Complexity of Action Logic

Logic and Applications 2019, Dubrovnik

Stepan Kuznetsov

Steklov Mathematical Institute of the Russian Academy of Sciences, Moscow

History of Action Logic

History of Action Logic

- S. C. Kleene 1956: **iteration** (Kleene star) introduced.

History of Action Logic

- S. C. Kleene 1956: **iteration** (Kleene star) introduced.
- W. Krull 1924; M. Ward & L. P. Dilworth 1939: **residuated** partially ordered algebraic structures.

History of Action Logic

- S. C. Kleene 1956: **iteration** (Kleene star) introduced.
- W. Krull 1924; M. Ward & L. P. Dilworth 1939: **residuated** partially ordered algebraic structures.
- J. Lambek 1958: the **Lambek calculus** (the logic of residuated semigroups, introduced for linguistic purposes).

History of Action Logic

- S. C. Kleene 1956: **iteration** (Kleene star) introduced.
- W. Krull 1924; M. Ward & L. P. Dilworth 1939: **residuated** partially ordered algebraic structures.
- J. Lambek 1958: the **Lambek calculus** (the logic of residuated semigroups, introduced for linguistic purposes).
- V. Pratt 1991: **action algebras,** or residuated Kleene algebras (language: $\cdot, \vee, ^*, \backslash, /$)

History of Action Logic

- S. C. Kleene 1956: **iteration** (Kleene star) introduced.
- W. Krull 1924; M. Ward & L. P. Dilworth 1939: **residuated** partially ordered algebraic structures.
- J. Lambek 1958: the **Lambek calculus** (the logic of residuated semigroups, introduced for linguistic purposes).
- V. Pratt 1991: **action algebras,** or residuated Kleene algebras (language: $\cdot, \vee, ^*, \backslash, /$)
  - Motivation: action algebras form a finitely based variety (main axiom: $(p \,/\, p)^* = p \,/\, p$), while Kleene algebras do not [V. N. Redko 1964].

History of Action Logic

- S. C. Kleene 1956: **iteration** (Kleene star) introduced.
- W. Krull 1924; M. Ward & L. P. Dilworth 1939: **residuated** partially ordered algebraic structures.
- J. Lambek 1958: the **Lambek calculus** (the logic of residuated semigroups, introduced for linguistic purposes).
- V. Pratt 1991: **action algebras,** or residuated Kleene algebras (language: $\cdot, \vee, {}^*, \backslash, /$)
  - Motivation: action algebras form a finitely based variety (main axiom: $(p \,/\, p)^* = p \,/\, p$), while Kleene algebras do not [V. N. Redko 1964].
- D. Kozen 1994: **action lattices** (adding $\wedge$)

History of Action Logic

- S. C. Kleene 1956: **iteration** (Kleene star) introduced.
- W. Krull 1924; M. Ward & L. P. Dilworth 1939: **residuated** partially ordered algebraic structures.
- J. Lambek 1958: the **Lambek calculus** (the logic of residuated semigroups, introduced for linguistic purposes).
- V. Pratt 1991: **action algebras,** or residuated Kleene algebras (language: $\cdot, \vee, {}^*, \backslash, /$)
  - Motivation: action algebras form a finitely based variety (main axiom: $(p \,/\, p)^* = p \,/\, p$), while Kleene algebras do not [V. N. Redko 1964].
- D. Kozen 1994: **action lattices** (adding $\wedge$)
  - Motivation: action lattices are closed under matrix formation.

History of Action Logic

- S. C. Kleene 1956: **iteration** (Kleene star) introduced.
- W. Krull 1924; M. Ward & L. P. Dilworth 1939: **residuated** partially ordered algebraic structures.
- J. Lambek 1958: the **Lambek calculus** (the logic of residuated semigroups, introduced for linguistic purposes).
- V. Pratt 1991: **action algebras,** or residuated Kleene algebras (language: $\cdot, \vee, ^*, \backslash, /$)
  - Motivation: action algebras form a finitely based variety (main axiom: $(p \,/\, p)^* = p \,/\, p$), while Kleene algebras do not [V. N. Redko 1964].
- D. Kozen 1994: **action lattices** (adding $\wedge$)
  - Motivation: action lattices are closed under matrix formation.
- W. Buszkowski, E. Palka 2007: an infinitary sequent calculus for the inequational theory of ***-continuous** action lattices, cut elimination & complexity.

Standard Examples of Action Lattices

- Algebra of formal languages, $\mathcal{P}(\Sigma^*)$ (so-called L-models):
  - multiplication is pairwise concatenation:
    $A \cdot B = \{uv \mid u \in A, v \in B\}$;
  - Kleene star is language iteration:
    $A^* = \{u_1 \ldots u_k \mid k \geq 0, u_i \in A\}$;
  - residuals are Lambek-style language divisions:
    $A \setminus B = \{u \in \Sigma^* \mid (\forall v \in A)\, vu \in B\}$,
    $B \,/\, A = \{u \in \Sigma^* \mid (\forall v \in A)\, uv \in B\}$;
  - $\preceq$ is $\subseteq$; $\vee$ and $\wedge$ are interpreted as $\cup$ and $\cap$.

Standard Examples of Action Lattices

- Algebra of formal languages, $\mathcal{P}(\Sigma^*)$ (so-called L-models):
  - multiplication is pairwise concatenation:
    $A \cdot B = \{uv \mid u \in A, v \in B\}$;
  - Kleene star is language iteration:
    $A^* = \{u_1 \dots u_k \mid k \geq 0, u_i \in A\}$;
  - residuals are Lambek-style language divisions:
    $A \setminus B = \{u \in \Sigma^* \mid (\forall v \in A)\, vu \in B\}$,
    $B \,/\, A = \{u \in \Sigma^* \mid (\forall v \in A)\, uv \in B\}$;
  - $\preceq$ is $\subseteq$; $\vee$ and $\wedge$ are interpreted as $\cup$ and $\cap$.
- Algebra of relations, $\mathcal{P}(W \times W)$ (so-called R-models):
  - multiplication is composition of relations;
  - Kleene star is reflexive-transitive closure;
  - residuals are relation divisions:
    $A \setminus B = \{\langle y, z\rangle \in W \times W \mid (\forall \langle x, y\rangle \in A)\, \langle x, z\rangle \in B\}$,
    $B \,/\, A = \{\langle x, y\rangle \in W \times W \mid (\forall \langle y, z\rangle \in A)\, \langle x, z\rangle \in B\}$;
  - $\preceq$, $\vee$, and $\wedge$ are set-theoretic.

(In)equational Theories

- In this talk, we are interested in **inequational theories** of classes of algebraic systems with residuals and iteration.

(In)equational Theories

- In this talk, we are interested in **inequational theories** of classes of algebraic systems with residuals and iteration.
- The inequational (atomic) theory is the set of all statements of the form $\alpha \preceq \beta$, where α and β are formulae (terms) in the signature of the algebra, which are generally true in all algebras of the given class.

(In)equational Theories

- In this talk, we are interested in **inequational theories** of classes of algebraic systems with residuals and iteration.
- The inequational (atomic) theory is the set of all statements of the form $\alpha \preceq \beta$, where α and β are formulae (terms) in the signature of the algebra, which are generally true in all algebras of the given class.
- Why inequational theories?

(In)equational Theories

- In this talk, we are interested in **inequational theories** of classes of algebraic systems with residuals and iteration.
- The inequational (atomic) theory is the set of all statements of the form $\alpha \preceq \beta$, where α and β are formulae (terms) in the signature of the algebra, which are generally true in all algebras of the given class.
- Why inequational theories?
  - Kozen 2002: the Horn theory (talking about statements of the form $\alpha_1 \preceq \beta_1 \,\&\, \ldots \,\&\, \alpha_n \preceq \beta_n \Rightarrow \gamma \preceq \delta$) is Π_1^1-complete already for the Kleene algebra signature $({}^*, \vee, \preceq)$, for L-models.

