Formal approach to stratification in NF/NFU

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joint work with Vedran Čačić

Syntax

Alphabet is a collection of:

- (individual) variables $v_0, v_1, v_2, \ldots$
- logical symbols (connectives and a quantifier) $\neg$, $\wedge$, $\vee$, $\exists$
- non-logical (relation) symbols $\in$, $=$; *set*
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$\varphi ::= x \in y \mid x = y \mid \mathit{set}(x) \mid \neg\varphi \mid (\varphi_1 \wedge \varphi_2) \mid (\varphi_1 \vee \varphi_2) \mid \exists x \varphi$

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A formula φ is **stratified** if there exists a mapping $type_\varphi$ from the variables of φ to the integers such that: if $x = y$ is subformula of φ, then $type_\varphi(x) = type_\varphi(y)$, and if $x \in y$ is subformula of φ, then $type_\varphi(y) = type_\varphi(x) + 1$.

Type mappings

An ordering on type mappings for a stratified formula φ can be defined as:

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By convention, we will usually fix the least type of a variable to 1.

The least type mapping

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Let φ be a stratified formula with type mappings $type_\varphi$ and $type'_\varphi$. Then for every $x, y \in \mathrm{Var}\,\varphi$, if $x \parallel^* y$, then
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If φ is a stratified formula, then there exists the least type mapping $mintype_\varphi$ of φ.

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Stratified comprehension: If $\varphi(z, x_1, \ldots, x_n)$ is stratified, then

$$\forall x_1 \ldots \forall x_n \exists y \big(set(y) \wedge \forall z \big(z \in y \leftrightarrow \varphi(z, x_1, \ldots, x_n) \big) \big).$$

Abstraction terms

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  $\forall y\big(y \in x \leftrightarrow y \in \{z \mid \varphi(z, x_1, \ldots, x_n)\}\big)$
- $\{z \mid \varphi(z, x_1, \ldots, x_n)\} \in x :\Leftrightarrow (\exists y \in x)\big(y = \{z \mid \varphi(z, x_1, \ldots, x_n)\}\big)$.
- $set(\{z \mid \varphi(z, x_1, \ldots, x_n)\}) :\Leftrightarrow \exists t\big(t = \{z \mid \varphi(z, x_1, \ldots, x_n)\} \wedge set(t)\big)$

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Some sets in NFU: $\emptyset$, SET, V, $x \cup y$, $x \setminus y$, $\bigcap x$, $\{x\}$, $\mathscr{P}(x)$, $\mathscr{P}_1(x) := \{\{t\} \mid t \in x\}$ ...

Types of abstraction terms

Let ψ' be a formula in the language extended by abstraction terms, ψ the corresponding formula in the basic language, and let ψ be a stratified formula with type mapping $type_\psi$. Let $t = \{z \mid \varphi(z, x_1, \ldots, x_n)\}$ be an abstraction term in formula ψ', where $\varphi(x, x_1, \ldots, x_n)$ is stratified formula, and let $type_\varphi$ be a restriction of mapping $type_\psi$ on variables of φ. We extend $type_\psi$ to the mapping $type_{\psi'}$ so that $type_{\psi'}(t) := 1 + type_\varphi(z)$.

Nested terms

Let $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and $\psi(w, x_1, \ldots, x_n)$ be formulas.
By a **nested abstraction term** we mean a term of the form $\{f(x_1, \ldots, x_n) \mid \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)\}$, where $f(x_1, \ldots, x_n)$ itself is an abstraction term.
We eliminate nested abstraction terms in the following way:

$$\big\{\{w \mid \psi(w, x_1, \ldots, x_n)\} \;\big|\; \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)\big\} :=$$
$$\big\{z \mid \exists x_1 \ldots \exists x_n \big(\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m) \land z = \{w \mid \psi(w, x_1, \ldots, x_n)\}\big)\big\}$$

Types of nested terms

Let $\psi(w, x_1, \ldots, x_n)$ be a stratified formula, $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ a stratified formula with underlying type mapping $type_\varphi$, let $s = \{w \mid \psi(w, x_1, \ldots, x_n)\}$ and let $t = \{s \mid \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)\}$ be a nested abstraction term. If $z \in s \leftrightarrow \psi(z, x_1, \ldots, x_n)$ is stratified, where types of $x_1, \ldots, x_n$ are determined by the mapping $type_\varphi$, then the type of a nested term t is one greater than the type of s.

