Soft Subexponentials and Multiplexing

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Logical Frameworks

Logical Specifications allow for the specification of deductive systems, logics, and operational semantics.

- **Linear Logical Frameworks**: Specify state conscious systems;

$$\Theta; \Gamma \longrightarrow G$$

Unbounded Formulas: Interpreted as a **set** of formulas.

Linear Formulas: Interpreted as a **multiset** of formulas.

Logical Frameworks

Two extensions of **Linear Logical Frameworks**:

Subexponentials

[Nigam,Olarte,Pimentel, Reis]

$$\Theta_1; \ldots; \Theta_n; \Gamma_1; \ldots; \Gamma_m \longrightarrow G$$

Allows for **many unbounded and linear contexts**.

- **Extended expressiveness:** specificaiton of systems with several contexts: logics, concurrent programming, etc.

Ordered Logics

[Pfenning,Simmons,Polakow]

$$\Theta; \Gamma; L \longrightarrow G$$

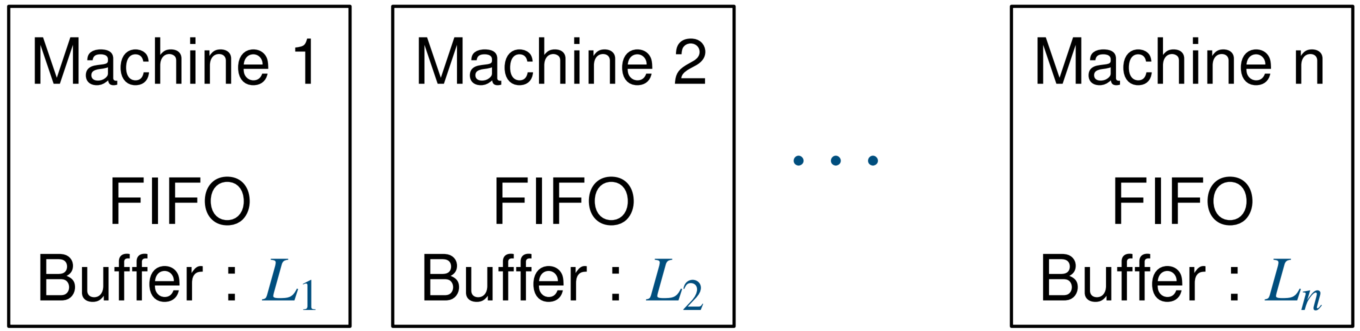
L - **Ordered Formulas:** Interpreted as a **list** of formulas.

- **Extended expressiveness:** specificaiton of systems with some order (PL evaluation strategies, systems with lists, etc.)

Contribution 1: A logical framework with commutative and non-commutative subexponentials.

Application

Example: Distributed System Semantics



List of formulas

$$\Theta; [\text{start}, \Gamma_1, \text{end}]_{\text{m1}}; [\text{start}, \Gamma_2, \text{end}]_{\text{m2}}; \cdots ; [\text{start}, \Gamma_n, \text{end}]_{\text{m}n} \longrightarrow G$$

Specification of the behavior of the system.

Lambek Proof System

$$\frac{}{F \to F}\ I \qquad \frac{\Gamma_1, \Gamma_2 \to C}{\Gamma_1, \mathbf{1}, \Gamma_2 \to C}\ \mathbf{1}_L \qquad \frac{}{\to \mathbf{1}}\ \mathbf{1}_R$$

Initial and Unit

$$\frac{\Pi \to G \quad \Gamma_1, F, \Gamma_2 \to C}{\Gamma_1, F \,/\, G, \Pi, \Gamma_2 \to C}\ /_L \qquad \frac{\Pi, F \to G}{\Pi \to G \,/\, F}\ /_R$$

Right Division

$$\frac{\Pi \to F \quad \Gamma_1, G, \Gamma_2 \to C}{\Gamma_1, \Pi, F \setminus G, \Gamma_2 \to C}\ \setminus_L \qquad \frac{F, \Pi \to G}{\Pi \to F \setminus G}\ \setminus_R$$

Left Division

$$\frac{\Gamma_1, F, G, \Gamma_2 \to C}{\Gamma_1, F \cdot G, \Gamma_2 \to C}\ \cdot_L \qquad \frac{\Gamma_1 \to F \quad \Gamma_2 \to G}{\Gamma_1, \Gamma_2 \to F \cdot G}\ \cdot_R$$

Product

$$\frac{\Pi \to F\{e/x\}}{\Pi \to \forall x.F}\ \forall_R \qquad \frac{\Gamma_1, F\{t/x\}, \Gamma_2 \to C}{\Gamma_1, \forall x.F, \Gamma_2 \to C}\ \forall_L$$

Quantifier

The order of formulas is important.

