Constructive cut elimination in geometric logic

Sara Negri [1,2] & Eugenio Orlandelli [2,3]

[1] Univ. Genova [2] Univ. Helsinki [3] Univ. Bologna

LAP 2020 – 22 September 2020

Outline

Structural proof theory

- Proof-theoretical semantics (PTS) builds on the goals of general proof theory: shift from the so called reductionist study of mathematics (Hilbert's program) to the analysis of proofs in their own right (Gentzen, Prawitz).
- Basic requirements to achieve these goals include
  1. A precise definition of formal systems of derivation
  2. Establishing structural properties, subformula property
  3. Establishing the meaning-conferring nature of the rules of deduction
- Achieved already by Gentzen for sequent calculi for *purely logical systems* (1933) and for *arithmetic* (1935), and by Prawitz (1973) for natural deduction [also, Gentzen 2008].
- Considered an impossibility for *extra-logical* axioms (Girard 1987).

Structural proof theory

- Proof-theoretical semantics (PTS) builds on the goals of general proof theory: shift from the so called reductionist study of mathematics (Hilbert's program) to the analysis of proofs in their own right (Gentzen, Prawitz).
- Basic requirements to achieve these goals include
  1. A precise definition of formal systems of derivation
  2. Establishing structural properties, subformula property
  3. Establishing the meaning-conferring nature of the rules of deduction
- Achieved already by Gentzen for sequent calculi for *purely logical systems* (1933) and for *arithmetic* (1935), and by Prawitz (1973) for natural deduction [also, Gentzen 2008].
- Considered an impossibility for *extra-logical* axioms (Girard 1987).

Structural proof theory

- Proof-theoretical semantics (PTS) builds on the goals of general proof theory: shift from the so called reductionist study of mathematics (Hilbert's program) to the analysis of proofs in their own right (Gentzen, Prawitz).
- Basic requirements to achieve these goals include
  1. A precise definition of formal systems of derivation
  2. Establishing structural properties, subformula property
  3. Establishing the meaning-conferring nature of the rules of deduction
- Achieved already by Gentzen for sequent calculi for *purely logical systems* (1933) and for *arithmetic* (1935), and by Prawitz (1973) for natural deduction [also, Gentzen 2008].
- Considered an impossibility for *extra-logical* axioms (Girard 1987).

Structural proof theory

- Proof-theoretical semantics (PTS) builds on the goals of general proof theory: shift from the so called reductionist study of mathematics (Hilbert's program) to the analysis of proofs in their own right (Gentzen, Prawitz).
- Basic requirements to achieve these goals include
  1. A precise definition of formal systems of derivation
  2. Establishing structural properties, subformula property
  3. Establishing the meaning-conferring nature of the rules of deduction
- Achieved already by Gentzen for sequent calculi for *purely logical systems* (1933) and for *arithmetic* (1935), and by Prawitz (1973) for natural deduction [also, Gentzen 2008].
- Considered an impossibility for *extra-logical* axioms (Girard 1987).

Proof analysis

- The program of *proof analysis* extends the goals of PTS to elementary mathematical theories such as
  1. Theories with universal axioms (N and von Plato 1998)
  2. Coherent theories (N 2003, Simpson 1994 in ND-style)
  3. Arbitrary first-order theories (Dyckhoff and N 2015)
- Geometric theories based on infinitary logic (N 202X)

Proof analysis

- The program of *proof analysis* extends the goals of PTS to elementary mathematical theories such as
  1. Theories with universal axioms (N and von Plato 1998)
  2. Coherent theories (N 2003, Simpson 1994 in ND-style)
  3. Arbitrary first-order theories (Dyckhoff and N 2015)
- Geometric theories based on infinitary logic (N 202X)

Proof analysis

- The program of *proof analysis* extends the goals of PTS to elementary mathematical theories such as
  1. Theories with universal axioms (N and von Plato 1998)
  2. Coherent theories (N 2003, Simpson 1994 in ND-style)
  3. Arbitrary first-order theories (Dyckhoff and N 2015)
- Geometric theories based on infinitary logic (N 202X)

Proof analysis

- The program of *proof analysis* extends the goals of PTS to elementary mathematical theories such as
  1. Theories with universal axioms (N and von Plato 1998)
  2. Coherent theories (N 2003, Simpson 1994 in ND-style)
  3. Arbitrary first-order theories (Dyckhoff and N 2015)
- Geometric theories based on infinitary logic (N 202X)

Infinitary logic and cut elimination

- (N 202X) introduces **G3**-style calculi for geometric theories based on (classical and intuitionistic) infinitary logic
- it shows that the structural rules of inference are admissible
- BUT the proof of cut-elimination is not constructive
- This is an instance of a common problem of cut-elimination procedures for infinitary logics:[1]

> *the proof uses the 'natural' (or Hessenberg) commutative sum of ordinals $\alpha\#\beta$, [whose] definition uses the Cantor normal form of ordinals to base ω. This normal form is not available in* **CZF** *(or* **IZF***) and thus a different approach is called for.*
> *(Rathjen 2016: 369)*

- We give a simple constructive proof of cut elimination fot geometric logic

[1] E.g., in Feferman (1968), Tait (1968), Takeuti (1975), Lopez-Escobar (1965)

Infinitary logic and cut elimination

- (N 202X) introduces **G3**-style calculi for geometric theories based on (classical and intuitionistic) infinitary logic
- it shows that the structural rules of inference are admissible
- BUT the proof of cut-elimination is not constructive
- This is an instance of a common problem of cut-elimination procedures for infinitary logics:[1]

> *the proof uses the 'natural' (or Hessenberg) commutative sum of ordinals $\alpha\#\beta$, [whose] definition uses the Cantor normal form of ordinals to base ω. This normal form is not available in* **CZF** *(or* **IZF***) and thus a different approach is called for.*
> *(Rathjen 2016: 369)*

- We give a simple constructive proof of cut elimination fot geometric logic

[1] E.g., in Feferman (1968), Tait (1968), Takeuti (1975), Lopez-Escobar (1965)

Outline

Coherent implications

A formula is Horn iff built from atoms (and $\top$) using only $\wedge$.

A formula is coherent, *or "positive"*, iff built from atoms (and $\top, \bot$) using only $\vee$, $\wedge$ and $\exists$.