(In)equational Theories

- In this talk, we are interested in **inequational theories** of classes of algebraic systems with residuals and iteration.
- The inequational (atomic) theory is the set of all statements of the form $\alpha \preceq \beta$, where α and β are formulae (terms) in the signature of the algebra, which are generally true in all algebras of the given class.
- Why inequational theories?
  - Kozen 2002: the Horn theory (talking about statements of the form $\alpha_1 \preceq \beta_1 \,\&\, \ldots \,\&\, \alpha_n \preceq \beta_n \Rightarrow \gamma \preceq \delta$) is Π^1_1-complete already for the Kleene algebra signature $({}^*, \vee, \preceq)$, for L-models.
  - For inequational theories, interesting complexity results can be obtained.

(In)equational Theories

- In this talk, we are interested in **inequational theories** of classes of algebraic systems with residuals and iteration.
- The inequational (atomic) theory is the set of all statements of the form $\alpha \preceq \beta$, where α and β are formulae (terms) in the signature of the algebra, which are generally true in all algebras of the given class.
- Why inequational theories?
  - Kozen 2002: the Horn theory (talking about statements of the form $\alpha_1 \preceq \beta_1 \,\&\, \ldots \,\&\, \alpha_n \preceq \beta_n \Rightarrow \gamma \preceq \delta$) is Π^1_1-complete already for the Kleene algebra signature $({}^*, \vee, \preceq)$, for L-models.
  - For inequational theories, interesting complexity results can be obtained.
- In presence of $\vee$, inequational theories are essentially the same as equational ones, by $\alpha \preceq \beta \iff \alpha \vee \beta = \beta$.

ACT_ω: Infinitary Action Logic

$$\frac{}{\alpha \vdash \alpha} \qquad \frac{\Gamma, \Delta \vdash \gamma}{\Gamma, \mathbf{1}, \Delta \vdash \gamma} \qquad \frac{}{\Lambda \vdash \mathbf{1}}$$

$$\frac{\Pi \vdash \alpha \quad \Gamma, \beta, \Delta \vdash \gamma}{\Gamma, \Pi, \alpha \setminus \beta, \Delta \vdash \gamma} \qquad \frac{\alpha, \Pi \vdash \beta}{\Pi \vdash \alpha \setminus \beta} \qquad \frac{\Gamma, \alpha, \beta, \Delta \vdash \gamma}{\Gamma, \alpha \cdot \beta, \Delta \vdash \gamma}$$

$$\frac{\Pi \vdash \alpha \quad \Gamma, \beta, \Delta \vdash \gamma}{\Gamma, \beta / \alpha, \Pi, \Delta \vdash \gamma} \qquad \frac{\Pi, \alpha \vdash \beta}{\Pi \vdash \beta / \alpha} \qquad \frac{\Gamma \vdash \alpha \quad \Delta \vdash \beta}{\Gamma, \Delta \vdash \alpha \cdot \beta}$$

$$\frac{\Gamma, \alpha_1, \Delta \vdash \gamma \quad \Gamma, \alpha_2, \Delta \vdash \gamma}{\Gamma, \alpha_1 \vee \alpha_2, \Delta \vdash \gamma} \qquad \frac{\Pi \vdash \alpha_i}{\Pi \vdash \alpha_1 \vee \alpha_2}$$

$$\frac{\Gamma, \alpha_i, \Delta \vdash \gamma}{\Gamma, \alpha_1 \wedge \alpha_2, \Delta \vdash \gamma} \qquad \frac{\Pi \vdash \alpha_1 \quad \Pi \vdash \alpha_2}{\Pi \vdash \alpha_1 \wedge \alpha_2}$$

$$\frac{\left(\Gamma, \alpha^n, \Delta \vdash \gamma\right)_{n=0}^{\infty}}{\Gamma, \alpha^*, \Delta \vdash \gamma} \qquad \frac{\Pi_1 \vdash \alpha \quad \dots \quad \Pi_n \vdash \alpha}{\Pi_1, \dots, \Pi_n \vdash \alpha^*} \; n \geq 0$$

$$\left.\begin{array}{ccc}
\dfrac{}{\alpha \vdash \alpha} & \dfrac{\Gamma, \Delta \vdash \gamma}{\Gamma, \mathbf{1}, \Delta \vdash \gamma} & \dfrac{}{\Lambda \vdash \mathbf{1}} \\[2ex]
\dfrac{\Pi \vdash \alpha \quad \Gamma, \beta, \Delta \vdash \gamma}{\Gamma, \Pi, \alpha \backslash \beta, \Delta \vdash \gamma} & \dfrac{\alpha, \Pi \vdash \beta}{\Pi \vdash \alpha \backslash \beta} & \dfrac{\Gamma, \alpha, \beta, \Delta \vdash \gamma}{\Gamma, \alpha \cdot \beta, \Delta \vdash \gamma} \\[2ex]
\dfrac{\Pi \vdash \alpha \quad \Gamma, \beta, \Delta \vdash \gamma}{\Gamma, \beta / \alpha, \Pi, \Delta \vdash \gamma} & \dfrac{\Pi, \alpha \vdash \beta}{\Pi \vdash \beta / \alpha} & \dfrac{\Gamma \vdash \alpha \quad \Delta \vdash \beta}{\Gamma, \Delta \vdash \alpha \cdot \beta}
\end{array}\right\} \mathbf{L_1}$$

$$\dfrac{\Gamma, \alpha_1, \Delta \vdash \gamma \quad \Gamma, \alpha_2, \Delta \vdash \gamma}{\Gamma, \alpha_1 \vee \alpha_2, \Delta \vdash \gamma} \qquad \dfrac{\Pi \vdash \alpha_i}{\Pi \vdash \alpha_1 \vee \alpha_2}$$

$$\dfrac{\Gamma, \alpha_i, \Delta \vdash \gamma}{\Gamma, \alpha_1 \wedge \alpha_2, \Delta \vdash \gamma} \qquad \dfrac{\Pi \vdash \alpha_1 \quad \Pi \vdash \alpha_2}{\Pi \vdash \alpha_1 \wedge \alpha_2}$$

$$\dfrac{\big(\Gamma, \alpha^n, \Delta \vdash \gamma\big)_{n=0}^{\infty}}{\Gamma, \alpha^*, \Delta \vdash \gamma} \qquad \dfrac{\Pi_1 \vdash \alpha \quad \ldots \quad \Pi_n \vdash \alpha}{\Pi_1, \ldots, \Pi_n \vdash \alpha^*} \; n \geq 0$$

$\mathbf{ACT}_\omega$: Infinitary Action Logic

$$\frac{}{\alpha \vdash \alpha} \qquad \frac{\Gamma, \Delta \vdash \gamma}{\Gamma, \mathbf{1}, \Delta \vdash \gamma} \qquad \frac{}{\Lambda \vdash \mathbf{1}}$$

$$\frac{\Pi \vdash \alpha \quad \Gamma, \beta, \Delta \vdash \gamma}{\Gamma, \Pi, \alpha \setminus \beta, \Delta \vdash \gamma} \qquad \frac{\alpha, \Pi \vdash \beta}{\Pi \vdash \alpha \setminus \beta} \qquad \frac{\Gamma, \alpha, \beta, \Delta \vdash \gamma}{\Gamma, \alpha \cdot \beta, \Delta \vdash \gamma}$$

$$\frac{\Pi \vdash \alpha \quad \Gamma, \beta, \Delta \vdash \gamma}{\Gamma, \beta / \alpha, \Pi, \Delta \vdash \gamma} \qquad \frac{\Pi, \alpha \vdash \beta}{\Pi \vdash \beta / \alpha} \qquad \frac{\Gamma \vdash \alpha \quad \Delta \vdash \beta}{\Gamma, \Delta \vdash \alpha \cdot \beta}$$

$\Big\}\ \mathbf{L_1}$

$$\frac{\Gamma, \alpha_1, \Delta \vdash \gamma \quad \Gamma, \alpha_2, \Delta \vdash \gamma}{\Gamma, \alpha_1 \vee \alpha_2, \Delta \vdash \gamma} \qquad \frac{\Pi \vdash \alpha_i}{\Pi \vdash \alpha_1 \vee \alpha_2}$$

$$\frac{\Gamma, \alpha_i, \Delta \vdash \gamma}{\Gamma, \alpha_1 \wedge \alpha_2, \Delta \vdash \gamma} \qquad \frac{\Pi \vdash \alpha_1 \quad \Pi \vdash \alpha_2}{\Pi \vdash \alpha_1 \wedge \alpha_2}$$

$\Big\}\ \mathbf{MALC}$

$$\frac{\big(\Gamma, \alpha^n, \Delta \vdash \gamma\big)_{n=0}^{\infty}}{\Gamma, \alpha^*, \Delta \vdash \gamma} \qquad \frac{\Pi_1 \vdash \alpha \quad \ldots \quad \Pi_n \vdash \alpha}{\Pi_1, \ldots, \Pi_n \vdash \alpha^*}\ n \geq 0$$

Properties of $\mathbf{ACT}_\omega$

Properties of $\mathbf{ACT}_\omega$

- W. Buszkowski & E. Palka 2008: the derivability problem for $\mathbf{ACT}_\omega$ is Π^0_1-complete.