Proof System with Subexponentials

Subexponential Signature

$$\Sigma = \langle \mathcal{I}, \preceq, \mathcal{W}, \mathcal{C}, \mathcal{E} \rangle$$

SNILL_Σ proof system.

- $\mathcal{I}$ is a set of lables, $\mathcal{W}, \mathcal{C}, \mathcal{E} \subseteq \mathcal{I}$
- $\preceq$ is a pre-order relation over $\mathcal{I}$ upwardly closed w.r.t. $\mathcal{W}, \mathcal{C}, \mathcal{E}$.

For each $\mathsf{s} \in \mathcal{I}$:

$$\frac{\Gamma_1, F, \Gamma_2 \to G}{\Gamma_1, !^{\mathsf{s}} F, \Gamma_2 \to G}\ Der \qquad \frac{!^{\mathsf{s}_1} F_1, \ldots, !^{\mathsf{s}_n} F_n \longrightarrow F}{!^{\mathsf{s}_1} F_1, \ldots, !^{\mathsf{s}_n} F_n \longrightarrow !^{\mathsf{s}} F}\ !^{\mathsf{s}}{}_R, \text{provided, } \mathsf{s} \preceq \mathsf{s}_i, 1 \leq i \leq n$$

For each $\mathsf{w} \in \mathcal{W}$ and $\mathsf{c} \in \mathcal{C}$:

$$\frac{\Gamma, \Delta \longrightarrow G}{\Gamma, !^{\mathsf{w}} F, \Delta \longrightarrow G}\ W \qquad \frac{\Gamma_1, !^{\mathsf{c}} F, \Delta, !^{\mathsf{c}} F, \Gamma_2 \to G}{\Gamma_1, !^{\mathsf{c}} F, \Delta, \Gamma_2 \to G}\ C_1 \qquad \frac{\Gamma_1, !^{\mathsf{c}} F, \Delta, !^{\mathsf{c}} F, \Gamma_2 \to G}{\Gamma_1, \Delta, !^{\mathsf{c}} F, \Gamma_2 \to G}\ C_2$$

For each $\mathsf{e} \in \mathcal{E}$:

$$\frac{\Gamma_1, \Delta, !^{\mathsf{e}} F, \Gamma_2 \to C}{\Gamma_1, !^{\mathsf{e}} F, \Delta, \Gamma_2 \to C}\ E_1 \qquad \frac{\Gamma_1, !^{\mathsf{e}} F, \Delta, \Gamma_2 \to C}{\Gamma_1, \Delta, !^{\mathsf{e}} F, \Gamma_2 \to C}\ E_2$$

Proof System with Subexponentials

Subexponential Signature

$$\Sigma = \langle \mathcal{I}, \preceq, \mathcal{W}, \mathcal{C}, \mathcal{E} \rangle$$

SNILL$_\Sigma$ proof system.

- $\mathcal{I}$ is a set of lables, $\mathcal{W}, \mathcal{C}, \mathcal{E} \subseteq \mathcal{I}$
- $\preceq$ is a pre-order relation over $\mathcal{I}$ upwardly closed w.r.t. $\mathcal{W}, \mathcal{C}, \mathcal{E}$.

- **Theorem** For any well formed Σ, SNILL$_\Sigma$ admits cut-elimination.

Proof Extends our previous results [Dale-Fest, MSCS 18] with quantifiers.

Kinds of Formulas

Assumption:

- $\mathcal{W} \subseteq \mathcal{E}$
- $\mathcal{C} \subseteq \mathcal{E}$

These assumptions are enough for our examples and facilitate proof search (focused proof system for SNILL).