A sentence is a coherent implication iff of the form

$$\forall \mathbf{x}. C \supset D$$

where C, D are coherent [$\forall \mathbf{x}. D$ is a coherent implication, with $\top \equiv C$]

Theorem (Normal form)

Any coherent implication is equivalent to a finite conjunction of sentences of the form

$$\forall \mathbf{x}. C \supset D$$

where C is Horn and D is a (finite) disjunction of existentially quantified Horn formulae.

Coherent implications

A formula is Horn iff built from atoms (and $\top$) using only $\wedge$.

A formula is coherent, *or "positive"*, iff built from atoms (and $\top, \bot$) using only $\vee$, $\wedge$ and $\exists$.

A sentence is a coherent implication iff of the form

$$\forall \mathbf{x}. C \supset D$$

where C, D are coherent [$\forall \mathbf{x}. D$ is a coherent implication, with $\top \equiv C$]

Theorem (Normal form)

Any coherent implication is equivalent to a finite conjunction of sentences of the form

$$\forall \mathbf{x}. C \supset D$$

where C is Horn and D is a (finite) disjunction of existentially quantified Horn formulae.

Coherent implications

A formula is Horn iff built from atoms (and $\top$) using only $\wedge$.

A formula is coherent, *or "positive"*, iff built from atoms (and $\top, \bot$) using only $\vee$, $\wedge$ and $\exists$.

A sentence is a coherent implication iff of the form

$$\forall \mathbf{x}. C \supset D$$

where C, D are coherent [$\forall \mathbf{x}. D$ is a coherent implication, with $\top \equiv C$]

Theorem (Normal form)

Any coherent implication is equivalent to a finite conjunction of sentences of the form

$$\forall \mathbf{x}. C \supset D$$

where C is Horn and D is a (finite) disjunction of existentially quantified Horn formulae.

Coherent implications

A formula is Horn iff built from atoms (and $\top$) using only $\wedge$.

A formula is coherent, *or "positive"*, iff built from atoms (and $\top, \bot$) using only $\vee$, $\wedge$ and $\exists$.

A sentence is a coherent implication iff of the form

$$\forall \mathbf{x}. C \supset D$$

where C, D are coherent [$\forall \mathbf{x}. D$ is a coherent implication, with $\top \equiv C$]

Theorem (Normal form)

Any coherent implication is equivalent to a finite conjunction of sentences of the form

$$\forall \mathbf{x}. C \supset D$$

where C is Horn and D is a (finite) disjunction of existentially quantified Horn formulae.

Examples of coherent theories

Theory of *fields* : $\forall x(x = 0 \vee \exists y(xy = 1))$.

Theory of *local rings* : $\forall x.\exists y(xy = 1) \vee \exists y((1 - x)y = 1)$

Theory of *transitive relations* : $\forall xyz.(xRy \wedge yRz) \supset xRz$

Theory of *partial order* : $\forall xy.\ (x \leq y \wedge y \leq x) \supset x = y$

Theory of *strongly directed relations* : $\forall xyz.(xRy \wedge xRz) \supset \exists u.yRu \wedge zRu$

Examples of coherent theories

Theory of *fields* : $\forall x(x = 0 \vee \exists y(xy = 1))$.

Theory of *local rings* : $\forall x.\exists y(xy = 1) \vee \exists y((1 - x)y = 1)$

Theory of *transitive relations* : $\forall xyz.(xRy \wedge yRz) \supset xRz$

Theory of *partial order* : $\forall xy.\ (x \leq y \wedge y \leq x) \supset x = y$

Theory of *strongly directed relations* : $\forall xyz.(xRy \wedge xRz) \supset \exists u.yRu \wedge zRu$

Examples of coherent theories

Theory of *fields* : $\forall x(x = 0 \vee \exists y(xy = 1))$.

Theory of *local rings* : $\forall x. \exists y(xy = 1) \vee \exists y((1 - x)y = 1)$

Theory of *transitive relations* : $\forall xyz.(xRy \wedge yRz) \supset xRz$

Theory of *partial order* : $\forall xy.\ (x \leq y \wedge y \leq x) \supset x = y$

Theory of *strongly directed relations* : $\forall xyz.(xRy \wedge xRz) \supset \exists u.yRu \wedge zRu$

Examples of coherent theories

Theory of *fields* : $\forall x(x = 0 \vee \exists y(xy = 1))$.

Theory of *local rings* : $\forall x.\exists y(xy = 1) \vee \exists y((1 - x)y = 1)$

Theory of *transitive relations* : $\forall xyz.(xRy \wedge yRz) \supset xRz$

Theory of *partial order* : $\forall xy.\ (x \leq y \wedge y \leq x) \supset x = y$

Theory of *strongly directed relations* : $\forall xyz.(xRy \wedge xRz) \supset \exists u.yRu \wedge zRu$

Examples of coherent theories

Theory of *fields* : $\forall x(x = 0 \vee \exists y(xy = 1))$.

Theory of *local rings* : $\forall x.\exists y(xy = 1) \vee \exists y((1 - x)y = 1)$

Theory of *transitive relations* : $\forall xyz.(xRy \wedge yRz) \supset xRz$

Theory of *partial order* : $\forall xy.\ (x \leq y \wedge y \leq x) \supset x = y$

Theory of *strongly directed relations* : $\forall xyz.(xRy \wedge xRz) \supset \exists u.yRu \wedge zRu$

Geometric implications

A formula is Horn iff built from atoms (and $\top$) using only $\land$.

A formula is geometric iff built from atoms (and $\top, \bot$) using only $\lor$, $\land$, $\exists$ and infinitary disjunctions.

A sentence is a geometric implication iff of the form

$$\forall \mathbf{x}. C \supset D$$

where C, D are geometric.

Theorem (Normal form)

Any geometric implication is equivalent to a (possibly infinite) conjunction of sentences of the form

$$\forall \mathbf{x}. C \supset D$$

where C is Horn and D is a (possibly infinite) disjunction of existentially quantified Horn formulae.

Geometric implications

A formula is Horn iff built from atoms (and $\top$) using only $\wedge$.

A formula is geometric iff built from atoms (and $\top, \bot$) using only $\vee$, $\wedge$, $\exists$ and infinitary disjunctions.

A sentence is a geometric implication iff of the form

$$\forall \mathbf{x}. C \supset D$$

where C, D are geometric.

Theorem (Normal form)

Any geometric implication is equivalent to a (possibly infinite) conjunction of sentences of the form

$$\forall \mathbf{x}. C \supset D$$

where C is Horn and D is a (possibly infinite) disjunction of existentially quantified Horn formulae.

