A proof-theoretic approach to formal epistemology

Sara Negri[1,2] & Edi Pavlović[2]

UNIVERSITÀ DEGLI STUDI DI GENOVA

HELSINGIN YLIOPISTO
HELSINGFORS UNIVERSITET
UNIVERSITY OF HELSINKI

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The main goal

Fully formal epistemology.

Desiderata:

- Epistemic and doxastic modalities
- Conditionals in the object language, allowing for nesting
- Notion of justification
- Treatment of belief revision
- Valid inference, absolute and multi-agent
- Semantic and syntactic approach, completeness theorems
- Computation

In this talk we address most of these.

Assumption: familiarity with sequent calculi.

Plan of the talk

1. Neighborhood semantics
2. Conditional doxastic logic CDL
3. Knowledge and simple belief
4. Sequent calculus G3SBK
5. Properties of knowledge
6. Proof theory and paradox control

Neighborhood semantics

Definition 1 (Neighborhood frame)

A neighborhood frame has the form $\langle W, I \rangle$ where W is a nonempty set and $I : W \to \mathcal{P}(\mathcal{P}(W))$ is a neighborhood function.

We form the neighborhood model $\mathcal{M} = \langle W, I, [\![\ ]\!] \rangle$ by adding the propositional evaluation function $[\![\ ]\!]$:

Definition 2 (Evaluation function $[\![\ ]\!]$)

$[\![\ ]\!] : Atm \to \mathcal{P}(W)$ is the evaluation for atomic formulas. Truth conditions for formulas extend $[\![\ ]\!]$ inductively as:

$[\![\neg A]\!] = W - [\![A]\!]$
$[\![A \,\&\, B]\!] = [\![A]\!] \cap [\![B]\!]$
$[\![A \vee B]\!] = [\![A]\!] \cup [\![B]\!]$
$[\![A \supset B]\!] = (W - [\![A]\!]) \cup [\![B]\!]$

A formulas A is valid in $\mathcal{M}$ if $[\![A]\!] = W$. We write $x \in [\![A]\!]$ as $\mathcal{M}, x \Vdash A$, and further omit $\mathcal{M}$ if no ambiguity arises.

Neighborhood and Kripke frames

To represent a Kripke frame $\langle W, R \rangle$ in a neighborhood frame Negri (2017b), one defines the neighborhood function I^R as

Definition 3 (Neighborhood function I^R)

$I^R(x) = \{a \mid R(x) \subseteq a\}$

i.e. we take all supersets of worlds accessible from x as its neighborhood.

Likewise, one can represent a neighborhood frame $\langle W, I \rangle$ in a relational one by defining the accessibility relation R^I as

Definition 4 (Accessibility relation R^I)

$xR^I y \equiv y \in \bigcap I(x)$

i.e. y is accessible from x if it is in all of its neighborhoods.

Neighborhood and Kripke frames

To establish correspondence between neighborhood and Kripke frames, we first define *augmented* neighborhoods Chellas (1980):

Definition 5 (Augmented neighborhood frame)

A neighborhood frame is augmented iff for every a and x,

$$a \in I(x) \equiv \bigcap I(x) \subseteq a$$

We can now show, following Chellas (1980), that

Lemma 6

For every Kripke model $\mathcal{M} = \langle W, R, \mathcal{V} \rangle$ there is an augmented neighborhood model $\mathcal{M}^R = \langle W, I^R, [\![\ ]\!] \rangle$ such that for any w, if $\mathcal{M}, w \Vdash A$, then $\mathcal{M}^R, w \Vdash A$.

Neighborhood and Kripke frames

Proof.

First, by Definition 3 (I^R), $a \in I^R(x) \equiv R(x) \subseteq a$. Then, since $\bigcap I^R(x) = R(x)$, it follows $\mathcal{M}^R$ is augmented.