A formula of the form $!^{\mathsf{s}}F$ is

- **Linear Formulas** if $\mathsf{s} \notin \mathcal{W} \cup \mathcal{C}$. They can be **non-commutative** if $\mathsf{s} \notin \mathcal{E}$ and **commutative** otherwise if $\mathsf{s} \in \mathcal{E}$;
- **Unbounded Formulas** if $\mathsf{s} \in \mathcal{W} \cap \mathcal{C}$;
- **Affine Formulas** if $\mathsf{s} \in \mathcal{W}$ and $\mathsf{s} \notin \mathcal{C}$;
- **Relevant Formulas** if $\mathsf{s} \in \mathcal{C}$ and $\mathsf{s} \notin \mathcal{W}$;

Kinds of Formulas

A formula of the form $!^{s}F$ is

- **Linear Formulas** if $s \notin \mathcal{W} \cup \mathcal{C}$. They can be **non-commutative** if $s \notin \mathcal{E}$ and **commutative** otherwise if $s \in \mathcal{E}$;
- **Unbounded Formulas** if $s \in \mathcal{W} \cap \mathcal{C}$;
- **Affine Formulas** if $s \in \mathcal{W}$ and $s \notin \mathcal{C}$;
- **Relevant Formulas** if $s \in \mathcal{C}$ and $s \notin \mathcal{W}$;

Logical frameworks have been proposed with unbounded, linear and affine formulas, but without relevant formulas.

Kinds of Formulas

Logical frameworks have been proposed with unbounded, linear and affine formulas, but without relevant formulas.

Safe to contract unbounded formulas as one does not lose provability.

$$\dfrac{\dfrac{!^{\mathsf{u}}F, !^{\mathsf{r}}H, \Gamma \longrightarrow G_1 \qquad !^{\mathsf{u}}F, \Delta \longrightarrow G_2}{!^{\mathsf{u}}F, !^{\mathsf{r}}H, \Gamma, !^{\mathsf{u}}F, \Delta \longrightarrow G_1 \cdot G_2}\ \otimes_R}{!^{\mathsf{u}}F, !^{\mathsf{r}}H, \Gamma, \Delta \longrightarrow G_1 \cdot G_2}\ C$$

Not always safe to contract relevant formulas as one may lose provability.

$$\dfrac{\dfrac{!^{\mathsf{u}}F, !^{\mathsf{r}}H, \Gamma \longrightarrow G_1 \qquad !^{\mathsf{u}}F, !^{\mathsf{r}}H, \Delta \longrightarrow G_2}{!^{\mathsf{u}}F, !^{\mathsf{r}}H, \Gamma, !^{\mathsf{u}}F, !^{\mathsf{r}}H, \Delta \longrightarrow G_1 \cdot G_2}\ \otimes_R}{!^{\mathsf{u}}F, !^{\mathsf{r}}H, \Gamma, \Delta \longrightarrow G_1 \cdot G_2}\ 2 \times C$$

Contribution 2: Logical framework with relevant formulas.

Application: Type-Logical Grammar

Assign logical formulas (or types) to sentences.

$N \backslash S / N$
"John loves Mary."
N N

$$\cfrac{N \to N \qquad \cfrac{N \to N \quad S \to S}{N, N \backslash S \to S}}{N, N \backslash S / N, N \to S}$$

The proof of formulas for sentences **may have contraction**: parasitic extraction.

"John signed the paper without reading it"

"The paper that John signed without reading."

"It" has been omitted twice.

Application: Type-Logical Grammar

"The paper that John signed without reading."

$$\dfrac{\dfrac{\dfrac{\dfrac{N, N \backslash S / N, N, (N \backslash S) \backslash (N \backslash S) / GC, GC / N, N \rightarrow S}{N, N \backslash S / N, !^{s}N, (N \backslash S) \backslash (N \backslash S) / GC, GC / N, !^{s}N \rightarrow S}\ Der}{N, N \backslash S / N, (N \backslash S) \backslash (N \backslash S) / GC, GC / N, !^{s}N \rightarrow S}\ C_L}{N, N \backslash S / N, (N \backslash S) \backslash (N \backslash S) / GC, GC / N \rightarrow S / !^{s}N} \qquad N / CN, CN, CN \backslash CN \rightarrow N}{N / CN, CN, (CN \backslash CN) / (S / !^{s}N), N, N \backslash S / N, (N \backslash S) \backslash (N \backslash S) / GC, GC / N \rightarrow N}$$

Contraction to fill the gap.