Natural numbers and ordered pairs

Zero: $0 := \{\emptyset\}$

Successor: $S(x) := \{y \mid (\exists z \in y)(y \setminus \{z\} \in x)\}$

Natural numbers: $\mathbb{N} := \bigcap\{x \mid 0 \in x \land (y \in x \to S(y) \in x)\}$

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Quine's functions:

- $Q_1(x) := (x \setminus \mathbb{N}) \cup \{S(y) \mid y \in x \cap \mathbb{N}$
- $Q_2(x) := Q_1(x) \cup \{0\}$

Ordered pairs: $(x, y) := \{Q_1(z) \mid z \in x\} \cup \{Q_2(z) \mid z \in y\}$

Relations and functions

Cartesian product: $X \times Y := \{(x, y) \mid x \in X \wedge y \in Y\}$

Binary relation: $R \subseteq X \times Y$, written $\mathit{rel}(R, X, Y)$.

Function: $\mathit{rel}(f, X, Y) \wedge (\forall x \in X)(\exists! y \in Y)(x\ f\ y)$, written $\mathit{func}(f, X, Y)$

Equivalence relation: reflexive, symmetric and transitive

Equivalence class: $[x]_R := \{y \mid \mathit{rel}(R, X, X) \wedge x\ R\ y\}$

Quotient set: $X/R := \{[x]_R \mid x \in X\}$

Orders

Partial order:
$rel(<, X, X) \land \forall x(x \not< x) \land \forall x \forall y \forall z(x < y \land y < z \to x < z)$, written $Po(X, <)$

Well-order: $Po(X, <) \land (\forall x \in X)(\forall y \in X)(x < y \lor x = y \lor y < x) \land \forall Y\big(Y \subseteq X \land Y \neq \emptyset \to (\exists m \in Y)(\forall y \in Y)(m < y \lor m = y)\big)$, written $Wo(X, <)$

Preserving well order:
$Wo(X, <) \land Wo(Y, \prec) \land func(f, X, Y) \land \forall x \forall y\big(x < y \to f(x) \prec f(y)\big)$, written $wop(f, X, <, Y, \prec)$

Similarity: $bij(f, X, Y) \land wop(f, X, <, Y, \prec) \ \land wop(f^{-1}, Y, \prec, X, <)$, written $sim(f, X, <, Y, \prec)$

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Class of cardinal numbers: $CARD := SET/(\sim)$ is a set

Ordinal numbers – a harder approach

$$\{[(X,<)]_{\simeq} \mid Wo(X,<)\} = \{t \mid \exists X \exists < \big(t = [(X,<)]_{\simeq} \land Wo(X,<)\big)\}$$

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$\exists X \exists < (\forall z(z \in t \leftrightarrow \exists Y \exists \prec (\forall u(u \in z \leftrightarrow ((\exists t \in Y)(u = Q_1(t)) \vee (\exists t \in \prec)(u = Q_2(t)))) \wedge \exists f((\forall t \in f \to (\exists x \in X)(\exists y \in Y)(\forall u(u \in t \leftrightarrow ((\exists v \in x)(u = Q_1(v)) \vee (\exists v \in y)(u = Q_2(v)))))) \wedge (\forall x \in X)(\exists! y \in Y)((x,y) \in f)) \wedge (\forall y \in Y)(\exists x \in X)((x,y) \in f) \wedge (\forall x_1 \in X)(\forall x_2 \in X)(f(x_1) = f(x_2) \to x_1 = x_2) \wedge Wo(X,<) \wedge Wo(Y,\prec) \wedge \forall x \forall y(x < y \to f(x) \prec f(y)) \wedge \forall u \forall v(u \prec v \to f^{-1}(u) < f^{-1}(v))))) \wedge Wo(X,<))$

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