Institute of Mathematics
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Proofs and surfaces

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Incidence theorems

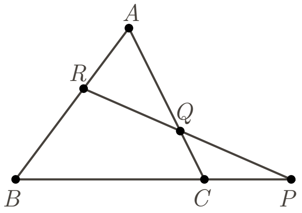
For three mutually distinct colinear points X, Y and Z in $\mathbf{R}^2$, let

$$(X, Y; Z) =_{df} \begin{cases} \frac{XZ}{YZ}, & \text{if } Z \text{ is between } X \text{ and } Y, \\ -\frac{XZ}{YZ}, & \text{otherwise.} \end{cases}$$

Menelaus' theorem

For a triangle ABC and points P, Q and R (different from the vertices) on the lines BC, CA and AB respectively, it holds that

$$P, Q, R \textit{ are colinear} \quad \textit{iff} \quad (B, C; P) \cdot (C, A; Q) \cdot (A, B; R) = -1.$$

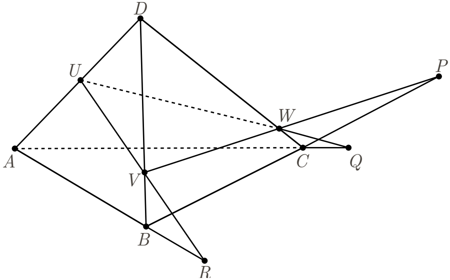


The sextuple $ABCPQR$ is a Menelaus configuration.

Desargues' theorem

If ABC and UVW are two triangles such that $A \neq U$, $B \neq V$ and $C \neq W$, if $BC \cap VW = \{P\}$, $AC \cap UW = \{Q\}$ and $AB \cap UV = \{R\}$, then

AU, BV and CW are concurrent iff P, Q and R are colinear.



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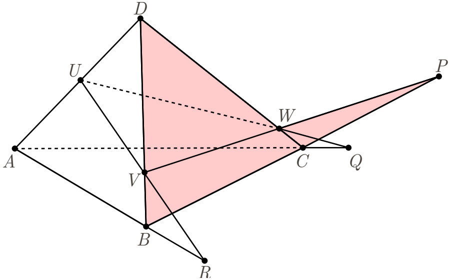

$$(C, D; W) \cdot (D, B; V) \cdot (B, C; P) = -1$$

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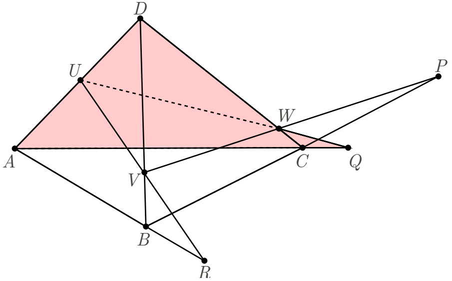

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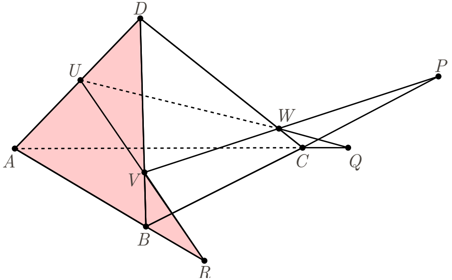

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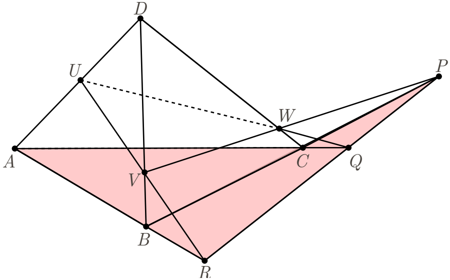

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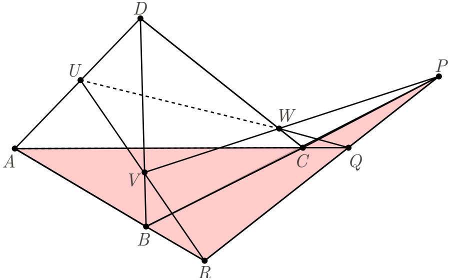

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Cyclic sequent:

$$\vdash ABCPQR, ABDVUR, ACDWUQ, BCDWVP$$

Whenever 3 out of 4 sextuples is a Menelaus configuration, then so is the fourth.