Geometric implications

A formula is Horn iff built from atoms (and $\top$) using only $\wedge$.

A formula is geometric iff built from atoms (and $\top, \bot$) using only $\vee$, $\wedge$, $\exists$ and infinitary disjunctions.

A sentence is a geometric implication iff of the form

$$\forall \mathbf{x}. C \supset D$$

where C, D are geometric.

Theorem (Normal form)

Any geometric implication is equivalent to a (possibly infinite) conjunction of sentences of the form

$$\forall \mathbf{x}. C \supset D$$

where C is Horn and D is a (possibly infinite) disjunction of existentially quantified Horn formulae.

Geometric implications

A formula is Horn iff built from atoms (and $\top$) using only $\wedge$.

A formula is geometric iff built from atoms (and $\top, \bot$) using only $\vee$, $\wedge$, $\exists$ and infinitary disjunctions.

A sentence is a geometric implication iff of the form

$$\forall \mathbf{x}. C \supset D$$

where C, D are geometric.

Theorem (Normal form)

Any geometric implication is equivalent to a (possibly infinite) conjunction of sentences of the form

$$\forall \mathbf{x}. C \supset D$$

where C is Horn and D is a (possibly infinite) disjunction of existentially quantified Horn formulae.

Examples of geometric theories

(Infinitary) theory of *torsion abelian groups* : $\forall x. \bigvee_{n>1}(nx = 0)$

Theory of *Archimedean ordered fields* : $\forall x. \bigvee_{n\geq 1}(x < n)$

Theory of *connected graphs* :

$\forall xy.x = y \vee \bigvee_{n\geq 1} \exists z_0 \ldots \exists z_n(x = z_0 \,\&\, y = z_n \,\&\, z_0 R z_1 \,\&\, \ldots \,\&\, z_{n-1} R z_n)$

Examples of geometric theories

(Infinitary) theory of *torsion abelian groups* : $\forall x. \bigvee_{n>1}(nx = 0)$

Theory of *Archimedean ordered fields* : $\forall x. \bigvee_{n\geq 1}(x < n)$

Theory of *connected graphs* :

$\forall xy. x = y \vee \bigvee_{n\geq 1} \exists z_0 \ldots \exists z_n (x = z_0 \,\&\, y = z_n \,\&\, z_0 R z_1 \,\&\, \ldots \,\&\, z_{n-1} R z_n)$

Examples of geometric theories

(Infinitary) theory of *torsion abelian groups* : $\forall x. \bigvee_{n>1} (nx = 0)$

Theory of *Archimedean ordered fields* : $\forall x. \bigvee_{n \geq 1} (x < n)$

Theory of *connected graphs* :

$\forall xy. x = y \vee \bigvee_{n \geq 1} \exists z_0 \ldots \exists z_n (x = z_0 \,\&\, y = z_n \,\&\, z_0 R z_1 \,\&\, \ldots \,\&\, z_{n-1} R z_n)$

Outline

Infinitary logics: syntax

- Countably many predicates and function symbols, and identity
- *Formulas* A are built up using, the standard connectives and quantifiers, and countable disjunctions $\bigvee_{n>0} A_n$ and countable conjunctions $\bigwedge_{n>0} A_n$

The *weight* $w(A)$ of a formula is defined inductively on the formation of A:

- $w(\bot) = w(P) \equiv 1$ for P atomic
- For compound formulas A,

$$w(A) \equiv sup_{B \in IS(A)} w(B) + 1$$

where $B \in IS(A)$ iff B is an *immediate subformula* of A.

If B is a proper subformula of A, then $w(B) < w(A)$.

Infinitary logics: syntax

- Countably many predicates and function symbols, and identity
- *Formulas* A are built up using, the standard connectives and quantifiers, and countable disjunctions $\bigvee_{n>0} A_n$ and countable conjunctions $\bigwedge_{n>0} A_n$

The *weight* $w(A)$ of a formula is defined inductively on the formation of A:

- $w(\bot) = w(P) \equiv 1$ for P atomic
- For compound formulas A,

$$w(A) \equiv sup_{B \in IS(A)} w(B) + 1$$

where $B \in IS(A)$ iff B is an *immediate subformula* of A.

If B is a proper subformula of A, then $w(B) < w(A)$.

Infinitary logics as contraction-free sequent calculi

Sequents are expressions of the form $\Gamma \Rightarrow \Delta$ where Γ, Δ are *finite multisets* of formulas.

Infinitary rules for disjunction:

$$\frac{\{\Gamma, A_n \Rightarrow \Delta \mid n > 0\}}{\Gamma, \bigvee_{n>0} A_n \Rightarrow \Delta}\ L\bigvee \qquad \frac{\Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n, A_k}{\Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n}\ R\bigvee_k.$$

- $L\bigvee$ has countably many premisses, one for each $n > 0$.
- Derivations built using these rules are, in general, infinite trees, with countable branching but where each branch has finite length.

Infinitary logics as contraction-free sequent calculi

Sequents are expressions of the form $\Gamma \Rightarrow \Delta$ where Γ, Δ are *finite multisets* of formulas.

Infinitary rules for disjunction:

$$\frac{\{\Gamma, A_n \Rightarrow \Delta \mid n > 0\}}{\Gamma, \bigvee_{n>0} A_n \Rightarrow \Delta}\ L\bigvee \qquad \frac{\Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n, A_k}{\Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n}\ R\bigvee_k.$$

- $L\bigvee$ has countably many premisses, one for each $n > 0$.
- Derivations built using these rules are, in general, infinite trees, with countable branching but where each branch has finite length.

Derivations and their height in infinitary sequent calculi

Definition (Derivation and its height)

1. Any sequent $\Gamma \Rightarrow \Delta$, where some atomic formula occurs in both Γ and Δ or $\perp$ occurs in Γ), is a derivation, of *height* 0 ;
2. If each $\mathcal{D}_n$ is a derivation, of height α_n, with end-sequent $\Gamma_n \Rightarrow \Delta_n$ and

$$\frac{\ldots \quad \Gamma_n \Rightarrow \Delta_n \quad \ldots}{\Gamma \Rightarrow \Delta}\ R$$

is an inference (i.e. an instance of a rule), then

$$\frac{\ldots \quad \dfrac{\mathcal{D}_n}{\Gamma_n \Rightarrow \Delta_n} \quad \ldots}{\Gamma \Rightarrow \Delta}\ R$$

is a derivation, of *height* the countable ordinal $sup_n(\alpha_n) + 1$.

