $$\frac{A \qquad A \to B}{B}$$ read as $$\frac{A\bot \qquad (A \to B)\top}{B\top}$$

  $(A \to B)\top$ implies $A\bot$ or $B\top$

Granularity of MCDs

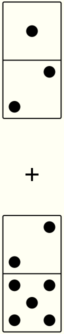
- Formula-level
- Inference-level

  View: MCDs are assembled from inferences (instances)
- Bigger blocks/proof fragments

  Analytic examples are interesting

Proof assembly/disassembly

The idea:



Provided by the local inference rules.

$\Pi_1/A + A/\Pi_2$

Consequences for MCD calculus

- Subformula property, Normal form theorem

  Trivial

- Simple proof search

  (naivé, greedy)

  Explained as semantic analysis (of α and β formulas)

  Synthesis explained as Robinson's resolution

Proof search of Δ from Γ

Sketch:

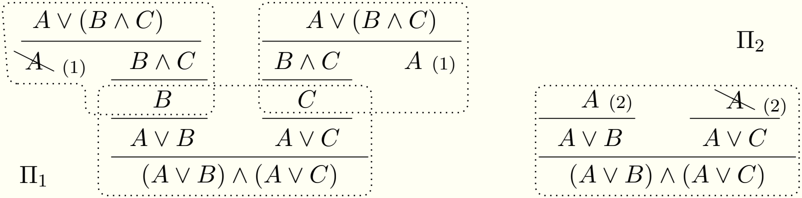
- Grow analytic MCDs from Γ downward (and from Δ upward)
- Simplify leaves (until atomic)
- Assemble proof from analytic parts

- Output

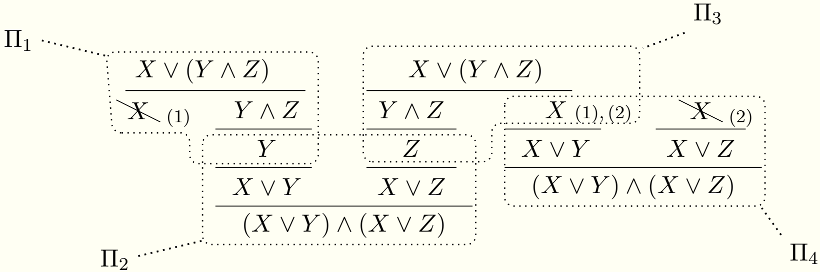
  A set of “analytic” MCDs

  Byproduct is a clausal form (CNF) of $\Gamma, \neg\Delta$

$A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$



Example



A final proof is formed by appropriate joins of analytic MCDs

Robinson's resolution

Disassembly

- Split MCD along *cut vertex*

  Plus contraction cleanup

  Component subgraphs are MCDs
- Compare with NK

$$
\dfrac{A \vee B \qquad \begin{matrix} \cancel{A} \\ \vdots \\ C \end{matrix} \qquad \begin{matrix} \cancel{B} \\ \vdots \\ C \end{matrix}}{C}\,(\mathrm{E}\vee)_G
$$

- Is given Π an MCD?

  Check inferences and contractions.

Quantification rules, CFL

$$(\mathrm{I}\exists)\ \frac{A(a)}{\exists x A(x)} \qquad (\mathrm{E}\exists)\ \frac{\exists x A(x)}{A(a)}$$

$$(\mathrm{I}\forall)\ \frac{A(a)}{\forall x A(x)} \qquad (\mathrm{E}\forall)\ \frac{\forall x A(x)}{A(a)}$$

With proviso: *NK: No a-connected path between* (E∃) *and* (I∀)*!*

Because we don't want $\exists x P(x) \vdash \forall x P(x) \ldots$

Necessary bookkeeping of parameters introduced by δ formula.

Proof search works as before (with minor tweaks).

Transpose example

$$\frac{\dfrac{\forall x A(x)}{A(a)}}{\exists x A(x)} \quad \xrightarrow{\text{transpose + dual}} \quad \frac{\dfrac{(\exists x A(x))^d}{(A(a))^d}}{(\forall x A(x))^d} \quad = \quad \ldots$$

It is self-dual.

Conclusion

On MCD

- ND system desiderata (informal)
  simple inference rules, easy to follow proofs (readable for humans), applicability of simple proof search strategies, ...
- Argue for the *Simplicity and accessibility to humans* for MCD calculus
- Relation to other established formalisms
  Discussed at LAP before

Acknowledgments

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`http://formals.ufzg.hr/`

References

Based on:

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