The lemma is proven by induction on the weight of A. We illustrate just for the interesting case of $\Box$, where

$$\mathcal{M}^R, w \Vdash \Box A \equiv [\![A]\!] \in I(w)$$

If $\mathcal{M}, w \Vdash \Box A$, then $\forall y(wRy \supset \mathcal{M}, y \Vdash A)$. So, by inductive hypothesis, $R(w) \subseteq [\![A]\!]$. Therefore, by Definition 3, $[\![A]\!] \in I^R(w)$, and finally $\mathcal{M}^R, w \Vdash \Box A$. QED

Neighborhood and Kripke frames

Lemma 7

For every augmented neighborhood model $\mathcal{M} = \langle W, I, [\![\ ]\!] \rangle$ there is a Kripke model $\mathcal{M}^I = \langle W, R^I, \mathcal{V} \rangle$ such that for any w, if $\mathcal{M}, w \Vdash A$, then $\mathcal{M}^I, w \Vdash A$.

Proof.

Essentially runs the previous proof in reverse, using instead of proving that the neighborhood frame is augmented. QED

Combined, these lemmas show that

Theorem 8 (Equivalence of Kripke and neighborhood models)

For every Kripke model, there is an augmented neighborhood model that validates the same formulas, and vice versa.

Conditional doxastic logic CDL

CDL uses the primitive epistemic operator of conditional belief $Bel_i(C|B)$ – "*agent i believes C, given B*".

Definition 9 (Formula of *CDL*)

$$A \;:=\; P \mid \bot \mid \neg A \mid A \wedge A \mid A \vee A \mid A \supset A \mid Bel_i(A|A)$$

The axiomatization of CDL Board (2004) contains the rules:

Definition 10 (Inference rules)

(1) If $\vdash B$, then $\vdash Bel_i(B|A)$ (epistemization rule)

(2) If $\vdash A \supset\subset B$, then $\vdash Bel_i(C|A) \supset\subset Bel_i(C|B)$ (rule of logical equivalence)

Conditional doxastic logic CDL

CDL is then axiomatized as:

Definition 11 (Axioms of *CDL*)

Any axiomatization of the classical propositional calculus, plus:

(3) $(Bel_i(B|A) \wedge Bel_i(B \supset C|A)) \supset Bel_i(C|A)$ (distribution axiom)

(4) $Bel_i(A|A)$ (success axiom)

(5) $Bel_i(B|A) \supset (Bel_i(C|A \wedge B) \supset\subset Bel_i(C|A))$ (minimal change principle 1)

(6) $\neg Bel_i(\neg B|A) \supset (Bel_i(C|A \wedge B) \supset\subset Bel_i(B \supset C|A))$ (minimal change principle 2)

(7) $Bel_i(B|A) \supset Bel_i(Bel_i(B|A)|C)$ (positive introspection)

(8) $\neg Bel_i(B|A) \supset Bel_i(\neg Bel_i(B|A)|C)$ (negative introspection)

(9) $A \supset \neg Bel_i(\bot|A)$ (consistency axiom)

Neighborhood models of CDL

Definition 12 (Multi-agent neighbourhood models)

Let $\mathcal{A}$ be a set of agents; a *multi-agent neighbourhood model* (*NM*) has the form

$$\mathcal{M} = \langle W, \{I\}_{i \in \mathcal{A}}, [\![\ ]\!] \rangle$$

where

W is a non empty set of elements called “worlds”,

$[\![\ ]\!] : Atm \to \mathcal{P}(W)$ is the evaluation for atomic formulas,

for each $i \in \mathcal{A}$, $I_i : W \to \mathcal{P}(\mathcal{P}(W))$ is the neighbourhood function, satisfying the following properties:

- *Non-emptiness*: $\forall \alpha \in I_i(x), \alpha \neq \emptyset$
- *Nesting*: $\forall \alpha, \beta \in I_i(x), \alpha \subseteq \beta$ or $\beta \subseteq \alpha$
- *Total reflexivity*: $\exists \alpha \in I_i(x)$ such that $x \in \alpha$
- *Local absoluteness*: If $\alpha \in I_i(x)$ and $y \in \alpha$ then $I_i(x) = I_i(y)$
- *Closure under intersection*: If $S \subseteq I_i(x)$ and $S \neq \emptyset$ then $\bigcap S \in S$ (always holds in finite models)

Neighborhood models of CDL

Conditional belief is defined as

Definition 13 (Conditional Belief)

$$x \Vdash Bel_i(B|A) \quad \textit{iff } \forall \alpha \in I_i(x)(\alpha \cap [\![A]\!] = \emptyset); \textit{ or}$$
$$\exists \beta \in I_i(x)(\beta \cap [\![A]\!] \neq \emptyset \textit{ and } \beta \cap [\![A]\!] \subseteq [\![B]\!])$$