Derivations in infinitary sequent calculi (cont.)

It follows from the definition that:

- Each derivation has a countable ordinal *height* (the successor of the supremum of the heights of its immediate subderivations).
- If $\mathcal{D}'$ is a subderivation of $\mathcal{D}$, then $ht(\mathcal{D}') < ht(\mathcal{D})$.[2]

[2] The definitions of depth and height differ from those in (Feferman 1968): we use the successor of a supremum rather than the supremum of the successors: note that $sup_{n>0}(n+1) = \omega \neq \omega + 1 = (sup_{n>0}(n)) + 1$

The calculus $\mathbf{G3c}_\omega$

$$P, \Gamma \Rightarrow \Delta, P$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ L\vee$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta}\ L\supset$$

$$\frac{A(t/x), \forall x A, \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\ L\forall$$

$$\frac{A(y/x), \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta}\ L\exists\ (y\ \textit{fresh})$$

$$\frac{}{\bot, \Gamma \Rightarrow \Delta}\ L\bot$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ R\wedge$$

$$\frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ R\vee$$

$$\frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \supset B}\ R\supset$$

$$\frac{\Gamma \Rightarrow \Delta, A(y/x)}{\Gamma \Rightarrow \Delta, \forall x A}\ R\forall\ (y\ \textit{fresh})$$

$$\frac{\Gamma \Rightarrow \Delta, \exists x A, A(t/x)}{\Gamma \Rightarrow \Delta, \exists x A}\ R\exists$$

$$\frac{A_k, \bigwedge_{n>0} A_n, \Gamma \Rightarrow \Delta}{\bigwedge_{n>0} A_n, \Gamma \Rightarrow \Delta}\ L\bigwedge$$

$$\frac{\{\Gamma, A_n \Rightarrow \Delta \mid n > 0\}}{\Gamma, \bigvee_{n>0} A_n \Rightarrow \Delta}\ L\bigvee$$

$$\frac{\{\Gamma \Rightarrow \Delta, A_n \mid n > 0\}}{\Gamma \Rightarrow \Delta, \bigwedge_{n>0} A_n}\ R\bigwedge$$

$$\frac{\Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n, A_k}{\Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n}\ R\bigvee$$

The calculus $\mathbf{G3c}_\omega$

$$P, \Gamma \Rightarrow \Delta, P$$

$$\frac{}{\bot, \Gamma \Rightarrow \Delta}\ L\bot$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ R\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ L\vee$$

$$\frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta}\ L\supset$$

$$\frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \supset B}\ R\supset$$

$$\frac{A(t/x), \forall x A, \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\ L\forall$$

$$\frac{\Gamma \Rightarrow \Delta, A(y/x)}{\Gamma \Rightarrow \Delta, \forall x A}\ R\forall\ (y\ \textit{fresh})$$

$$\frac{A(y/x), \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta}\ L\exists\ (y\ \textit{fresh})$$

$$\frac{\Gamma \Rightarrow \Delta, \exists x A, A(t/x)}{\Gamma \Rightarrow \Delta, \exists x A}\ R\exists$$

$$\frac{A_k, \bigwedge_{n>0} A_n, \Gamma \Rightarrow \Delta}{\bigwedge_{n>0} A_n, \Gamma \Rightarrow \Delta}\ L\bigwedge$$

$$\frac{\{\Gamma \Rightarrow \Delta, A_n \mid n > 0\}}{\Gamma \Rightarrow \Delta, \bigwedge_{n>0} A_n}\ R\bigwedge$$

$$\frac{\{\Gamma, A_n \Rightarrow \Delta \mid n > 0\}}{\Gamma, \bigvee_{n>0} A_n \Rightarrow \Delta}\ L\bigvee$$

$$\frac{\Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n, A_k}{\Gamma \Rightarrow \Delta, \bigvee_{n>0} A_n}\ R\bigvee$$

Extensions with rules for geometric theories

Extension of **G3c** with rules for finitary coherent theories (N 2003) and infinitary geometric ones (N202X) maintains the structural properties of the ground calculus.

Recall that a *geometric implication* is a sentence G of the form

$$\forall \mathbf{x}.\, P_1 \wedge \cdots \wedge P_k \supset \bigvee E_n$$

$$\text{where } E_n \equiv \exists \mathbf{y}_n (Q_{n1} \wedge \cdots \wedge Q_{nm_n})$$

Such a sentence G determines a (finitary or infinitary) *geometric rule*:

$$\frac{\ldots \quad Q_{n1}(\mathbf{x}, \mathbf{y}_n), \ldots, Q_{nm_n}(\mathbf{x}, \mathbf{y}_n), P_1(\mathbf{x}), \ldots, P_k(\mathbf{x}), \Gamma \Rightarrow \Delta \quad \ldots}{P_1(\mathbf{x}), \ldots, P_k(\mathbf{x}), \Gamma \Rightarrow \Delta}\; R_G$$

with one premiss for each of the countably many disjuncts E_n of D. The variables in $\mathbf{y}_n$ are *fresh*, i.e. are not in the conclusion.

Extensions with rules for geometric theories

Extension of **G3c** with rules for finitary coherent theories (N 2003) and infinitary geometric ones (N202X) maintains the structural properties of the ground calculus.

Recall that a *geometric implication* is a sentence G of the form

$$\forall \mathbf{x}.\, P_1 \wedge \cdots \wedge P_k \supset \bigvee E_n$$

$$\text{where } E_n \equiv \exists \mathbf{y}_n (Q_{n1} \wedge \cdots \wedge Q_{nm_n})$$

Such a sentence G determines a (finitary or infinitary) *geometric rule*:

$$\frac{\ldots \quad Q_{n1}(\mathbf{x},\mathbf{y}_n),\ldots,Q_{nm_n}(\mathbf{x},\mathbf{y}_n),P_1(\mathbf{x}),\ldots,P_k(\mathbf{x}),\Gamma \Rightarrow \Delta \quad \ldots}{P_1(\mathbf{x}),\ldots,P_k(\mathbf{x}),\Gamma \Rightarrow \Delta}\; R_G$$

with one premiss for each of the countably many disjuncts E_n of D. The variables in $\mathbf{y}_n$ are *fresh*, i.e. are not in the conclusion.