Properties of $\mathbf{ACT}_\omega$

- W. Buszkowski & E. Palka 2008: the derivability problem for $\mathbf{ACT}_\omega$ is Π_1^0-complete.
- E. Palka 2008: cut elimination for $\mathbf{ACT}_\omega$

Properties of $\mathbf{ACT}_\omega$

- W. Buszkowski & E. Palka 2008: the derivability problem for $\mathbf{ACT}_\omega$ is Π^0_1-complete.
- E. Palka 2008: cut elimination for $\mathbf{ACT}_\omega$

$$\frac{\Pi \vdash \alpha \quad \Gamma, \alpha, \Delta \vdash \gamma}{\Gamma, \Pi, \Delta \vdash \gamma}\ \mathrm{Cut}$$

Properties of $\mathbf{ACT}_\omega$

- W. Buszkowski & E. Palka 2008: the derivability problem for $\mathbf{ACT}_\omega$ is Π_1^0-complete.
- E. Palka 2008: cut elimination for $\mathbf{ACT}_\omega$

$$\frac{\Pi \vdash \alpha \quad \Gamma, \alpha, \Delta \vdash \gamma}{\Gamma, \Pi, \Delta \vdash \gamma} \ \mathrm{Cut}$$

- $\mathbf{ACT}_\omega$ is complete w.r.t. a general class of algebraic models, namely, ***-continuous residuated Kleene lattices**

Properties of $\mathbf{ACT}_\omega$

- W. Buszkowski & E. Palka 2008: the derivability problem for $\mathbf{ACT}_\omega$ is Π^0_1-complete.
- E. Palka 2008: cut elimination for $\mathbf{ACT}_\omega$

$$\frac{\Pi \vdash \alpha \quad \Gamma, \alpha, \Delta \vdash \gamma}{\Gamma, \Pi, \Delta \vdash \gamma}\ \mathrm{Cut}$$

- $\mathbf{ACT}_\omega$ is complete w.r.t. a general class of algebraic models, namely, ***-continuous residuated Kleene lattices:**
  - $\cdot$ and $\mathbf{1}$ impose a monoid structure;
  - $\preceq$ (in sequents, $\vdash$) is a lattice preorder, $\vee$ and $\wedge$ being join and meet;
  - $\backslash$ and $/$ are residuals of $\cdot$ w.r.t. $\preceq$:

$$\beta \preceq \alpha \mathop{\backslash} \gamma \iff \alpha \cdot \beta \preceq \gamma \iff \alpha \preceq \gamma \mathop{/} \beta;$$

  - $\alpha^* = \sup_{\preceq}\{\alpha^n \mid n \geq 0\}$.

Incompleteness

- However, $\mathbf{ACT}_\omega$ is incomplete w.r.t. L-models and R-models.

Incompleteness

- However, $\mathbf{ACT}_\omega$ is incomplete w.r.t. L-models and R-models.
- The reason is the distributivity law,

$$\alpha \wedge (\beta \vee \gamma) \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma).$$

Incompleteness

- However, $\mathbf{ACT}_\omega$ is incomplete w.r.t. L-models and R-models.
- The reason is the distributivity law,

$$\alpha \wedge (\beta \vee \gamma) \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma).$$

- There are also corollaries of this law which yield incompleteness for restricted languages:
  - $(s / (r / r)) \wedge (s / (p^+ \wedge q^+)) \vdash s / (p^* \wedge q^*)$ [S. K. 2018]
  - $((x / y) \vee x) / ((x / y) \vee (x / z) \vee x), (x / y) \vee x, ((x / y) \vee x) \backslash ((x / z) \vee x) \vdash (x / (y \vee z)) \vee x$
    [M. Kanovich, S. K., A. Scedrov 2019]

Completeness Results

- Completeness results for fragments:
  - H. Andréka & Sz. Mikulás 1994: R-completeness for $\backslash, /, \cdot, \wedge$
  - W. Buszkowski 1982: L-completeness for $\backslash, /, \wedge$
  - M. Pentus 1995: L-completeness for $\backslash, /, \cdot$
  - N. Ryzhkova & S. K. 2015: L-completeness for $\backslash, /, \wedge$, and * restricted to subformulae of the form $\alpha^* \backslash \beta$ or β / α^*
  - S. K. 2018: R-completeness for $\backslash, /, \cdot, \wedge$, and restricted *.

Completeness Results

- Completeness results for fragments:
  - H. Andréka & Sz. Mikulás 1994: R-completeness for $\backslash, /, \cdot, \wedge$
  - W. Buszkowski 1982: L-completeness for $\backslash, /, \wedge$
  - M. Pentus 1995: L-completeness for $\backslash, /, \cdot$
  - N. Ryzhkova & S. K. 2015: L-completeness for $\backslash, /, \wedge$, and * restricted to subformulae of the form $\alpha^* \setminus \beta$ or $\beta \,/\, \alpha^*$
  - S. K. 2018: R-completeness for $\backslash, /, \cdot, \wedge$, and restricted *.
- Restricted fragments are still Π_1^0-hard [S. K. 2019]. Thus, we get Π_1^0-hardness for inequational theories of L- and R-models.

Completeness Results

- Completeness results for fragments:
  - H. Andréka & Sz. Mikulás 1994: R-completeness for $\backslash, /, \cdot, \wedge$
  - W. Buszkowski 1982: L-completeness for $\backslash, /, \wedge$
  - M. Pentus 1995: L-completeness for $\backslash, /, \cdot$
  - N. Ryzhkova & S. K. 2015: L-completeness for $\backslash, /, \wedge$, and * restricted to subformulae of the form $\alpha^* \backslash \beta$ or β / α^*
  - S. K. 2018: R-completeness for $\backslash, /, \cdot, \wedge$, and restricted *.
- Restricted fragments are still Π_1^0-hard [S. K. 2019]. Thus, we get Π_1^0-hardness for inequational theories of L- and R-models.
- In whole, $\mathbf{ACT}_\omega$ is complete w.r.t. **syntactic concept lattices** introduced by C. Wurm in 2015–17 [D. Makarov 2019].

Syntactic Concept Lattices

- Syntactic concept lattices are a modification of L-models which avoid distributivity

Syntactic Concept Lattices

- Syntactic concept lattices are a modification of L-models which avoid distributivity… and are in a sense more natural from the linguistic point of view!

Syntactic Concept Lattices

- Syntactic concept lattices are a modification of L-models which avoid distributivity... and are in a sense more natural from the linguistic point of view!
- Fix a language L ("all correct sentences").