Homology meets Menelaus

J. Richter-Gebert
Meditations on Ceva's theorem
The Coxeter Legacy: Reflections and Projections (C. Davis and E.W. Ellers, editors), American Mathematical Society and Fields Institute, Providence, 2006, pp. 227-254

We consider compact, orientable 2-manifolds without boundary and subdivisions by CW-complexes whose faces are triangles. Consider such a cycle as being interpreted by flat triangles. The presence of Menelaus configurations on all but one of the faces implies the existence of a Menelaus configuration on the final face.

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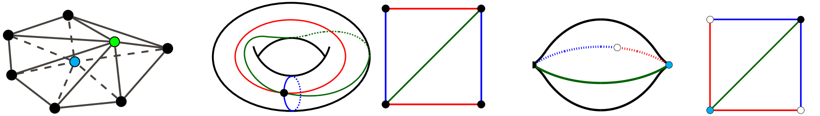
$\mathcal{M}$-complexes:

finite, homogeneous 2-dimensional, regular, linked, orientable Δ-complexes



Permutations of vertices and switching of triangles

- If $A_1A_2A_3B_1B_2B_3$ makes a Menelaus configuration and π is a permutation of the set $\{1,2,3\}$, then the sextuple

$$A_{\pi(1)}A_{\pi(2)}A_{\pi(3)}B_{\pi(1)}B_{\pi(2)}B_{\pi(3)}$$

makes a Menelaus configuration, too.

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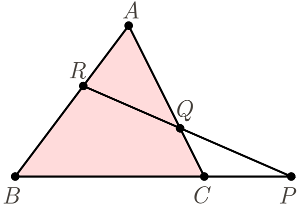

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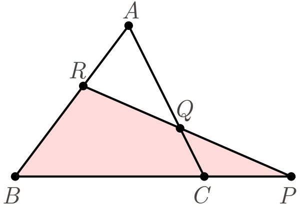

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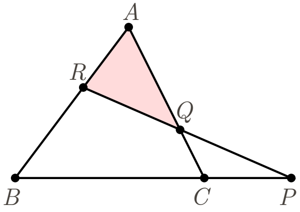

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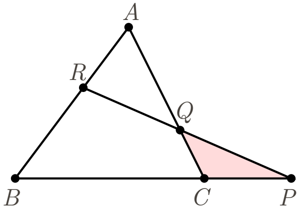

The Menelaus system

- Atomic formulae: $F^6(W) = W^6 - \{X_1 \dots X_6 \in W \mid X_i \neq X_j \text{ for } i \neq j\}$
- Sequents: finite multisets of formulae; denoted by $\vdash \Gamma$
- For an $\mathcal{M}$-complex L such that $L_0 \cup L_1 \subseteq W$, let $\nu \colon L_2 \to F^6(W)$ be defined as $\nu x = d_1 d_2 x \;\; d_0 d_2 x \;\; d_0 d_0 x \;\; d_0 x \;\; d_1 x \;\; d_2 x$.

$$\frac{}{\vdash \{\nu x \mid x \in L_2\}}$$

$$\frac{}{\vdash ABCPQR, BCAQRP} \qquad \frac{}{\vdash ABCPQR, ARQPCB}$$

$$\frac{\vdash \Gamma, \varphi \qquad \vdash \Delta, \varphi}{\vdash \Gamma, \Delta}$$

The Menelaus system is sound

- A Euclidean interpretation is a function from W to $\mathbf{R}^2$.
- An interpretation satisfies the atomic formula $ABCPQR$ when the sextuple $ABCPQR$ of points in $\mathbf{R}^2$ is a Menelaus configuration.
- Let $\Gamma \models_E \varphi$ mean that every Euclidean interpretation that satisfies every formula in Γ also satisfies φ.