Extensions with rules for geometric theories

Extension of **G3c** with rules for finitary coherent theories (N 2003) and infinitary geometric ones (N202X) maintains the structural properties of the ground calculus.

Recall that a *geometric implication* is a sentence G of the form

$$\forall \mathbf{x}.\, P_1 \wedge \cdots \wedge P_k \supset \bigvee E_n$$

$$\text{where } E_n \equiv \exists \mathbf{y}_n (Q_{n1} \wedge \cdots \wedge Q_{nm_n})$$

Such a sentence G determines a (finitary or infinitary) *geometric rule*:

$$\frac{\ldots \quad Q_{n1}(\mathbf{x}, \mathbf{y}_n), \ldots, Q_{nm_n}(\mathbf{x}, \mathbf{y}_n), P_1(\mathbf{x}), \ldots, P_k(\mathbf{x}), \Gamma \Rightarrow \Delta \quad \ldots}{P_1(\mathbf{x}), \ldots, P_k(\mathbf{x}), \Gamma \Rightarrow \Delta}\, R_G$$

with one premiss for each of the countably many disjuncts E_n of D. The variables in $\mathbf{y}_n$ are *fresh*, i.e. are not in the conclusion.

Closure condition

To ensure the hp-admissibility of contraction geometric rules must respect the closure condition

Definition

- If the calculus contain a geometric rule with repetition of some principal formula:

$$\frac{\dots \quad \vec{Q}, P_1, \dots, P, P, \dots, P_k, \Gamma \Rightarrow \Delta \quad \dots}{P_1, \dots, P, P, \dots, P_k, \Gamma \Rightarrow \Delta}$$

- Then it contains the corresponding contracted instance:

$$\frac{\dots \quad \vec{Q}, P_1, \dots, P, \dots, P_k, \Gamma \Rightarrow \Delta \quad \dots}{P_1, \dots, P, \dots, P_k, \Gamma \Rightarrow \Delta}$$

Coherent rules for identity

- The rules introduced in (N & von Plato 1998)

$$\frac{s = s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Ref}$$

$$\frac{P(t/x), s = t, P(s/x), \Gamma \Rightarrow \Delta}{s = t, P(s/x), \Gamma \Rightarrow \Delta}\ \mathit{Repl}$$

- to derive the following theorem of FOL:

$$x = f(x) \supset x = f(f(x))$$

- we add contracted instance of rule *Repl*:

$$\frac{t = f(\dots, f^n(\dots, t, \dots), \dots), t = f(\dots, t, \dots), \Gamma \Rightarrow \Delta}{t = f(\dots, t, \dots), \Gamma \Rightarrow \Delta}\ \mathit{Repl}^c$$

Coherent rules for identity

- The rules introduced in (N & von Plato 1998)

$$\frac{s = s, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Ref}$$

$$\frac{P(t/x), s = t, P(s/x), \Gamma \Rightarrow \Delta}{s = t, P(s/x), \Gamma \Rightarrow \Delta}\ \mathit{Repl}$$

- to derive the following theorem of FOL:

$$x = f(x) \supset x = f(f(x))$$

- we add contracted instance of rule *Repl*:

$$\frac{t = f(\dots, f^n(\dots, t, \dots), \dots), t = f(\dots, t, \dots), \Gamma \Rightarrow \Delta}{t = f(\dots, t, \dots), \Gamma \Rightarrow \Delta}\ \mathit{Repl}^c$$

Examples of geometric rules

Axiom of **torsion abelian groups**, $\forall x. \bigvee_{n>1}(nx = 0)$, becomes the rule

$$\frac{\ldots \quad nx = 0, \Gamma \Rightarrow \Delta \quad \ldots}{\Gamma \Rightarrow \Delta}\ R_{Tor}$$

Axiom of **Archimedean ordered fields**, $\forall x. \bigvee_{n\geq 1}(x < n)$, becomes the rule

$$\frac{\ldots \quad x < n, \Gamma \Rightarrow \Delta \quad \ldots}{\Gamma \Rightarrow \Delta}\ R_{Arc}$$

Axiom of **connected graphs**,

$$\forall xy. x = y \vee \bigvee_{n\geq 1} \exists z_0 \ldots \exists z_n (x = z_0 \,\&\, y = z_n \,\&\, z_0 R z_1 \,\&\, \ldots \,\&\, z_{n-1} R z_n)$$

becomes the rule

$$\frac{x = y, \Gamma \Rightarrow \Delta \quad xRy, \Gamma \Rightarrow \Delta \quad \ldots \quad x = z_0, y = z_n, z_0 R z_1, \ldots, z_{n-1} R z_n, \Gamma \Rightarrow \Delta \quad \ldots}{\Gamma \Rightarrow \Delta}$$

Examples of geometric rules

Axiom of **torsion abelian groups**, $\forall x. \bigvee_{n>1}(nx = 0)$, becomes the rule

$$\frac{\ldots \quad nx = 0, \Gamma \Rightarrow \Delta \quad \ldots}{\Gamma \Rightarrow \Delta}\ R_{Tor}$$

Axiom of **Archimedean ordered fields**, $\forall x. \bigvee_{n\geq 1}(x < n)$, becomes the rule

$$\frac{\ldots \quad x < n, \Gamma \Rightarrow \Delta \quad \ldots}{\Gamma \Rightarrow \Delta}\ R_{Arc}$$

Axiom of **connected graphs**,

$$\forall xy. x = y \vee \bigvee_{n\geq 1} \exists z_0 \ldots \exists z_n (x = z_0 \,\&\, y = z_n \,\&\, z_0 R z_1 \,\&\, \ldots \,\&\, z_{n-1} R z_n)$$

becomes the rule

$$\frac{x = y, \Gamma \Rightarrow \Delta \quad xRy, \Gamma \Rightarrow \Delta \quad \ldots \quad x = z_0, y = z_n, z_0 R z_1, \ldots, z_{n-1} R z_n, \Gamma \Rightarrow \Delta \quad \ldots}{\Gamma \Rightarrow \Delta}$$

Examples of geometric rules

Axiom of **torsion abelian groups**, $\forall x. \bigvee_{n>1}(nx = 0)$, becomes the rule

$$\frac{\dots \quad nx = 0, \Gamma \Rightarrow \Delta \quad \dots}{\Gamma \Rightarrow \Delta}\ R_{Tor}$$