Syntactic Concept Lattices

- Syntactic concept lattices are a modification of L-models which avoid distributivity... and are in a sense more natural from the linguistic point of view!
- Fix a language L ("all correct sentences").
- For an arbitrary language A, let

$$A^{\triangleright} = \{\langle u, v \rangle \mid (\forall w \in A)\, uwv \in L\}$$

Syntactic Concept Lattices

- Syntactic concept lattices are a modification of L-models which avoid distributivity... and are in a sense more natural from the linguistic point of view!
- Fix a language L ("all correct sentences").
- For an arbitrary language A, let

$$A^{\triangleright} = \{\langle u, v\rangle \mid (\forall w \in A)\, uwv \in L\}$$
$$A^{\triangleright\triangleleft} = \{w \mid (\forall \langle u, v\rangle \in A^{\triangleright})\, uwv \in L\}$$

Syntactic Concept Lattices

- Syntactic concept lattices are a modification of L-models which avoid distributivity... and are in a sense more natural from the linguistic point of view!
- Fix a language L ("all correct sentences").
- For an arbitrary language A, let

$$A^{\triangleright} = \{\langle u, v\rangle \mid (\forall w \in A)\, uwv \in L\}$$
$$A^{\triangleright\triangleleft} = \{w \mid (\forall \langle u, v\rangle \in A^{\triangleright})\, uwv \in L\}$$

- ${}^{\triangleright\triangleleft}$ is a closure operation: $A \subseteq A^{\triangleright\triangleleft}$ and $A^{\triangleright\triangleleft\triangleright\triangleleft} = A^{\triangleright\triangleleft}$.

Syntactic Concept Lattices

- Syntactic concept lattices are a modification of L-models which avoid distributivity... and are in a sense more natural from the linguistic point of view!
- Fix a language L ("all correct sentences").
- For an arbitrary language A, let

$$A^{\triangleright} = \{\langle u, v \rangle \mid (\forall w \in A)\, uwv \in L\}$$
$$A^{\triangleright\triangleleft} = \{w \mid (\forall \langle u, v \rangle \in A^{\triangleright})\, uwv \in L\}$$

- $^{\triangleright\triangleleft}$ is a closure operation: $A \subseteq A^{\triangleright\triangleleft}$ and $A^{\triangleright\triangleleft\triangleright\triangleleft} = A^{\triangleright\triangleleft}$.
- We use only closed sets (such that $A = A^{\triangleright\triangleleft}$), which correspond to linguistic concepts: say, if A included the word "dog," $A^{\triangleright\triangleleft}$ is likely to include "cat" also.

Syntactic Concept Lattices

- Syntactic concept lattices are a modification of L-models which avoid distributivity... and are in a sense more natural from the linguistic point of view!
- Fix a language L ("all correct sentences").
- For an arbitrary language A, let

$$A^{\rhd} = \{\langle u, v\rangle \mid (\forall w \in A)\, uwv \in L\}$$
$$A^{\rhd\lhd} = \{w \mid (\forall \langle u, v\rangle \in A^{\rhd})\, uwv \in L\}$$

- ${}^{\rhd\lhd}$ is a closure operation: $A \subseteq A^{\rhd\lhd}$ and $A^{\rhd\lhd\rhd\lhd} = A^{\rhd\lhd}$.
- We use only closed sets (such that $A = A^{\rhd\lhd}$), which correspond to linguistic concepts: say, if A included the word "dog," $A^{\rhd\lhd}$ is likely to include "cat" also.
- When interpreting operations of $\mathbf{ACT}_{\omega}$, we add closure, if needed.

Proof of Π^0_1-hardness

Buszkowski's proof of Π^0_1-hardness for $\mathbf{ACT}_\omega$ goes via the totality problem for context-free grammars.

$$\boxed{\begin{array}{c}\mathfrak{M}, x\\ \text{Turing machine}\\ \text{and its input}\end{array}} \longrightarrow \boxed{\begin{array}{c}\mathcal{G}_{\mathfrak{M},x}\\ \text{context-free}\\ \text{grammar}\end{array}} \longrightarrow \boxed{\begin{array}{c}\psi_{\mathfrak{M},x}\\ \text{formula}\\ \text{of } \mathbf{ACT}_\omega\end{array}}$$

$$\begin{array}{c}\mathfrak{M} \text{ does } \textbf{not}\\ \text{halt on } x\end{array} \iff \begin{array}{c}\mathcal{G}_{\mathfrak{M},x} \text{ generates}\\ \textbf{all} \text{ non-empty words}\end{array} \iff \begin{array}{c}\psi^+_{\mathfrak{M},x} \vdash S \text{ is}\\ \text{derivable in } \mathbf{ACT}_\omega\end{array}$$

Proof of Π^0_1-hardness

$\mathfrak{M}, x$	$\longrightarrow$	$\mathcal{G}_{\mathfrak{M},x}$	$\longrightarrow$	$\psi_{\mathfrak{M},x}$
Turing machine and its input		context-free grammar		formula of $\mathbf{ACT}_\omega$

Proof of Π^0_1-hardness

$\mathfrak{M}, x$ Turing machine and its input	$\longrightarrow$	$\mathcal{G}_{\mathfrak{M},x}$ context-free grammar	$\longrightarrow$	$\psi_{\mathfrak{M},x}$ formula of $\mathbf{ACT}_\omega$

- The construction of $\mathcal{G}_{\mathfrak{M},x}$ is standard (see textbooks).

Proof of Π^0_1-hardness

$$
\begin{tikzcd}
\begin{array}{c} \mathfrak{M}, x \\ \text{Turing machine} \\ \text{and its input} \end{array} \arrow[r] & \begin{array}{c} \mathcal{G}_{\mathfrak{M},x} \\ \text{context-free} \\ \text{grammar} \end{array} \arrow[r] & \begin{array}{c} \psi_{\mathfrak{M},x} \\ \text{formula} \\ \text{of } \mathbf{ACT}_\omega \end{array}
\end{tikzcd}
$$

- The construction of $\mathcal{G}_{\mathfrak{M},x}$ is standard (see textbooks).
- For $\psi_{\mathfrak{M},x}$, we translate $\mathcal{G}_{\mathfrak{M},x}$ into *Lambek categorial grammar*.

Proof of Π^0_1-hardness

$\boxed{\begin{array}{c}\mathfrak{M}, x \\ \text{Turing machine} \\ \text{and its input}\end{array}} \longrightarrow \boxed{\begin{array}{c}\mathcal{G}_{\mathfrak{M},x} \\ \text{context-free} \\ \text{grammar}\end{array}} \longrightarrow \boxed{\begin{array}{c}\psi_{\mathfrak{M},x} \\ \text{formula} \\ \text{of } \mathbf{ACT}_\omega\end{array}}$

- The construction of $\mathcal{G}_{\mathfrak{M},x}$ is standard (see textbooks).
- For $\psi_{\mathfrak{M},x}$, we translate $\mathcal{G}_{\mathfrak{M},x}$ into *Lambek categorial grammar*.
    - Let $\mathcal{G}_{\mathfrak{M},x}$ be in Greibach normal form, and for each rule $A \Rightarrow aB_1 \ldots B_k$ take $A\,/(B_1 \cdot \ldots \cdot B_k)$.

Proof of Π^0_1-hardness

$$
\begin{array}{|c|}\hline \mathfrak{M}, x \\ \text{Turing machine} \\ \text{and its input} \\ \hline \end{array}
\longrightarrow
\begin{array}{|c|}\hline \mathcal{G}_{\mathfrak{M},x} \\ \text{context-free} \\ \text{grammar} \\ \hline \end{array}
\longrightarrow
\begin{array}{|c|}\hline \psi_{\mathfrak{M},x} \\ \text{formula} \\ \text{of } \mathbf{ACT}_\omega \\ \hline \end{array}
$$

- The construction of $\mathcal{G}_{\mathfrak{M},x}$ is standard (see textbooks).
- For $\psi_{\mathfrak{M},x}$, we translate $\mathcal{G}_{\mathfrak{M},x}$ into *Lambek categorial grammar*.
    - Let $\mathcal{G}_{\mathfrak{M},x}$ be in Greibach normal form, and for each rule $A \Rightarrow aB_1 \ldots B_k$ take $A\,/(B_1 \cdot \ldots \cdot B_k)$.
    - Let φ_a be the conjunction of all such formulae for a particular $a \in \Sigma$.