Type-Logical Grammars

Our previous work [**IJCAR18**] proposed a Subexponential Non-Commutative Linear Logical Framework for Type-Logical Grammars (and distributed systems):

"The paper that John signed without reading."

$$\dfrac{\dfrac{\dfrac{\dfrac{N, N \backslash S / N, N, (N \backslash S) \backslash (N \backslash S) / GC, GC / N, N \rightarrow S}{N, N \backslash S / N, !^{\mathsf{s}}N, (N \backslash S) \backslash (N \backslash S) / GC, GC / N, !^{\mathsf{s}}N \rightarrow S}\ Der}{N, N \backslash S / N, (N \backslash S) \backslash (N \backslash S) / GC, GC / N, !^{\mathsf{s}}N \rightarrow S}\ C_L}{N, N \backslash S / N, (N \backslash S) \backslash (N \backslash S) / GC, GC / N \rightarrow S / !^{\mathsf{s}}N} \qquad N / CN, CN, CN \backslash CN \rightarrow N}{N / CN, CN, (CN \backslash CN) / (S / !^{\mathsf{s}}N), N, N \backslash S / N, (N \backslash S) \backslash (N \backslash S) / GC, GC / N \rightarrow N}$$

Type-Logical Grammars

However, as shown recently [**JLC 2020**], our proposed logical framework does not satisfy **Lambek's Non Emptiness Property**.

"All sequent antecendents shall not be empty."

This means that our previous logical framework may type sentences that are not grammatically correct.

very interesting book — corresponds → $(N / N) / (N / N), N / N, N \rightarrow N$

very book — corresponds → $(N / N) / (N / N), N \rightarrow N$

Does not satisfy Lambek's Non-Emptiness Property

Key Inspiration

Subexponentials

$!$ is non-commutative

Two types of subexponentials $!$ and ∇

from Soft Linear Logic

- Multiplexing rule instead of contraction:

$$\frac{\Gamma, F, \ldots, F, \Delta \to G}{\Gamma, !F, \Delta \to G}\ !_L$$

from Light Linear Logic

$$\frac{F \to G}{!F \to !G}\ !_R \qquad \frac{F \to G}{\nabla F \to \nabla G}\ \nabla_R$$

Exactly one formula in the antecedent.

Our proposed non-commutative logical framework SLLM contains subexponentials with behavior from soft and light linear logics..

Key Contributions

Our new logical framework SLLM contains subexponentials with behavior from soft and light linear logics..

- **Cut Rule Admissibility**, implying that SLLM is consistent and admits the sub-formula property.
- **Lambek's Non-Emptiness Condition**: no sequent antecedents are empty in a proof.
- **Focused Proof System**: Prove soundness and completeness of a focused proof, SLLMF, for SLLM.
- **Complexity**: Provability problem is undecidable in general, and we identify a fragment that is decidable.

Lambek Proof System

$$\frac{}{F \to F}\ I$$

Initial

$$\frac{\Pi \to G \quad \Gamma_1, F, \Gamma_2 \to C}{\Gamma_1, F / G, \Pi, \Gamma_2 \to C}\ /_L \qquad \frac{\Pi, F \to G}{\Pi \to G / F}\ /_R,\ \text{where } \Pi \neq \emptyset$$

Right Division

$$\frac{\Pi \to F \quad \Gamma_1, G, \Gamma_2 \to C}{\Gamma_1, \Pi, F \backslash G, \Gamma_2 \to C}\ \backslash_L \qquad \frac{F, \Pi \to G}{\Pi \to F \backslash G}\ \backslash_R,\ \text{where } \Pi \neq \emptyset$$

Left Division

$$\frac{\Gamma_1, F, G, \Gamma_2 \to C}{\Gamma_1, F \cdot G, \Gamma_2 \to C}\ \cdot_L \qquad \frac{\Gamma_1 \to F \quad \Gamma_2 \to G}{\Gamma_1, \Gamma_2 \to F \cdot G}\ \cdot_R$$

Product

$$\frac{\Pi \to F\{e/x\}}{\Pi \to \forall x.F}\ \forall_R \qquad \frac{\Gamma_1, F\{t/x\}, \Gamma_2 \to C}{\Gamma_1, \forall x.F, \Gamma_2 \to C}\ \forall_L$$

Quantifier

The order of formulas is important.