Axiom of **Archimedean ordered fields**, $\forall x. \bigvee_{n\geq 1}(x < n)$, becomes the rule

$$\frac{\dots \quad x < n, \Gamma \Rightarrow \Delta \quad \dots}{\Gamma \Rightarrow \Delta}\ R_{Arc}$$

Axiom of **connected graphs**,

$$\forall xy.x = y \vee \bigvee_{n\geq 1} \exists z_0 \dots \exists z_n(x = z_0 \,\&\, y = z_n \,\&\, z_0Rz_1 \,\&\, \dots \,\&\, z_{n-1}Rz_n)$$

becomes the rule

$$\frac{x = y, \Gamma \Rightarrow \Delta \quad xRy, \Gamma \Rightarrow \Delta \quad \dots \quad x = z_0, y = z_n, z_0Rz_1, \dots, z_{n-1}Rz_n, \Gamma \Rightarrow \Delta \quad \dots}{\Gamma \Rightarrow \Delta}$$

Structural properties (N 202X)

Lemma (α-conv)

If $\mathcal{S}$ is n-derivable then each bound alphabetic variant of $\mathcal{S}$ is n-derivable.

Lemma (hp-substitution)

If $\vdash^{\alpha} \Gamma \Rightarrow \Delta$ then $\vdash^{\alpha} \Gamma(t/x) \Rightarrow \Delta(t/x)$ (for t free for x in Γ, Δ).

Theorem (hp-weakening)

The left and right rules of weakening are hp-admissible.

Lemma (hp-invertibility)

Each rule is hp-invertible.

Theorem (hp-contraction)

The left and right rules of contraction are hp-admissible.

Structural properties (N 202X)

Lemma (α-conv)

If $\mathcal{S}$ is n-derivable then each bound alphabetic variant of $\mathcal{S}$ is n-derivable.

Lemma (hp-substitution)

If $\vdash^\alpha \Gamma \Rightarrow \Delta$ then $\vdash^\alpha \Gamma(t/x) \Rightarrow \Delta(t/x)$ (for t free for x in Γ, Δ).

Theorem (hp-weakening)

The left and right rules of weakening are hp-admissible.

Lemma (hp-invertibility)

Each rule is hp-invertible.

Theorem (hp-contraction)

The left and right rules of contraction are hp-admissible.

Admissibility of cut for $\mathbf{G3c}^*_\omega$

Admissibility of

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ Cut$$

(N 202X) and Here: finite *multisets* and extension with rules for geometric implications.[3]

Inductive parameters

Rank $\pi(I)$ of an instance I of *Cut* with cut-free premisses $\mathcal{D}$ and $\mathcal{D}'$ is the (*lexicographically* ordered) pair (δ, σ) where

- $\delta \equiv w(A) \equiv$ weight of A
- $\sigma \equiv h(\mathcal{D}) \# h(\mathcal{D}') \equiv$ natural sum of the heights of the premisses

Here $\#$ is the standard notion of (natural or Hessenberg) commutative sum $\alpha \# \beta$ for ordinals α and β

[3] for the infinitary calculus proved using finite *sets* is shown by a Gentzen-style argument in Feferman (1968) and by Tait (1968) using single-sided sequents. Takeuti (1975) uses infinitary sequents. Lopez-Escobar (1965) infinitary sequents as sets.

Admissibility of cut for $\mathbf{G3c}^*_\omega$

Admissibility of

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ Cut$$

(N 202X) and Here: finite *multisets* and extension with rules for geometric implications.[3]

Inductive parameters

Rank $\pi(I)$ of an instance I of *Cut* with cut-free premisses $\mathcal{D}$ and $\mathcal{D}'$ is the (*lexicographically* ordered) pair (δ, σ) where

- $\delta \equiv w(A) \equiv$ weight of A
- $\sigma \equiv h(\mathcal{D}) \# h(\mathcal{D}') \equiv$ natural sum of the heights of the premisses

Here $\#$ is the standard notion of (natural or Hessenberg) commutative sum $\alpha \# \beta$ for ordinals α and β

[3] for the infinitary calculus proved using finite *sets* is shown by a Gentzen-style argument in Feferman (1968) and by Tait (1968) using single-sided sequents. Takeuti (1975) uses infinitary sequents. Lopez-Escobar (1965) infinitary sequents as sets.

Outline

A proof strategy avoiding cut-height

We use a cut-elimination strategy that is often used for hypersequents and substructural logics (Metcalfe, Olivetti, Gabbay 2008):

Definition (Cut rank)

Let the cut rank of a derivation $\mathcal{D}$ – $\mathtt{cr}(\mathcal{D})$ – be the maximal weight of cut formulas in $\mathcal{D}$.

- The proof is by lexicographical induction on the pair

$$(\mathtt{cr}(\mathcal{D}), n(\mathtt{cr}(\mathcal{D}))$$

  the latter being the number of cuts of maximal rank occurring in $\mathcal{D}$.
- It is based on three lemmas:
  1. Cut-substitutivity takes care of non-principal cuts;
  2. Right reduction takes care of cases with cut formula principal in the left premiss
  3. Left reduction covers all other cases.

A proof strategy avoiding cut-height

We use a cut-elimination strategy that is often used for hypersequents and substructural logics (Metcalfe, Olivetti, Gabbay 2008):

Definition (Cut rank)

Let the cut rank of a derivation $\mathcal{D}$ – $\mathtt{cr}(\mathcal{D})$ – be the maximal weight of cut formulas in $\mathcal{D}$.

- The proof is by lexicographical induction on the pair

$$(\mathtt{cr}(\mathcal{D}), n(\mathtt{cr}(\mathcal{D}))$$

  the latter being the number of cuts of maximal rank occurring in $\mathcal{D}$.
- It is based on three lemmas:
  1. Cut-substitutivity takes care of non-principal cuts;
  2. Right reduction takes care of cases with cut formula principal in the left premiss
  3. Left reduction covers all other cases.

A proof strategy avoiding cut-height

We use a cut-elimination strategy that is often used for hypersequents and substructural logics (Metcalfe, Olivetti, Gabbay 2008):

Definition (Cut rank)

Let the cut rank of a derivation $\mathcal{D}$ – $\mathtt{cr}(\mathcal{D})$ – be the maximal weight of cut formulas in $\mathcal{D}$.