Proof of Π^0_1-hardness

$\mathfrak{M}, x$ Turing machine and its input	$\longrightarrow$	$\mathcal{G}_{\mathfrak{M},x}$ context-free grammar	$\longrightarrow$	$\psi_{\mathfrak{M},x}$ formula of $\mathbf{ACT}_\omega$

- The construction of $\mathcal{G}_{\mathfrak{M},x}$ is standard (see textbooks).
- For $\psi_{\mathfrak{M},x}$, we translate $\mathcal{G}_{\mathfrak{M},x}$ into *Lambek categorial grammar*.
    - Let $\mathcal{G}_{\mathfrak{M},x}$ be in Greibach normal form, and for each rule $A \Rightarrow aB_1 \ldots B_k$ take $A \,/ (B_1 \cdot \ldots \cdot B_k)$.
    - Let φ_a be the conjunction of all such formulae for a particular $a \in \Sigma$.
    - Let $\psi_{\mathfrak{M},x} = \bigvee_{a \in \Sigma} \varphi_a$.

Proof of Π^0_1-hardness

$\mathfrak{M}, x$ Turing machine and its input $\longrightarrow$ $\mathcal{G}_{\mathfrak{M},x}$ context-free grammar $\longrightarrow$ $\psi_{\mathfrak{M},x}$ formula of $\mathbf{ACT}_\omega$

- The construction of $\mathcal{G}_{\mathfrak{M},x}$ is standard (see textbooks).
- For $\psi_{\mathfrak{M},x}$, we translate $\mathcal{G}_{\mathfrak{M},x}$ into *Lambek categorial grammar*.
  - Let $\mathcal{G}_{\mathfrak{M},x}$ be in Greibach normal form, and for each rule $A \Rightarrow aB_1 \ldots B_k$ take $A \,/ (B_1 \cdot \ldots \cdot B_k)$.
  - Let φ_a be the conjunction of all such formulae for a particular $a \in \Sigma$.
  - Let $\psi_{\mathfrak{M},x} = \bigvee_{a \in \Sigma} \varphi_a$.
  - $\psi^+_{\mathfrak{M},x} \vdash S$ means exactly "any non-empty word is derivable from the starting symbol S."

Π^0_1-hardness without $\vee$ and $\wedge$

- One can prove Π^0_1-hardness without $\vee$ and $\wedge$ [S. K. 2019].

Π_1^0-hardness without $\vee$ and $\wedge$

- One can prove Π_1^0-hardness without $\vee$ and $\wedge$ [S. K. 2019].
- This is achieved by using Lambek grammars **with unique type assignment** [A. Safiullin 2007]. In such a grammar, each letter receives a syntactic type φ_a without $\wedge$.

Π_1^0-hardness without $\vee$ and $\wedge$

- One can prove Π_1^0-hardness without $\vee$ and $\wedge$ [S. K. 2019].
- This is achieved by using Lambek grammars **with unique type assignment** [A. Safiullin 2007]. In such a grammar, each letter receives a syntactic type φ_a without $\wedge$.
- Next, we restrict ourselves to $\Sigma = \{a, b\}$ and use the sequent $(\varphi_a^+ \cdot \varphi_b^+)^+ \vdash s$.

Π^0_1-hardness without $\vee$ and $\wedge$

- One can prove Π^0_1-hardness without $\vee$ and $\wedge$ [S. K. 2019].
- This is achieved by using Lambek grammars **with unique type assignment** [A. Safiullin 2007]. In such a grammar, each letter receives a syntactic type φ_a without $\wedge$.
- Next, we restrict ourselves to $\Sigma = \{a, b\}$ and use the sequent $(\varphi_a^+ \cdot \varphi_b^+)^+ \vdash s$.
- Safiullin's construction originally worked for the Lambek calculus with non-empty antecedent restriction, but can be modified for $\mathbf{L_1}$ also.

Action Lattices

- In general, Kleene star is not required to be *-continuous, but is rather defined as a **least fixpoint:**

$$a^* = \min\{b \mid \mathbf{1} \preceq b \,\&\, a \cdot b \preceq b\}.$$

Action Lattices

- In general, Kleene star is not required to be *-continuous, but is rather defined as a **least fixpoint:**

$$a^* = \min\{b \mid \mathbf{1} \preceq b \,\&\, a \cdot b \preceq b\}.$$

- Existence of non-*-continuous residuated Kleene algebras can be shown by Gödel–Mal'cev compactness theorem. Explicit examples also available [S. K. 2018].

Action Lattices

- In general, Kleene star is not required to be *-continuous, but is rather defined as a **least fixpoint:**

$$a^* = \min\{b \mid \mathbf{1} \preceq b \,\&\, a \cdot b \preceq b\}.$$

- Existence of non-*-continuous residuated Kleene algebras can be shown by Gödel–Mal'cev compactness theorem. Explicit examples also available [S. K. 2018].
- Yet, no *natural* classes of non-*-continuous RKLs known.

Action Lattices

- In general, Kleene star is not required to be *-continuous, but is rather defined as a **least fixpoint:**

$$a^* = \min\{b \mid \mathbf{1} \preceq b \,\&\, a \cdot b \preceq b\}.$$

- Existence of non-*-continuous residuated Kleene algebras can be shown by Gödel–Mal'cev compactness theorem. Explicit examples also available [S. K. 2018].
- Yet, no *natural* classes of non-*-continuous RKLs known.
- The inequational theory of all RKLs [V. Pratt 1991]:

$$\mathbf{ACT} \quad = \quad \mathbf{MALC} \quad + \quad \mathbf{1} \vee \alpha \vee \alpha^* \cdot \alpha^* \vdash \alpha^* \quad + \\ \alpha^* \vdash (\alpha + \beta)^* \quad + \quad (\alpha \setminus \alpha)^* \vdash \alpha \setminus \alpha \quad + \quad \mathrm{Cut}$$

Action Lattices

- In general, Kleene star is not required to be *-continuous, but is rather defined as a **least fixpoint:**

$$a^* = \min\{b \mid \mathbf{1} \preceq b \,\&\, a \cdot b \preceq b\}.$$

- Existence of non-*-continuous residuated Kleene algebras can be shown by Gödel–Mal'cev compactness theorem. Explicit examples also available [S. K. 2018].
- Yet, no *natural* classes of non-*-continuous RKLs known.
- The inequational theory of all RKLs [V. Pratt 1991]:

$$\textbf{ACT} \quad = \quad \textbf{MALC} \quad + \quad \mathbf{1} \vee \alpha \vee \alpha^* \cdot \alpha^* \vdash \alpha^* \quad + \quad \alpha^* \vdash (\alpha + \beta)^* \quad + \quad (\alpha \setminus \alpha)^* \vdash \alpha \setminus \alpha \quad + \quad \mathrm{Cut}$$

- **Pozor!** No good (cut-free) sequent calculus known, thus we do not know conservativity of elementary fragments in **ACT**.

Complexity of ACT

- **ACT is Σ^0_1-complete** [S. K. 2019]

Complexity of ACT

- **ACT is Σ^0_1-complete** [S. K. 2019] (solving a problem left open by D. Kozen, P. Jipsen, W. Buszkowski).
- We also feature Σ^0_1-completeness of fragments $\backslash, /, \cdot, \vee, {}^*$ (original action algebras by Pratt) and $\backslash, /, \cdot, \wedge, {}^*$.