We can now introduce the local forcing relation, due to Negri (2017a):

Definition 14 (Local forcing relations, $\Vdash^{\forall}, \Vdash^{\exists}$)

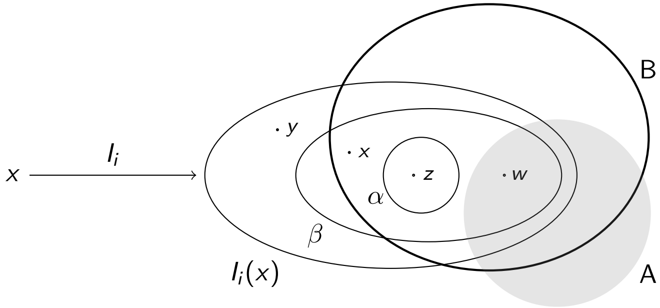
$a \Vdash^{\forall} A$ *iff* $\forall y \in a.y \Vdash A$
$a \Vdash^{\exists} A$ *iff* $\exists y \in a.y \Vdash A$

Neighborhood models of CDL

Using these, we can then render the definition as:

$$x \Vdash Bel_i(B|A) \text{ iff } \quad (\forall a \in I_i(x).a \Vdash^{\forall} \neg A) \text{ or}$$
$$\exists b \in I_i(x).b \Vdash^{\exists} A \text{ and } b \Vdash^{\forall} A \supset B$$

Graphically, these truth conditions can be represented as



Knowledge and simple belief

Due to Stalnaker (1998), knowledge and simple (non-conditional) belief can be defined as

Definition 15 (Knowledge and simple belief in *CDL*)

Knowledge: $K_i A \equiv Bel_i(\bot | \neg A)$
Simple belief: $Bel_i A \equiv Bel_i(A | \top)$

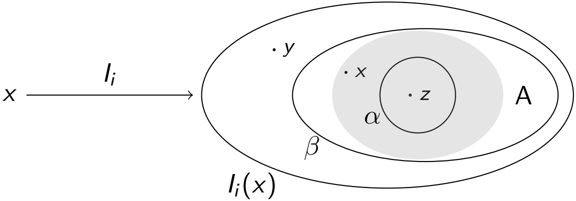
We unpack these definitions to obtain the truth conditions for each.

Simple belief

Definition 16 (Belief)

$$x \Vdash Bel_i A \quad \textit{iff} \quad \exists \alpha \in I_i(x)\ (\alpha \subseteq [\![A]\!]) \quad \textit{iff} \quad \exists a \in I_i(x)\ (a \Vdash^{\forall} A)$$

Graphically these conditions can be represented as:

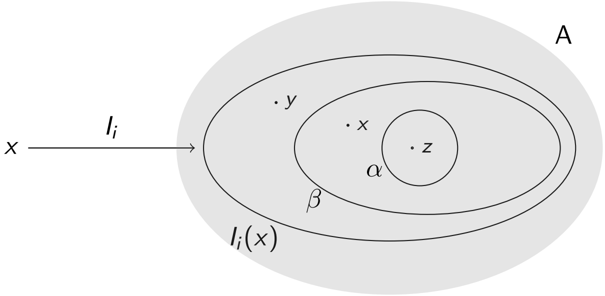


Knowledge

Definition 17 (Knowledge)

$x \Vdash K_i A \quad \text{iff} \quad \forall \beta \in I_i(x)\ (\beta \subseteq [\![A]\!]) \quad \text{iff} \quad \forall b \in I_i(x)\ (b \Vdash^{\forall} A)$

Graphically these conditions can be represented as:



Sequent calculus G3SBK

We retain the rules of G3CDL, and extend them with rules for simple belief and knowledge, which adhere to these definitions, to obtain the sequent calculus G3SBK:

Initial sequents $x : P, \Gamma \Rightarrow \Delta, x : P$

Propositional rules: rules of **G3K** Negri (2005)