- The proof is by lexicographical induction on the pair

$$(\mathtt{cr}(\mathcal{D}), n(\mathtt{cr}(\mathcal{D}))$$

  the latter being the number of cuts of maximal rank occurring in $\mathcal{D}$.
- It is based on three lemmas:
  1. Cut-substitutivity takes care of non-principal cuts;
  2. Right reduction takes care of cases with cut formula principal in the left premiss
  3. Left reduction covers all other cases.

Lemma for non-principal cuts

Lemma (Cut-substitutivity)

Each rule of $\mathbf{G3C}^*_\omega$ *is cut-substitutive: each instance of cut with cut formula not principal in the last rule instance Rule of one of the premisses of cut can be permuted upwards w.r.t. Rule.*

Proof.

By inspecting the rules (using Lemma 6 for rules $L\exists$, $R\forall$, and L_Q) it is immediate to realise that each rule is cut-substitutive. □

Lemma for cut formula principal in the left premiss

Lemma (Right reduction)

If all of the following hold:

1. $\mathcal{D}_1 \vdash \Gamma \Rightarrow \Delta, A$
2. $\mathcal{D}_2 \vdash A, \Pi \Rightarrow \Sigma$
3. $cr(\mathcal{D}_1, \mathcal{D}_2) < w(A)$
4. *A is principal in the last rule instance applied in $\mathcal{D}_1$*
5. *If $A \equiv \exists x B$ or $A \equiv \bigvee B_i$, then A is not principal in the last rule instance applied in $\mathcal{D}_2$*

Then there is a derivation $\mathcal{D}$ concluding $\Pi, \Gamma \Rightarrow \Delta, \Sigma$ and such that $cr(\mathcal{D}) < w(A)$.

Proof.

By induction on the derivation of the right premiss. □

Lemma for cut formula principal in the left premiss

Lemma (Right reduction)

If all of the following hold:

1. $\mathcal{D}_1 \vdash \Gamma \Rightarrow \Delta, A$
2. $\mathcal{D}_2 \vdash A, \Pi \Rightarrow \Sigma$
3. $cr(\mathcal{D}_1, \mathcal{D}_2) < w(A)$
4. *A is principal in the last rule instance applied in $\mathcal{D}_1$*
5. *If $A \equiv \exists x B$ or $A \equiv \bigvee B_i$, then A is not principal in the last rule instance applied in $\mathcal{D}_2$*

Then there is a derivation $\mathcal{D}$ concluding $\Pi, \Gamma \Rightarrow \Delta, \Sigma$ and such that $cr(\mathcal{D}) < w(A)$.

Proof.

By induction on the derivation of the right premiss. □

Lemma for all other cases

Lemma (Left reduction)

If all of the following hold:

1. $\mathcal{D}_1 \vdash \Gamma \Rightarrow \Delta, A$
2. $\mathcal{D}_2 \vdash A, \Pi \Rightarrow \Sigma$
3. $cr(\mathcal{D}_1, \mathcal{D}_2) < w(A)$

Then there is a derivation $\mathcal{D}$ concluding $\Pi, \Gamma \Rightarrow \Delta, \Sigma$ and such that $cr(\mathcal{D}) < w(A)$.

Proof.

By induction on the derivation of the left premiss. □

Lemma for all other cases

Lemma (Left reduction)

If all of the following hold:

1. $\mathcal{D}_1 \vdash \Gamma \Rightarrow \Delta, A$
2. $\mathcal{D}_2 \vdash A, \Pi \Rightarrow \Sigma$
3. $cr(\mathcal{D}_1, \mathcal{D}_2) < w(A)$

Then there is a derivation $\mathcal{D}$ concluding $\Pi, \Gamma \Rightarrow \Delta, \Sigma$ and such that $cr(\mathcal{D}) < w(A)$.

Proof.

By induction on the derivation of the left premiss. □

Constructive cut elimination

Theorem (Cut elimination)

Cut is admissible in $\mathbf{G3C}^*_\omega$.

Proof.

The proof is by lexicographical induction on $\mathtt{cr}(\mathcal{D}), n(\mathtt{cr}(\mathcal{D}))$.
We consider an uppermost application of *Cut* whose rank is $\mathtt{cr}(\mathcal{D})$ and we apply the Left-reduction lemma to it.
This decreases either $\mathtt{cr}(\mathcal{D})$ or $n(\mathtt{cr}(\mathcal{D}))$, and the theorem holds by inductive hypothesis. □

Outline

$\mathbf{G3i}_\omega$, an infinitary intuitionistic calculus

The intuitionistic infinitary calculus is obtained from the classical one by

1. making implication intuitionistic.

$$\frac{A \supset B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta}\, L\supset \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow \Delta, A \supset B}\, R\supset$$

2. making the universal quantifier intuitionistic.

$$\frac{\forall x A, A(t/x), \Gamma \Rightarrow \Delta}{\Gamma, \forall x A \Rightarrow \Delta}\, L\forall \qquad \frac{\Gamma \Rightarrow A(y/x)}{\Gamma \Rightarrow \Delta, \forall x A}\, R\forall$$

3. Making infinitary conjunction intuitionistic (like $\forall$).

$$\frac{A_k, \bigwedge_{n>0} A_n, \Gamma \Rightarrow \Delta}{\bigwedge_{n>0} A_n, \Gamma \Rightarrow \Delta}\, L\bigwedge_k \qquad \frac{\{\Gamma \Rightarrow A_n \mid n > 0\}}{\Gamma \Rightarrow \Delta, \bigwedge_{n>0} A_n}\, R\bigwedge$$

$\mathbf{G3i}_\omega$, an infinitary intuitionistic calculus

The intuitionistic infinitary calculus is obtained from the classical one by

1. making implication intuitionistic.

$$\frac{A \supset B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta}\ L\supset \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow \Delta, A \supset B}\ R\supset$$

2. making the universal quantifier intuitionistic.

$$\frac{\forall x A, A(t/x), \Gamma \Rightarrow \Delta}{\Gamma, \forall x A \Rightarrow \Delta}\ L\forall \qquad \frac{\Gamma \Rightarrow A(y/x)}{\Gamma \Rightarrow \Delta, \forall x A}\ R\forall$$

3. Making infinitary conjunction intuitionistic (like $\forall$).

$$\frac{A_k, \bigwedge_{n>0} A_n, \Gamma \Rightarrow \Delta}{\bigwedge_{n>0} A_n, \Gamma \Rightarrow \Delta}\ L\bigwedge_k \qquad \frac{\{\Gamma \Rightarrow A_n \mid n > 0\}}{\Gamma \Rightarrow \Delta, \bigwedge_{n>0} A_n}\ R\bigwedge$$