Inspiration: Circular Proofs for ACT

- $\mathbf{ACT}_\omega$ can be equivalently reformulated as a system with non-well-founded proofs [A. Das & D. Pous 2018]:

$$\frac{\Gamma, \Delta \vdash \gamma \quad \Gamma, \alpha, \alpha^*, \Delta \vdash \gamma}{\Gamma, \alpha^*, \Delta \vdash \gamma} \qquad \frac{}{\Lambda \vdash \alpha^*} \quad \frac{\Gamma \vdash \alpha \quad \Delta \vdash \alpha^*}{\Gamma, \Delta \vdash \alpha^*}$$

Inspiration: Circular Proofs for ACT

- **ACT**$_\omega$ can be equivalently reformulated as a system with non-well-founded proofs [A. Das & D. Pous 2018]:

$$\frac{\Gamma, \Delta \vdash \gamma \quad \Gamma, \alpha, \alpha^*, \Delta \vdash \gamma}{\Gamma, \alpha^*, \Delta \vdash \gamma} \qquad \frac{}{\Lambda \vdash \alpha^*} \quad \frac{\Gamma \vdash \alpha \quad \Delta \vdash \alpha^*}{\Gamma, \Delta \vdash \alpha^*}$$

  - Proofs are allowed to have infinite branches.

Inspiration: Circular Proofs for ACT

- **ACT**$_\omega$ can be equivalently reformulated as a system with non-well-founded proofs [A. Das & D. Pous 2018]:

$$\frac{\Gamma, \Delta \vdash \gamma \quad \Gamma, \alpha, \alpha^*, \Delta \vdash \gamma}{\Gamma, \alpha^*, \Delta \vdash \gamma} \qquad \frac{}{\Lambda \vdash \alpha^*} \quad \frac{\Gamma \vdash \alpha \quad \Delta \vdash \alpha^*}{\Gamma, \Delta \vdash \alpha^*}$$

  - Proofs are allowed to have infinite branches.
  - **Correctness condition:** each infinite branch has an application of the left rule for *, where the branch goes right.

Inspiration: Circular Proofs for ACT

- **ACT**$_\omega$ can be equivalently reformulated as a system with non-well-founded proofs [A. Das & D. Pous 2018]:

$$\frac{\Gamma, \Delta \vdash \gamma \quad \Gamma, \alpha, \alpha^*, \Delta \vdash \gamma}{\Gamma, \alpha^*, \Delta \vdash \gamma} \qquad \frac{}{\Lambda \vdash \alpha^*} \quad \frac{\Gamma \vdash \alpha \quad \Delta \vdash \alpha^*}{\Gamma, \Delta \vdash \alpha^*}$$

  - Proofs are allowed to have infinite branches.
  - **Correctness condition:** each infinite branch has an application of the left rule for *, where the branch goes right.
- **Circular fragment:** consider non-well-founded proofs, possibly with cut, with a finite number of non-isomorphic subtrees.

Inspiration: Circular Proofs for ACT

- **ACT**$_\omega$ can be equivalently reformulated as a system with non-well-founded proofs [A. Das & D. Pous 2018]:

$$\frac{\Gamma, \Delta \vdash \gamma \quad \Gamma, \alpha, \alpha^*, \Delta \vdash \gamma}{\Gamma, \alpha^*, \Delta \vdash \gamma} \qquad \frac{}{\Lambda \vdash \alpha^*} \quad \frac{\Gamma \vdash \alpha \quad \Delta \vdash \alpha^*}{\Gamma, \Delta \vdash \alpha^*}$$

  - Proofs are allowed to have infinite branches.
  - **Correctness condition:** each infinite branch has an application of the left rule for *, where the branch goes right.
- **Circular fragment:** consider non-well-founded proofs, possibly with cut, with a finite number of non-isomorphic subtrees. This fragment axiomatises **ACT**.

Inspiration: Circular Proofs for ACT

- $\mathbf{ACT}_\omega$ can be equivalently reformulated as a system with non-well-founded proofs [A. Das & D. Pous 2018]:

$$\frac{\Gamma, \Delta \vdash \gamma \quad \Gamma, \alpha, \alpha^*, \Delta \vdash \gamma}{\Gamma, \alpha^*, \Delta \vdash \gamma} \qquad \frac{}{\Lambda \vdash \alpha^*} \quad \frac{\Gamma \vdash \alpha \quad \Delta \vdash \alpha^*}{\Gamma, \Delta \vdash \alpha^*}$$

  - Proofs are allowed to have infinite branches.
  - **Correctness condition:** each infinite branch has an application of the left rule for *, where the branch goes right.
- **Circular fragment:** consider non-well-founded proofs, possibly with cut, with a finite number of non-isomorphic subtrees. This fragment axiomatises **ACT**.
  - **Caveat!** Adding symmetric versions of the rules yields a system $\mathbf{ACT}_{\mathrm{bicycle}}$ which is **stronger** than **ACT**: it derives $(p \wedge q \wedge (p \,/\, q) \wedge (p \setminus q))^+ \vdash p$, which is not derivable in **ACT** [S. K. 2018].

Inspiration: Circular Proofs for ACT

- In circular proofs, an argument supporting a sequent relies on the sequent itself. However, correctness conditions make this sound, since we have to use the left rule and thus perform an inductive step.

Inspiration: Circular Proofs for ACT

- In circular proofs, an argument supporting a sequent relies on the sequent itself. However, correctness conditions make this sound, since we have to use the left rule and thus perform an inductive step.
- **Idea:** while $\mathbf{ACT}_\omega$ can prove non-halting for an *arbitrary* Turing machine $\mathfrak{M}$ and input word x (if it is so), by deriving $\psi^+_{\mathfrak{M},x} \vdash S$, the **circular** fragment could prove it in the case when $\mathfrak{M}$ goes into a **cycle** while running on x.

Proving Σ^0_1-hardness for ACT

- Let q_c be a state of $\mathfrak{M}$ in which it goes into a cycle.

Proving Σ_1^0-hardness for ACT

- Let q_c be a state of $\mathfrak{M}$ in which it goes into a cycle.
- Then we add the following production rules to $\mathcal{G}_{\mathfrak{M},x}$, making this fact explicit:

$$U \Rightarrow a \qquad U \Rightarrow aU \qquad S \Rightarrow \#CU \qquad C \Rightarrow aC \qquad C \Rightarrow q_c U$$

Proving Σ_1^0-hardness for ACT

- Let q_c be a state of $\mathfrak{M}$ in which it goes into a cycle.
- Then we add the following production rules to $\mathcal{G}_{\mathfrak{M},x}$, making this fact explicit:

  $$U \Rightarrow a \qquad U \Rightarrow aU \qquad S \Rightarrow \#CU \qquad C \Rightarrow aC \qquad C \Rightarrow q_c U$$

  ($\#$ is the starting symbol of a protocol)

Proving Σ_1^0-hardness for ACT

- Let q_c be a state of $\mathfrak{M}$ in which it goes into a cycle.
- Then we add the following production rules to $\mathcal{G}_{\mathfrak{M},x}$, making this fact explicit:

  $$U \Rightarrow a \qquad U \Rightarrow aU \qquad S \Rightarrow \#CU \qquad C \Rightarrow aC \qquad C \Rightarrow q_c U$$

  ($\#$ is the starting symbol of a protocol)
- We also add $S \Rightarrow \#TU$, where T ("trash") generates all malformed protocols, which cannot be repaired by further extension.

Proving Σ_1^0-hardness for ACT

- Let q_c be a state of $\mathfrak{M}$ in which it goes into a cycle.
- Then we add the following production rules to $\mathcal{G}_{\mathfrak{M},x}$, making this fact explicit:

  $$U \Rightarrow a \qquad U \Rightarrow aU \qquad S \Rightarrow \#CU \qquad C \Rightarrow aC \qquad C \Rightarrow q_c U$$

  ($\#$ is the starting symbol of a protocol)
- We also add $S \Rightarrow \#TU$, where T ("trash") generates all malformed protocols, which cannot be repaired by further extension.
- Now, if a word starting with $\#$ is long enough (longer than the length of a partial protocol reaching q_c), it can be generated either by $S \Rightarrow \#CU$ (if it is a prefix of the correct infinite protocol), or by $S \Rightarrow \#TU$.