Theorem

Suppose that $\Gamma = \{x_1, \ldots, x_n\}$. If $\vdash \Gamma, x_i$ is derivable, then $\Gamma \models_E x_i$.

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- Suppose $\vdash \Gamma, x_i$ is an axiomatic sequent derived from an $\mathcal{M}$-complex L.

  Define $h\colon (C_1, +, 0) \to (\mathbb{R}\backslash\{0\}, \cdot, 1)$ by $h(a) = h(y_1)^{\alpha_1} \cdot \ldots \cdot h(y_m)^{\alpha_m}$, for

$$a = \sum_{i=1}^{m} \alpha_i y_i \quad \text{and} \quad h(y_i) = (vd_0\, y_i, vd_1\, y_i; vy_i).$$

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  Therefore, assuming that $\partial x_i = P - Q + R$, we have

$$\begin{aligned} h(\partial x_i) &= (C, B\,; P) \cdot (C, A\,; Q)^{-1} \cdot (B, A\,; R) \\ &= ((B, C; P) \cdot (C, A\,; Q) \cdot (A, B\,; R))^{-1}. \end{aligned}$$

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- Suppose $\vdash \Gamma, x_i$ is an axiomatic sequent derived from an $\mathcal{M}$-complex L.

  By orientability of L, we have that

$$\partial x_i = \sum_{j \neq i} \varepsilon_j \partial x_j,$$

  where $|\{j \,|\, j \neq i\}|$ is odd, and where $h(\varepsilon_j \partial x_j) = -1$. Therefore, $h(\partial x_i) = -1$.

The Menelaus system is decidable

For a multiset Γ of formulae, let $\lambda(\Gamma)$ be the set of elements of W occurring in Γ, and let $\kappa(\Gamma)$ be the number of elements of Γ.

Lemma (finiteness of the search space). For every sequent $\vdash \Delta$ that occurs in a derivation of $\vdash \Gamma$, we have that $\lambda(\Delta) \subseteq \lambda(\Gamma)$ and $2 \leq \kappa(\Delta) \leq \kappa(\Gamma)$.

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- Otherwise, let S_1 contain the elements of S_0 and all the sequents from S obtained from two S_0 sequents by a single application of cut.

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- If $\Gamma \subseteq S_0$, $\vdash \Gamma$ is derivable.
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  - → If $S_1 = S_0$, $\vdash \Gamma$ is not derivable.
  - → Otherwise, we proceed in the same manner...

From derivable sequents to incidence results

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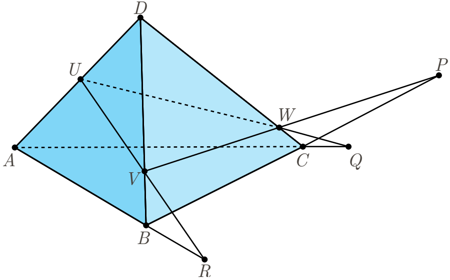
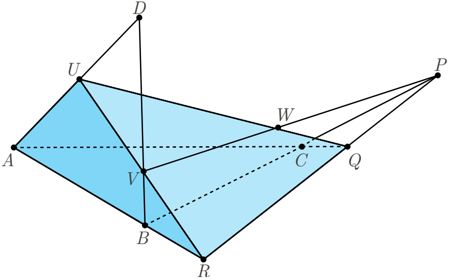