Subexponentials

Two subexponentials: $!$ and ∇.

$$\frac{\Gamma_1, \overbrace{A, A, \ldots, A}^{k \text{ times}}, \Gamma_2 \to C}{\Gamma_1, !A, \Gamma_2 \to C} \; !_L \; (k \geq 1) \qquad \frac{A \to C}{!A \to !C} \; !_R$$

$k > 0$

No weakening, no contraction and no exchange

$$\frac{\Gamma_1, A, \Gamma_2 \to C}{\Gamma_1, \nabla A, \Gamma_2 \to C} \; \nabla_L \qquad \frac{A \to C}{\nabla A \to \nabla C} \; \nabla_R$$

$$\frac{\Gamma_1, \Gamma_2, \nabla A, \Gamma_3 \to C}{\Gamma_1, \nabla A, \Gamma_2, \Gamma_3 \to C} \qquad \frac{\Gamma_1, \nabla A, \Gamma_2, \Gamma_3 \to C}{\Gamma_1, \Gamma_2, \nabla A, \Gamma_3 \to C} \; \nabla_E$$

No weakening and no contraction.

Basic Properties

Theorem

- The calculus SLLM enjoys admissibility of the Cut Rule.
- Given an atomic A and sequent $\Gamma(A) \longrightarrow C(A)$ derivable in SLLM, then for any formula B, $\Gamma(B) \longrightarrow C(B)$ is also derivable in SLLM.

What if we take a more general rule: $\dfrac{\Gamma \to C}{!\Gamma \to !C}\ !_R$

$$\dfrac{\dfrac{B, B \setminus C \to C}{!B, !(B \setminus C) \to !C} \quad !C \to C \cdot C}{!B, !(B \setminus C) \to C \cdot C}\ Cut$$

No cut-free proof.

Examples

The letter that Bob sent yesterday.

“Bob sent the letter yesterday”

$$\dfrac{\dfrac{\dfrac{N, V \,/\, N, N, V \setminus V \rightarrow S}{\nabla N, N, V \,/\, N, V \setminus V \rightarrow S}\ \nabla_L, \nabla_E}{N, V \,/\, N, V \setminus V \rightarrow S'} \qquad N, N \setminus N \rightarrow N}{N, (N \setminus N) \,/\, S', N, V \,/\, N, V \setminus V \rightarrow N}$$

$$S' = (\nabla N \setminus S)$$

Examples

The letter that Bob sent without reading.

"Bob sent the letter without reading it"

$$\dfrac{\dfrac{\dfrac{\dfrac{\Delta_1, N, \Delta_2, N \to S}{\nabla N, \nabla N, \Delta_1, \Delta_2 \to S}\ \nabla_L, \nabla_E}{!\nabla N, \Delta_1, \Delta_2 \to S}\ !_L}{\Delta_1, \Delta_2 \to S''} \qquad N, N \setminus N \to N}{N, (N \setminus N) \,/\, S'', \Delta_1, \Delta_2 \to N}$$

$$S' = (!\nabla N \setminus S)$$

Lambek's non-emptiness restriction

Theorem

- The calculus SLLM provides Lambek's non-emptiness restriction: If a sequent $\Gamma \longrightarrow C$ is provable, then list Γ is not empty.

- No weakening.
- The introduction of $!$ or ∇ never produces the empty list.

Focused Proof System

In the paper, we also propose a **focused proof system** for SLLM, thus enabling proof search.

Our previous work **[IJCAR 2018]** left open how to design focused proof system for subexponentials that do not allow both weakening and exchange.

Key Challenge

$$\frac{\Gamma, F, \ldots, F, \Delta \rightarrow G}{\Gamma, !F, \Delta \rightarrow G}$$

This rule has a great deal of non-determinism as one has to decide how many copies of F appears in the premise.