Proving Σ_1^0-hardness for ACT

- Let q_c be a state of $\mathfrak{M}$ in which it goes into a cycle.
- Then we add the following production rules to $\mathcal{G}_{\mathfrak{M},x}$, making this fact explicit:

  $$U \Rightarrow a \qquad U \Rightarrow aU \qquad S \Rightarrow \#CU \qquad C \Rightarrow aC \qquad C \Rightarrow q_c U$$

  ($\#$ is the starting symbol of a protocol)
- We also add $S \Rightarrow \#TU$, where T ("trash") generates all malformed protocols, which cannot be repaired by further extension.
- Now, if a word starting with $\#$ is long enough (longer than the length of a partial protocol reaching q_c), it can be generated either by $S \Rightarrow \#CU$ (if it is a prefix of the correct infinite protocol), or by $S \Rightarrow \#TU$.
- (If the word does not start with $\#$, we can use the rule $S \Rightarrow aU$, where $a \neq \#$.)

Proving Σ_1^0-hardness for ACT

- We encode this reasoning in **ACT** by using the **long rule:**

$$\frac{\psi \vdash S \quad \psi^2 \vdash S \quad \dots \quad \psi^n \vdash s \quad \psi^n, \psi^+ \vdash s}{\psi^+ \vdash S}$$

Proving Σ_1^0-hardness for ACT

- We encode this reasoning in **ACT** by using the **long rule:**

$$\frac{\psi \vdash S \quad \psi^2 \vdash S \quad \dots \quad \psi^n \vdash s \quad \psi^n, \psi^+ \vdash s}{\psi^+ \vdash S}$$

- The first n premises are derivable exactly as in $\mathbf{ACT}_\omega$ (they do not contain *).

Proving Σ_1^0-hardness for ACT

- We encode this reasoning in **ACT** by using the **long rule:**

$$\frac{\psi \vdash S \quad \psi^2 \vdash S \quad \dots \quad \psi^n \vdash s \quad \psi^n, \psi^+ \vdash s}{\psi^+ \vdash S}$$

- The first n premises are derivable exactly as in $\mathbf{ACT}_\omega$ (they do not contain *).
- For the last one, the derivation is as follows.

Proving Σ_1^0-hardness for ACT

- We encode this reasoning in **ACT** by using the **long rule:**

$$\frac{\psi \vdash S \quad \psi^2 \vdash S \quad \ldots \quad \psi^n \vdash s \quad \psi^n, \psi^+ \vdash s}{\psi^+ \vdash S}$$

- The first n premises are derivable exactly as in $\mathbf{ACT}_\omega$ (they do not contain *).
- For the last one, the derivation is as follows.
    - We derive all sequents of the form $\varphi_{a_1}, \ldots, \varphi_{a_n}, \psi^+ \vdash S$. Consider the case where one of the a_i is q_c, and we use $S \Rightarrow \#CU$ (others are similar).

Proving Σ_1^0-hardness for ACT

- We encode this reasoning in **ACT** by using the **long rule:**

$$\frac{\psi \vdash S \quad \psi^2 \vdash S \quad \ldots \quad \psi^n \vdash s \quad \psi^n, \psi^+ \vdash s}{\psi^+ \vdash S}$$

- The first n premises are derivable exactly as in $\mathbf{ACT}_\omega$ (they do not contain *).
- For the last one, the derivation is as follows.
  - We derive all sequents of the form $\varphi_{a_1}, \ldots, \varphi_{a_n}, \psi^+ \vdash S$. Consider the case where one of the a_i is q_c, and we use $S \Rightarrow \# CU$ (others are similar).
  - First we derive $\psi^+ \vdash U$ (see next slide).

Proving Σ_1^0-hardness for ACT

- We encode this reasoning in **ACT** by using the **long rule:**

$$\frac{\psi \vdash S \quad \psi^2 \vdash S \quad \ldots \quad \psi^n \vdash s \quad \psi^n, \psi^+ \vdash s}{\psi^+ \vdash S}$$

- The first n premises are derivable exactly as in $\mathbf{ACT}_\omega$ (they do not contain *).
- For the last one, the derivation is as follows.
  - We derive all sequents of the form $\varphi_{a_1}, \ldots, \varphi_{a_n}, \psi^+ \vdash S$. Consider the case where one of the a_i is q_c, and we use $S \Rightarrow \#CU$ (others are similar).
  - First we derive $\psi^+ \vdash U$ (see next slide).
  - Then we do the following (by cut):

$$\frac{\varphi_{a_2}, \ldots, \varphi_{a_n} \vdash C \quad S\,/(C \cdot U), C, U \vdash S}{S\,/(C \cdot U), \varphi_{a_2}, \ldots, \varphi_{a_n}, U \vdash S}$$

Derivation of $\psi^{+} \vdash U$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\psi \vdash U}{\Lambda \vdash \psi \setminus U} \qquad \dfrac{\psi \vdash \psi \quad \dfrac{\dfrac{\psi \vdash U \,/\, U}{\psi, U \vdash U}}{U \vdash \psi \setminus U}}{\psi, \psi \setminus U \vdash \psi \setminus U}}{\psi^{*} \vdash \psi \setminus U}}{\psi, \psi^{*} \vdash U}}{\psi^{+} \vdash U}
$$

Proving Σ^0_1-hardness of ACT

- In order to finish the proof, introduce the following notations:
  - $\mathcal{C} = \{\langle \mathfrak{M}, x\rangle \mid \mathfrak{M}$ reaches q_c on $x\}$
  - $\mathcal{H} = \{\langle \mathfrak{M}, x\rangle \mid \mathfrak{M}$ halts on $x\}$
  - $\mathcal{K}(\mathbf{ACT}) = \{\langle \mathfrak{M}, x\rangle \mid \mathbf{ACT}$ derives $\psi^+_{\mathfrak{M},x} \vdash s\}$,
    ditto for $\mathbf{ACT}_\omega$.

Proving Σ_1^0-hardness of ACT

- In order to finish the proof, introduce the following notations:
  - $\mathcal{C} = \{\langle \mathfrak{M}, x \rangle \mid \mathfrak{M} \text{ reaches } q_c \text{ on } x\}$
  - $\mathcal{H} = \{\langle \mathfrak{M}, x \rangle \mid \mathfrak{M} \text{ halts on } x\}$
  - $\mathcal{K}(\mathbf{ACT}) = \{\langle \mathfrak{M}, x \rangle \mid \mathbf{ACT} \text{ derives } \psi^{+}_{\mathfrak{M},x} \vdash s\}$, ditto for $\mathbf{ACT}_\omega$.
- We now know that

$$\mathcal{C} \subseteq \mathcal{K}(\mathbf{ACT}) \subseteq \mathcal{K}(\mathbf{ACT}_\omega) = \overline{\mathcal{H}}.$$

Proving Σ^0_1-hardness of ACT

- In order to finish the proof, introduce the following notations:
  - $\mathcal{C} = \{\langle \mathfrak{M}, x\rangle \mid \mathfrak{M} \text{ reaches } q_c \text{ on } x\}$
  - $\mathcal{H} = \{\langle \mathfrak{M}, x\rangle \mid \mathfrak{M} \text{ halts on } x\}$
  - $\mathcal{K}(\mathbf{ACT}) = \{\langle \mathfrak{M}, x\rangle \mid \mathbf{ACT} \text{ derives } \psi^+_{\mathfrak{M},x} \vdash s\}$, ditto for $\mathbf{ACT}_\omega$.
- We now know that

$$\mathcal{C} \subseteq \mathcal{K}(\mathbf{ACT}) \subseteq \mathcal{K}(\mathbf{ACT}_\omega) = \overline{\mathcal{H}}.$$

- $\mathcal{C}$ and $\mathcal{H}$ are **recursively inseparable,** thus **ACT** is undecidable.