Rules for local forcing

$$\frac{x \in a, \Gamma \Rightarrow \Delta, x : A}{\Gamma \Rightarrow \Delta, a \Vdash^{\forall} A} \; R\Vdash^{\forall} \; (x \;\; \textit{fresh}) \qquad \frac{x : A, x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta} \; L\Vdash^{\forall}$$

$$\frac{x \in a, \Gamma \Rightarrow \Delta, x : A, a \Vdash^{\exists} A}{x \in a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A} \; R\Vdash^{\exists} \qquad \frac{x \in a, x : A, \Gamma \Rightarrow \Delta}{a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta} \; L\Vdash^{\exists} \; (x \;\; \textit{fresh})$$

Sequent calculus G3SBK

Rules for inclusion

$$\frac{a \subseteq a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Ref} \qquad \frac{c \subseteq a, c \subseteq b, b \subseteq a, \Gamma \Rightarrow \Delta}{c \subseteq b, b \subseteq a, \Gamma \Rightarrow \Delta}\ \mathit{Tr}$$

$$\frac{x \in a, a \subseteq b, x \in b, \Gamma \Rightarrow \Delta}{x \in a, a \subseteq b, \Gamma \Rightarrow \Delta}\ L\subseteq$$

Sequent calculus G3SBK

Rules for semantic conditions

$$\frac{a \subseteq b, a \in I_i(x), b \in I_i(x), \Gamma \Rightarrow \Delta \quad b \subseteq a, a \in I_i(x), b \in I_i(x), \Gamma \Rightarrow \Delta}{a \in I_i(x), b \in I_i(x), \Gamma \Rightarrow \Delta} \; S$$

$$\frac{y \in a, a \in I_i(x), \Gamma \Rightarrow \Delta}{a \in I_i(x), \Gamma \Rightarrow \Delta} \; N \; (y \; fresh) \qquad \frac{x \in a, a \in I_i(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \; T \; (a \; fresh)$$

$$\frac{a \in I_i(x), y \in a, b \in I_i(x), b \in I_i(y), \Gamma \Rightarrow \Delta}{a \in I_i(x), y \in a, b \in I_i(x), \Gamma \Rightarrow \Delta} \; A_1$$

$$\frac{a \in I_i(x), y \in a, b \in I_i(x), b \in I_i(y), \Gamma \Rightarrow \Delta}{a \in I_i(x), y \in a, b \in I_i(y), \Gamma \Rightarrow \Delta} \; A_2$$

$$\frac{a \in I_i(x), y \in a, a \in I_i(y), \Gamma \Rightarrow \Delta}{a \in I_i(x), y \in a, \Gamma \Rightarrow \Delta} \; A_1^*$$

Sequent calculus G3SBK

Rules for knowledge and belief

$$\frac{a \in I_i(x), \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}{\Gamma \Rightarrow \Delta, x : K_i A} \; RK \; (a \textit{ fresh})$$

$$\frac{a \in I_i(x), x : K_i A, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{a \in I_i(x), x : K_i A, \Gamma \Rightarrow \Delta} \; LK$$

$$\frac{a \in I_i(x), \Gamma \Rightarrow \Delta, x : Bel_i A, a \Vdash^{\forall} A}{a \in I_i(x), \Gamma \Rightarrow \Delta, x : Bel_i A} \; RSB$$

$$\frac{a \in I_i(x), a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x : Bel_i A, \Gamma \Rightarrow \Delta} \; LSB \; (a \textit{ fresh})$$

Structural properties

Here we extend the proofs of structural properties from the Girlando et al. (2018) with added rules. We start with the notion of the weight of the formula:

Definition 18 (Weight of a labelled formula)

The weight of the labelled formula $\mathcal{F}$ is the pair $(\mathtt{w}(p(\mathcal{F})), \mathtt{w}(l(\mathcal{F})))$, where $l(\mathcal{F})$ is the label of $\mathcal{F}$, and

$$\mathtt{w}(x) = 0,\ \mathtt{w}(a) = 1,$$

and $p(\mathcal{F})$ is the part of $\mathcal{F}$ without the label and the forcing relation, and

$$\begin{aligned}
&\mathtt{w}(P) = \mathtt{w}(\top) = 1,\\
&\mathtt{w}(A \circ B) = \mathtt{w}(A) + \mathtt{w}(B) + 1,\ \circ \in \{\vee, \&, \supset\},\\
&\mathtt{w}(\neg A) = \mathtt{w}(A) + 2,\\
&\mathtt{w}(B|A) = \mathtt{w}(A) + \mathtt{w}(B) + 2\\
&\mathtt{w}(Bel_i(B|A)) = \mathtt{w}(B|A) + 1.\\
&\mathtt{w}(Bel_i A) = \mathtt{w}(A) + 4\\
&\mathtt{w}(K_i A) = \mathtt{w}(A) + 6
\end{aligned}$$

Weights of labelled formulas are ordered lexicographically.