Solution Idea

Introduce two new modalities:

Formula is used once or more times

Formula is used zero or more times

$$\frac{\Gamma, \sharp^+ F, \Delta}{\Gamma, !F, \Delta \rightarrow G} \qquad \frac{\Gamma, F, \sharp^* F, \Delta \rightarrow G}{\Gamma, \sharp^+ F, \Delta \rightarrow G} \qquad \frac{\Gamma, F, \Delta \rightarrow G}{\Gamma, \sharp^* F, \Delta \rightarrow G} \qquad \frac{\Gamma, \Delta \rightarrow G}{\Gamma, \sharp^* F, \Delta \rightarrow G}$$

Structural rules are incorporated into the introduction rules:

$$\frac{\sharp^* C, \Gamma_2 \rightarrow F \quad \Gamma_1, \sharp^+ C, G, \Gamma_3 \rightarrow H}{\Gamma_1, \sharp^+ C, \Gamma_2, F \setminus G, \Gamma_3 \rightarrow H} \qquad \frac{F \rightarrow G}{\Gamma_1^*, !F, \Gamma_2^* \rightarrow !G} \qquad \frac{F \rightarrow G}{\Gamma_1^*, \nabla F, \Gamma_2^* \rightarrow \nabla G}$$

Theorem

- Let Γ, G be formulas not containing $\#^+, \#^*$. A sequent $\Gamma \longrightarrow G$ is provable in SLLM# if and only if it is provable in SLLM.

Complexity

Provability in SLLM is undecidable in general.

Encoding of Turing Machines (TMs):

- A Turing Machine configuration is encoded in the sequent context:

  $[B_1, q_1, \xi, B_2]$ → TM state q_1, tape with B_1, ξ, B_2, and head looking at ξ.

- An instruction $I : q\xi \to q'\eta R$, for example, is encoded as the formula $!\nabla[(q \cdot \xi) \backslash (\eta \cdot q')]$:

  → The prefix enables the instruction to be used multiple times at any place of the tape.

- Strong correspondence (level of proofs): A deterministic TM M leads to a final configuration using instructions $I_1, \ldots, I_m$ only iff the following sequent is derivable in SLLM:

$$!\nabla A_{I_1},\ !\nabla A_{I_2},\ \ldots,\ !\nabla A_{I_m},\ B_1 \cdot q_1 \cdot \xi \cdot B_2 \ \longrightarrow\ q_0$$

- Focused proof system helps to prove this result.

Some decidable fragments:

Theorems

- If we bound k in the multiplexing rule in the calculus SLLM with a fixed constant k_0, such a fragment becomes decidable.

- In the case where we bound k in the multiplexing rule in the calculus SLLM with a fixed constant k_0, and, in addition, we bound the depth of nesting of $!A$, we get NP-completeness.

→ This result provides NP-procedures for parsing complex and compound sentences in many practically important cases.

Conclusions and Future Work

We proposed SLLM, a proof system for type-logical grammars that:

- admits cut-elimination;
- admits substitution;
- satisfies Lambek's non-emptiness restriction.

We proposed a sound and complete focused proof system for SLLM

We investigated the complexity for SLLM provability.

For future work:

- Classical logic versions of our logical framework;
- Extending systems with additives;
- Implementation of lazy forms of proof search.

Related Work

- R. J. Simmons and F. Pfenning. Weak Focusing for Ordered Linear Logic. Technical Report CMU-CS-10-147 2011.
- J. Polakow. Linear logic programming with an ordered context. In PPDP 2000.
- F. Pfenning and R. J. Simmons. Substructural operational semantics as ordered logic programming. In LICS, pages 101–110, 2009.
- M. Kanovich, S. Kuznetsov, V. Nigam,and A. Scedrov. Subexponentials in non-commutative linear logic. In Mathematical Structures in Computer Science 2018. Dale Miller's Festschrift.
- M. Kanovich, S. Kuznetsov, V. Nigam,and A. Scedrov. A Logical Framework with Commutative and Non-commutative Subexponentials. In IJCAR 2018.
- G. Morrill and O. Valentin. Multiplicative-additive focusing for parsing as deduction. In First International Workshop on Focusing, 2015.
- C. Olarte, E. Pimentel, and V. Nigam. Subexponential concurrent constraint programming. Theor. Comput. Sci., 606:98–120, 2015.
- V. Nigam, E. Pimentel, and G. Reis. An extended framework for specifying and reasoning about proof systems. J. Log. Comput., 26(2):539–576, 2016.
- V. Nigam. A framework for linear authorization logics. TCS, 536:21–41, 2014.