Proving Σ^0_1-hardness of ACT

- In order to finish the proof, introduce the following notations:
  - $\mathcal{C} = \{\langle \mathfrak{M}, x \rangle \mid \mathfrak{M}$ reaches q_c on $x\}$
  - $\mathcal{H} = \{\langle \mathfrak{M}, x \rangle \mid \mathfrak{M}$ halts on $x\}$
  - $\mathcal{K}(\mathbf{ACT}) = \{\langle \mathfrak{M}, x \rangle \mid \mathbf{ACT}$ derives $\psi^+_{\mathfrak{M},x} \vdash s\}$, ditto for $\mathbf{ACT}_\omega$.
- We now know that

$$\mathcal{C} \subseteq \mathcal{K}(\mathbf{ACT}) \subseteq \mathcal{K}(\mathbf{ACT}_\omega) = \overline{\mathcal{H}}.$$

- $\mathcal{C}$ and $\mathcal{H}$ are **recursively inseparable,** thus **ACT** is undecidable.
- $\mathcal{C}$ and $\mathcal{H}$ are **effectively inseparable,** thus **ACT** is Σ^0_1-complete.

Σ^0_1-hardness with only one of $\vee$ and $\wedge$

- In order to leave only one of $\vee$ and $\wedge$, we use the **pseudo-double-negation** trick.

Σ^0_1-hardness with only one of $\vee$ and $\wedge$

- In order to leave only one of $\vee$ and $\wedge$, we use the **pseudo-double-negation** trick.
- For a variable b, let $\alpha^b = b \,/\, \alpha$.

Σ_1^0-hardness with only one of $\vee$ and $\wedge$

- In order to leave only one of $\vee$ and $\wedge$, we use the **pseudo-double-negation** trick.
- For a variable b, let $\alpha^b = b / \alpha$.
- In **MALC**, $\alpha_1, \ldots, \alpha_n \vdash \beta$ is equiderivable with $\alpha_1^{bb}, \ldots, \alpha_n^{bb} \vdash \beta^{bb}$.

Σ^0_1-hardness with only one of $\vee$ and $\wedge$

- In order to leave only one of $\vee$ and $\wedge$, we use the **pseudo-double-negation** trick.
- For a variable b, let $\alpha^b = b \,/\, \alpha$.
- In **MALC**, $\alpha_1, \ldots, \alpha_n \vdash \beta$ is equiderivable with $\alpha_1^{bb}, \ldots, \alpha_n^{bb} \vdash \beta^{bb}$.
- In order to get rid of $\wedge$, we replace φ_a with φ_a^{bb}, and use the fact that $(\gamma \vee \delta)^b \equiv \gamma^b \wedge \delta^b$. The sequent used now is $\tilde{\psi}^+ \vdash S^{bb}$.

Σ_1^0-hardness with only one of $\vee$ and $\wedge$

- In order to leave only one of $\vee$ and $\wedge$, we use the **pseudo-double-negation** trick.
- For a variable b, let $\alpha^b = b \,/\, \alpha$.
- In **MALC**, $\alpha_1, \ldots, \alpha_n \vdash \beta$ is equiderivable with $\alpha_1^{bb}, \ldots, \alpha_n^{bb} \vdash \beta^{bb}$.
- In order to get rid of $\wedge$, we replace φ_a with φ_a^{bb}, and use the fact that $(\gamma \vee \delta)^b \equiv \gamma^b \wedge \delta^b$. The sequent used now is $\tilde{\psi}^+ \vdash S^{bb}$.
- For the system without $\vee$, we take the original φ_a's, but replace ψ with ψ^{bb} (again, $\vee$ turns to $\wedge$).

Σ_1^0-hardness with only one of $\vee$ and $\wedge$

- In order to leave only one of $\vee$ and $\wedge$, we use the **pseudo-double-negation** trick.
- For a variable b, let $\alpha^b = b \,/\, \alpha$.
- In **MALC**, $\alpha_1, \ldots, \alpha_n \vdash \beta$ is equiderivable with $\alpha_1^{bb}, \ldots, \alpha_n^{bb} \vdash \beta^{bb}$.
- In order to get rid of $\wedge$, we replace φ_a with φ_a^{bb}, and use the fact that $(\gamma \vee \delta)^b \equiv \gamma^b \wedge \delta^b$. The sequent used now is $\tilde{\psi}^+ \vdash S^{bb}$.
- For the system without $\vee$, we take the original φ_a's, but replace ψ with ψ^{bb} (again, $\vee$ turns to $\wedge$).
- Since we do not know conservativity, we need to reprove everything (including the long rule)!

Complexity Synopsis

- For $\mathbf{ACT}_\omega$, we have Π^0_1-completeness, starting from the language of $\backslash, /, \cdot, {}^*$.
- For **ACT**, we have Σ^0_1-completeness, even with only one of $\vee$ and $\wedge$.

Complexity Synopsis

- For $\mathbf{ACT}_\omega$, we have Π^0_1-completeness, starting from the language of $\backslash, /, \cdot, ^*$.
- For $\mathbf{ACT}$, we have Σ^0_1-completeness, even with only one of $\vee$ and $\wedge$.
- In the language of $\cdot, \vee, ^*$ (Kleene algebras), the two systems coincide and the fragment is PSPACE-complete [D. Kozen 1994].

Complexity Synopsis

- For **ACT**$_\omega$, we have Π^0_1-completeness, starting from the language of $\backslash, /, \cdot, {}^*$.
- For **ACT**, we have Σ^0_1-completeness, even with only one of $\vee$ and $\wedge$.
- In the language of $\cdot, \vee, {}^*$ (Kleene algebras), the two systems coincide and the fragment is PSPACE-complete [D. Kozen 1994].
- **MALC** is PSPACE-complete [M. Kanovich 1994], even in the minimal fragments $(\backslash, \vee)$ and $(\backslash, \wedge)$ [M. Kanovich, S. K., A. Scedrov 2019].

Complexity Synopsis

- For $\mathbf{ACT}_\omega$, we have Π^0_1-completeness, starting from the language of $\backslash, /, \cdot, ^*$.
- For **ACT**, we have Σ^0_1-completeness, even with only one of $\vee$ and $\wedge$.
- In the language of $\cdot, \vee, ^*$ (Kleene algebras), the two systems coincide and the fragment is PSPACE-complete [D. Kozen 1994].
- **MALC** is PSPACE-complete [M. Kanovich 1994], even in the minimal fragments $(\backslash, \vee)$ and $(\backslash, \wedge)$ [M. Kanovich, S. K., A. Scedrov 2019].
- The Lambek calculus is NP-complete [M. Pentus 1996], even in fragments $(\backslash, /)$ and $(\backslash, \cdot)$ [Yu. Savateev 2008].

Complexity Synopsis

- For **ACT**$_\omega$, we have Π^0_1-completeness, starting from the language of $\backslash, /, \cdot, {}^*$.
- For **ACT**, we have Σ^0_1-completeness, even with only one of $\vee$ and $\wedge$.
- In the language of $\cdot, \vee, {}^*$ (Kleene algebras), the two systems coincide and the fragment is PSPACE-complete [D. Kozen 1994].
- **MALC** is PSPACE-complete [M. Kanovich 1994], even in the minimal fragments $(\backslash, \vee)$ and $(\backslash, \wedge)$ [M. Kanovich, S. K., A. Scedrov 2019].
- The Lambek calculus is NP-complete [M. Pentus 1996], even in fragments $(\backslash, /)$ and $(\backslash, \cdot)$ [Yu. Savateev 2008].
- The Lambek calculus with only one division is polytime decidable [Yu. Savateev 2007].

Open Questions

- Complexity:
  - Kleene lattices $(\cdot, \vee, \wedge, ^*)$.
  - **ACT** with only $\backslash, /, \cdot, ^*$ (Lambek calculus with iteration).

Open Questions

- Complexity:
  - Kleene lattices $(\cdot, \vee, \wedge, {}^*)$.
  - **ACT** with only $\backslash, /, \cdot, {}^*$ (Lambek calculus with iteration).
- L-/R-completeness: Lambek calculus with iteration.

Open Questions

- Complexity:
  - Kleene lattices $(\cdot, \vee, \wedge, {}^*)$.
  - **ACT** with only $\backslash, /, \cdot, {}^*$ (Lambek calculus with iteration).
- L-/R-completeness: Lambek calculus with iteration.
- Cut-free sequent calculus for **ACT** (maybe some circular approach?).

Thanks*