Structural properties

Lemma 19 (Axiom generalization)

For any labelled formula $\mathcal{F}$, the sequent $\mathcal{F}, \Gamma \Rightarrow \Delta, \mathcal{F}$ is derivable.

Lemma 20 (Substitution)

If $\vdash_n \Gamma \Rightarrow \Delta$ then $\vdash_n \Gamma(y/x) \Rightarrow \Delta(y/x)$; if $\vdash_n \Gamma \Rightarrow \Delta$ then $\vdash_n \Gamma(a/b) \Rightarrow \Delta(a/b)$.

Lemma 21 (Weakening)

Weakening is height-preserving admissible.

Lemma 22 (Invertibility)

All the rules of G3SBK are height-preserving invertible.

Lemma 23 (Contraction)

The rules of left and right contraction are height-preserving admissible.

Structural properties

Theorem 24 (Cut)

Cut is admissible.

Proof is by primary induction on the weight of the formula and secondary induction on the sum of the heights of the premises of cut. We illustrate for the case where the cut formula is principal in both premises and of the form $x : K_i A$.

Structural properties

Proof.

$$\dfrac{\dfrac{b \in I_i(x), \Gamma \Rightarrow \Delta, b \Vdash^{\forall} A}{\Gamma \Rightarrow \Delta, x : K_i A} RK \quad \dfrac{a \in I_i(x), x : K_i A, a \Vdash^{\forall} A, \Gamma' \Rightarrow \Delta'}{a \in I_i(x), x : K_i A, \Gamma' \Rightarrow \Delta'} LK}{a \in I_i(x), \Gamma', \Gamma \Rightarrow \Delta, \Delta'} Cut$$

This is transformed into:

$$\dfrac{\dfrac{\dfrac{b \in I_i(x), \Gamma \Rightarrow \Delta, b \Vdash^{\forall} A}{a \in I_i(x), \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A} Lm\ 20 \quad \dfrac{\dfrac{b \in I_i(x), \Gamma \Rightarrow \Delta, b \Vdash^{\forall} A}{\Gamma \Rightarrow \Delta, x : K_i A} RK \quad a \in I_i(x), x : K_i A, a \Vdash^{\forall} A, \Gamma' \Rightarrow \Delta'}{a \in I_i(x), a \Vdash^{\forall} A, \Gamma', \Gamma \Rightarrow \Delta, \Delta'} Cut_1}{a \in I_i(x), a \in I_i(x), \Gamma', \Gamma, \Gamma \Rightarrow \Delta, \Delta, \Delta'} Cut_2}{a \in I_i(x), \Gamma', \Gamma \Rightarrow \Delta, \Delta'} Lm\ 23$$

The application of the Cut rule labeled Cut_1 is of lower height, and that labeled Cut_2 is of lower weight (recall again the lexicographical ordering). QED

Properties of knowledge

We can show that:

Theorem 25 (K_i is S5)

K_i is (at least) an S5 operator. Specifically, the following hold of it:

(i) $K_iA \supset A$
(ii) $K_iA \supset K_iK_iA$
(iii) $\neg K_iA \supset K_i\neg K_iA$

In fact, we can be more fine-grained and relate semantic conditions to properties of K_i.

Factivity

Factivity *(i)* $K_iA \supset A$ of knowledge follows from **total reflexivity**:

(i)

$$\cfrac{\cfrac{\cfrac{x \in a, a \in I_i(x), x : K_iA, a \Vdash^{\forall} A, x : A \Rightarrow x : A}{x \in a, a \in I_i(x), x : K_iA, a \Vdash^{\forall} A \Rightarrow x : A}\; L\Vdash^{\forall}}{x \in a, a \in I_i(x), x : K_iA \Rightarrow x : A}\; LK}{x : K_iA \Rightarrow x : A}\; T$$

Positive introspection

Positive introspection *(ii)* $K_iA \supset K_iK_iA$ for knowledge follows from one direction of **local absoluteness**:

(ii)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{z : A, b \Vdash^{\forall} A, b \in I_i(x), z \in b, b \in I_i(y), y \in a, a \in I_i(x), x : K_iA \Rightarrow z : A}{b \Vdash^{\forall} A, b \in I_i(x), z \in b, b \in I_i(y), y \in a, a \in I_i(x), x : K_iA \Rightarrow z : A}\; L\Vdash^{\forall}}{b \in I_i(x), z \in b, b \in I_i(y), y \in a, a \in I_i(x), x : K_iA \Rightarrow z : A}\; LK}{z \in b, b \in I_i(y), y \in a, a \in I_i(x), x : K_iA \Rightarrow z : A}\; A_2}{b \in I_i(y), y \in a, a \in I_i(x), x : K_iA \Rightarrow b \Vdash^{\forall} A}\; R\Vdash^{\forall}}{y \in a, a \in I_i(x), x : K_iA \Rightarrow y : K_iA}\; RK}{a \in I_i(x), x : K_iA \Rightarrow a \Vdash^{\forall} K_iA}\; R\Vdash^{\forall}}{x : K_iA \Rightarrow x : K_iK_iA}\; RK$$

Negative introspection

Negative introspection *(iii)* $\neg K_i A \supset K_i \neg K_i A$ for knowledge follows from the other direction of **local absoluteness**:

(iii)

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{a \in I_i(z), z : K_i A, a \Vdash^{\forall} A, y : A, z \in b, y \in a, b \in I_i(x), a \in I_i(x) \Rightarrow y : A}{a \in I_i(z), z : K_i A, a \Vdash^{\forall} A, z \in b, y \in a, b \in I_i(x), a \in I_i(x) \Rightarrow y : A}\; L\Vdash^{\forall}}{a \in I_i(z), z : K_i A, z \in b, y \in a, b \in I_i(x), a \in I_i(x) \Rightarrow y : A}\; LK}{z : K_i A, z \in b, y \in a, b \in I_i(x), a \in I_i(x) \Rightarrow y : A}\; A_1}{z \in b, y \in a, b \in I_i(x), a \in I_i(x) \Rightarrow z : \neg K_i A, y : A}\; R\neg}{y \in a, b \in I_i(x), a \in I_i(x) \Rightarrow b \Vdash^{\forall} \neg K_i A, y : A}\; R\Vdash^{\forall}}{b \in I_i(x), a \in I_i(x) \Rightarrow b \Vdash^{\forall} \neg K_i A, a \Vdash^{\forall} A}\; R\Vdash^{\forall}}{a \in I_i(x) \Rightarrow x : K_i \neg K_i A, a \Vdash^{\forall} A}\; RK}{\Rightarrow x : K_i \neg K_i A, x : K_i A}\; RK}{x : \neg K_i A \Rightarrow x : K_i \neg K_i A}\; L\neg$$

Argument against a perfect believer

The paradox of the perfect believer is a derivation of an implication from *belief of knowledge* to *knowledge* using (apparently) reasonable assumption on the classical epistemic/doxastic operators:

Infallibility, $\neg Bel_i \perp$,
Knowledge implies belief, $K_i A \supset Bel_i A$ and
Introspection about belief, $Bel_i A \supset K_i Bel_i A$.

We take what is needed to have the same assumptions used in the puzzle.

Argument against a perfect believer

Infallibility, $\neg Bel_i \perp$, follows from N (non-emptiness):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{y : \perp, y \in a, a \in I(x), a \Vdash^{\forall} \perp \Rightarrow}\ L\perp}{y \in a, a \in I(x), a \Vdash^{\forall} \perp \Rightarrow}\ L\Vdash^{\forall}}{a \in I(x), a \Vdash^{\forall} \perp \Rightarrow}\ N}{x : Bel_i \perp \Rightarrow}\ LSB$$