$\vdash ABDVUR, BCDWVP, ACDWUQ, ABCPQR$

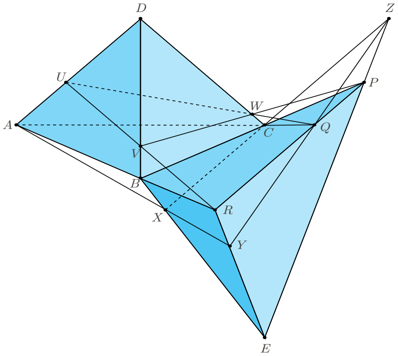


$\vdash ARUVDB, ARQPCB, URQPWV, AQUWDC$

From derivable sequents to incidence results

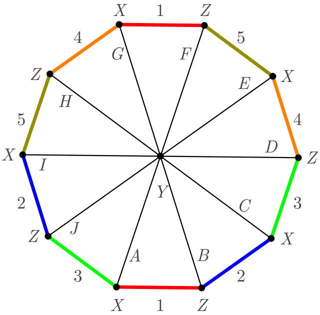
Theorem

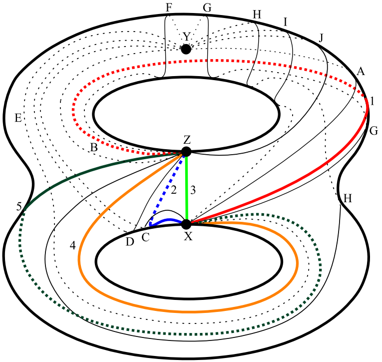
Let AU, BV and CW be concurrent lines in $\mathbb{R}^2$, and let X and E be such that B, X and E are colinear. For $\{P\} = BC \cap VW$, $\{Q\} = AC \cap UW$, $\{R\} = AB \cap UV$, $\{Y\} = AX \cap RE$, $\{Z\} = XC \cap EP$, the points Q, Y and Z are colinear.



$$\frac{\vdash ABDVUR, BCDWVP, ACDWUQ, ABCPQR \quad \vdash ABCPQR, BPRQAC \quad \vdash BREYXA, BPEZXC, RPEZYQ, BPRQAC}{\vdash ABDVUR, BCDWVP, ACDWUQ, BREYXA, BPEZXC, RPEZYQ}$$

From derivable sequents to incidence results

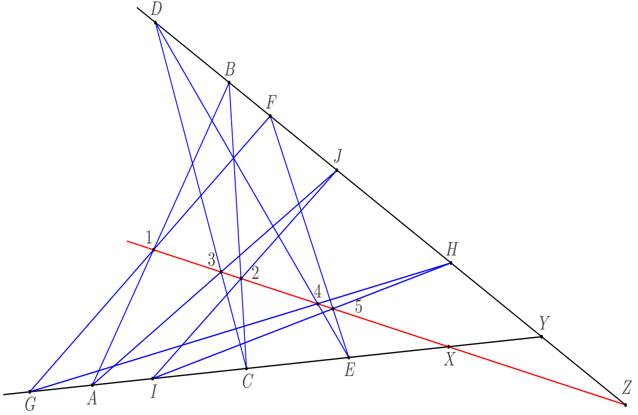




$$\vdash \quad (X, Y, Z, B, 1, A), (X, Y, Z, B, 2, C), (X, Y, Z, D, 3, C), (X, Y, Z, D, 4, E), (X, Y, Z, F, 5, E), (X, Y, Z, F, 1, G), (X, Y, Z, H, 4, G), (X, Y, Z, H, 5, I), (X, Y, Z, J, 2, I), (X, Y, Z, J, 3, A)$$

From derivable sequents to incidence results

$$\vdash \quad (X, Y, Z, B, 1, A), (X, Y, Z, B, 2, C), (X, Y, Z, D, 3, C), (X, Y, Z, D, 4, E),$$
$$(X, Y, Z, F, 5, E), (X, Y, Z, F, 1, G), (X, Y, Z, H, 4, G),$$
$$(X, Y, Z, H, 5, I), (X, Y, Z, J, 2, I), (X, Y, Z, J, 3, A)$$



Thank you!

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