$\mathbf{G3i}_\omega$, an infinitary intuitionistic calculus

The intuitionistic infinitary calculus is obtained from the classical one by

1. making implication intuitionistic.

$$\frac{A \supset B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta}\ L\supset \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow \Delta, A \supset B}\ R\supset$$

2. making the universal quantifier intuitionistic.

$$\frac{\forall x A, A(t/x), \Gamma \Rightarrow \Delta}{\Gamma, \forall x A \Rightarrow \Delta}\ L\forall \qquad \frac{\Gamma \Rightarrow A(y/x)}{\Gamma \Rightarrow \Delta, \forall x A}\ R\forall$$

3. Making infinitary conjunction intuitionistic (like $\forall$).

$$\frac{A_k, \bigwedge_{n>0} A_n, \Gamma \Rightarrow \Delta}{\bigwedge_{n>0} A_n, \Gamma \Rightarrow \Delta}\ L\bigwedge_k \qquad \frac{\{\Gamma \Rightarrow A_n \mid n > 0\}}{\Gamma \Rightarrow \Delta, \bigwedge_{n>0} A_n}\ R\bigwedge$$

Structural properties of $\mathbf{G3i}^*_\omega$

The proofs of the structural properties for $\mathbf{G3i}_\omega$ involve some "Dragalin-style" subtleties, similar to those in use for the *finitary* intuitionistic multisuccedent calculus.

- α-conversion and substitution are hp-admissible
- Left and right weakening are hp-admissible in $\mathbf{G3i}^*_\omega$
- All the rules of $\mathbf{G3i}^*_\omega$ **except** $R\bigwedge$**,** $R\supset$**, and** $R\forall$ are hp-invertible in $\mathbf{G3i}^*_\omega$
- Left and right contraction are hp-admissible in $\mathbf{G3i}^*_\omega$
- Cut is admissible in $\mathbf{G3i}^*_\omega$

Structural properties of $\mathbf{G3i}^*_\omega$

The proofs of the structural properties for $\mathbf{G3i}_\omega$ involve some "Dragalin-style" subtleties, similar to those in use for the *finitary* intuitionistic multisuccedent calculus.

- α-conversion and substitution are hp-admissible
- Left and right weakening are hp-admissible in $\mathbf{G3i}^*_\omega$
- All the rules of $\mathbf{G3i}^*_\omega$ **except** $R\bigwedge$**,** $R\supset$**, and** $R\forall$ are hp-invertible in $\mathbf{G3i}^*_\omega$
- Left and right contraction are hp-admissible in $\mathbf{G3i}^*_\omega$
- Cut is admissible in $\mathbf{G3i}^*_\omega$

Outline

A proof of the infinitary Barr theorem

First-order Barr's theorem: If a (finitary) geometric implication is provable classically in a geometric theory, it is provable also intuitionistically.

Several proofs in the literature for the finitary case: Orevkov (1968), Palmgren (1998), Coste and Coste (1975), Nadathur (2001) .... ; for the infinitary Rathjen (2016). We extend the method of Negri (2003).

1. Consider a classical theory T axiomatized by finitary or infinitary geometric implications.
2. Convert the geometric axioms into infinitary geometric rules
3. Transform the classical theory into a contraction- and cut-free sequent calculus, denoted by $\mathbf{G3c}_{\omega}\mathbf{T}$.
4. Denote by $\mathbf{G3i}_{\omega}\mathbf{T}$ the corresponding intuitionistic extension of $\mathbf{G3i}_{\omega}$.

A proof of the infinitary Barr theorem

First-order Barr's theorem: If a (finitary) geometric implication is provable classically in a geometric theory, it is provable also intuitionistically.

Several proofs in the literature for the finitary case: Orevkov (1968), Palmgren (1998), Coste and Coste (1975), Nadathur (2001) .... ; for the infinitary Rathjen (2016). We extend the method of Negri (2003).

1. Consider a classical theory T axiomatized by finitary or infinitary geometric implications.
2. Convert the geometric axioms into infinitary geometric rules
3. Transform the classical theory into a contraction- and cut-free sequent calculus, denoted by $\mathbf{G3c}_\omega\mathbf{T}$.
4. Denote by $\mathbf{G3i}_\omega\mathbf{T}$ the corresponding intuitionistic extension of $\mathbf{G3i}_\omega$.

A proof of the infinitary Barr theorem (cont.)

Theorem

If a finitary or infinitary geometric implication is derivable in $\mathbf{G3c}_{\omega}\mathbf{T}$*, it is derivable in* $\mathbf{G3i}_{\omega}\mathbf{T}$*.*

Proof.

Almost nothing to prove.

Any derivation in $\mathbf{G3c}_{\omega}\mathbf{T}$ uses only rules that follow the geometric rule scheme and logical rules. Because of the shape of the conclusion, no instance of the rules that violates the intuitionistic restrictions is used, so the derivation directly gives[a] a derivation in $\mathbf{G3i}_{\omega}\mathbf{T}$ of the same conclusion. □

[a] through the addition, where needed, of the missing implications in steps of $L\supset$.

References

- Feferman, S. *Lectures on Proof Theory*, Lecture Notes in Mathematics **70**, pp 1–108, 1968.
- Lopez-Escobar, E.G.F. *An interpolation theorem for denumerably long formulas*, Fundamenta Mathematicae **57**, pp 253–257, 1965.
- Metcalfe, Olivetti, Gabbay *Proof Theory for Fuzzy Logics*, Springer, 2009.
- Negri, S. *Contraction-free sequent calculi for geometric theories, with an application to Barr's theorem*, Archive for Mathematical Logic **42**, pp 389–401, 2003.
- Negri, S. *Proof analysis in modal logic*, Journal of Philosophical Logic **34**, pp 507–544, 2005.
- Negri, S. *Geometric rules in infinitary logic*, In "Arnon Avron on Semantics and Proof Theory of Non-Classical Logic", 202X.
- Negri, S. and von Plato, J. *Structural Proof Theory*, Cambridge University Press, 2001.
- Negri, S. and von Plato, J. *Proof Analysis*, CUP, 2011.
- Rathjen, M. *Remarks on Barr's theorem: Proofs in geometric theories*. In "Concepts of Proof in Mathematics, Philosophy, and Computer Science", 2016.