Knowledge implies belief $K_i A \supset Bel_i A$ is valid thanks to T (*total reflexivity*):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y \in a, x \in a, a \in I(x), x : K_i A, a \Vdash^{\forall} A, y : A \Rightarrow x : Bel_i A, y : A}{y \in a, x \in a, a \in I(x), x : K_i A, a \Vdash^{\forall} A \Rightarrow x : Bel_i A, y : A}\ L\Vdash^{\forall}}{x \in a, a \in I(x), x : K_i A, a \Vdash^{\forall} A \Rightarrow x : Bel_i A, a \Vdash^{\forall} A}\ R\Vdash^{\forall}}{x \in a, a \in I(x), x : K_i A, a \Vdash^{\forall} A \Rightarrow x : Bel_i A}\ RSB}{x \in a, a \in I(x), x : K_i A \Rightarrow x : Bel_i A}\ LK}{x : K_i A \Rightarrow x : Bel_i A}\ T$$

Argument against a perfect believer

Introspection about belief $Bel_i A \supset K_i Bel_i A$ is valid thanks to A rules (*local absoluteness*):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{z \in a, a \in I(y), y \in b, b \in I(x), a \in I(x), a \Vdash^{\forall} A, z : A \Rightarrow y : Bel_i A, z : A}{z \in a, a \in I(y), y \in b, b \in I(x), a \in I(x), a \Vdash^{\forall} A \Rightarrow y : Bel_i A, z : A} L\Vdash^{\forall}}{a \in I(y), y \in b, b \in I(x), a \in I(x), a \Vdash^{\forall} A \Rightarrow y : Bel_i A, a \Vdash^{\forall} A} R\Vdash^{\forall}}{a \in I(y), y \in b, b \in I(x), a \in I(x), a \Vdash^{\forall} A \Rightarrow y : Bel_i A} RSB}{y \in b, b \in I(x), a \in I(x), a \Vdash^{\forall} A \Rightarrow y : Bel_i A} A_1}{b \in I(x), a \in I(x), a \Vdash^{\forall} A \Rightarrow b \Vdash^{\forall} Bel_i A} R\Vdash^{\forall}}{a \in I(x), a \Vdash^{\forall} A \Rightarrow x : K_i Bel_i A} RK}{x : Bel_i A \Rightarrow x : K_i Bel_i A} LSB
$$

Argument against a perfect believer

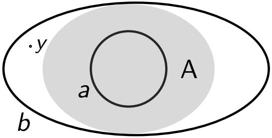
The derivation of the paradox proceeds as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\ldots, y : A \Rightarrow y : A}{\ldots, y \in b, b \in I_i(y), b \Vdash^{\forall} A \Rightarrow y : A}\, L\Vdash^{\forall}}{\ldots, b \in I_i(y), y : K_iA \Rightarrow y : A}\, LK_i}{\ldots, y : K_iA \Rightarrow y : A}\, A_1}{y \in a, \ldots a \Vdash^{\forall} K_iA \Rightarrow y : A}\, L\Vdash^{\forall}}{b \subseteq a, \ldots, y \in b, a \Vdash^{\forall} K_iA \Rightarrow y : A}\, L\subseteq
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\ldots, y : A \Rightarrow y : A}{b \in I_i(z), z \in a, \ldots, y \in b, b \Vdash^{\forall} A \Rightarrow y : A}\, L\Vdash^{\forall}}{b \in I_i(z), z \in a, \ldots, y \in b, z : K_iA \Rightarrow y : A}\, LK_i}{z \in a, \ldots, y \in b, z : K_iA \Rightarrow y : A}\, A_1}{a \subseteq b, a \in I_i(x), b \in I_i(x), y \in b, a \Vdash^{\forall} K_iA \Rightarrow y : A}\, N
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{b \subseteq a, \ldots, y \in b, a \Vdash^{\forall} K_iA \Rightarrow y : A \qquad a \subseteq b, a \in I_i(x), b \in I_i(x), y \in b, a \Vdash^{\forall} K_iA \Rightarrow y : A}{a \in I_i(x), b \in I_i(x), y \in b, a \Vdash^{\forall} K_iA \Rightarrow y : A}\, S}{a \in I_i(x), b \in I(x)_i, a \Vdash^{\forall} K_iA \Rightarrow a \Vdash^{\forall} A}\, R\Vdash^{\forall}}{a \in I_i(x), a \Vdash^{\forall} K_iA \Rightarrow x : K_iA}\, RK}{x : Bel_iK_iA \Rightarrow x : K_iA}\, LSB
$$

Obviously, without N proof search stops on the right and we obtain the countermodel from the failed proof search:



Hvala/Thank you!

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