Logics for Reasoning about Knowledge and Conditional Probability

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Overview

Goal:

We provide logics for reasoning about knowledge and conditional probabilities together

The main results:

- Sound and strongly complete axiomatization for logics
- Decision procedure for propositional case

Methods:

- Completeness: Henkin-style method
- Decidability: Method of filtration and reduction to a system of inequalities

Starting points

Epistemic logic + Probabilistic logic

$K_i(\alpha)$

$w(\alpha) \geq r$

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Epistemic logic + Probabilistic logic

$K_i(\alpha)$ $\quad$ $w(\alpha) \geq r$

- Fagin, R., Halpern, J.Y.: Reasoning about knowledge and probability
  - *Formula*: $K_i(w_i(\alpha) \geq r)$
  - *Linear Weight Formulas* (FHM90):
    $a_1 w_i(\alpha_1) + \cdots + a_n w_i(\alpha_n) \geq r$
    Not applicable for axiomatizing conditional probabilities

Our logic CKL

• CKL logic for reasoning about conditional probabilities and knowledge

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$$w_i(\alpha, K_j(\beta)) \geq r$$

"According to the agent i the conditional probability of α given that the agent j knows β, is at least r"

CKL - Syntax

- Formulas: $\alpha, \beta \dots$

- $\{p, q, r, \dots\}$

- $K_i\alpha$ "Agent i knows α"

- $a_1 w_i(\alpha_1, \alpha'_1) + \cdots + a_k w_i(\alpha_k, \alpha'_k) \geq r$,

  - $w_i(\alpha, \beta)$ "Conditional probability of α given β according to the agent i"

- $\neg\alpha, \alpha \wedge \beta$

Definition

$M = (W, \mathcal{K}, Prob, v)$

- W is a nonempty set of worlds,
- $v : W \times P \longrightarrow \{true, false\}$
- $\mathcal{K} = \{\mathcal{K}_i \mid i \in \mathbf{A}\}$ is a set of binary equivalence relations on W, where $\mathcal{K}_i(u) = \{u' \mid (u', u) \in \mathcal{K}_i\}$, and $u\mathcal{K}_i u'$ if $u' \in \mathcal{K}_i(u)$,
- $Prob(i, u) = (W_i(u), H_i(u), \mu_i(u))$ such that
  - $W_i(u)$ is a non-empty subset of W,
  - $H_i(u)$ is an algebra of subsets of $W_i(u)$
  - $\mu_i(u) : H_i(u) \longrightarrow [0, 1]$ is a finitely additive measure

Satisfiability

- $M, u \models K_i\alpha$ iff $M, u' \models \alpha$ for all $u' \in K_i(u)$

- $M, u \models w_i(\alpha, \beta) \geq r$ if $\mu_i(u)(\{u' \in W_i(u) \mid M, u' \models \beta\}) > 0$
and
$\mu_i(u)(\{u' \in W_i(u) \mid M, u' \models \alpha\}|\{u' \in W_i(u) \mid M, u' \models \beta\}) \geq r$

Axiomatization

- Propositional reasoning
- Reasoning about knowledge
- Reasoning about linear inequalities
- Reasoning about conditional probabilities

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- Reasoning about knowledge
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- Infinitary inference rule:
  - From the set of premises

$$\{w_i(\alpha, \beta) \geq r - \frac{1}{k} \mid k \in \mathcal{N}\}$$

infer $w_i(\alpha, \beta) \geq r$

If the conditional probability is arbitrary close to r, it is at least r.

Our results

Theorem (Soundness and completeness)

A set of formulas T is consistent iff T is satisfiable.

- Adaptation of Henkin's construction

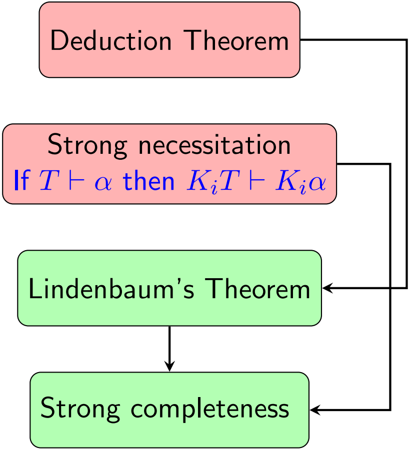
Theorem (Decidability)

Satisfiability problem for CKL is decidable

- Combination of the method of filtration and reduction to a system of inequalities

Technical details

- Henkin's construction



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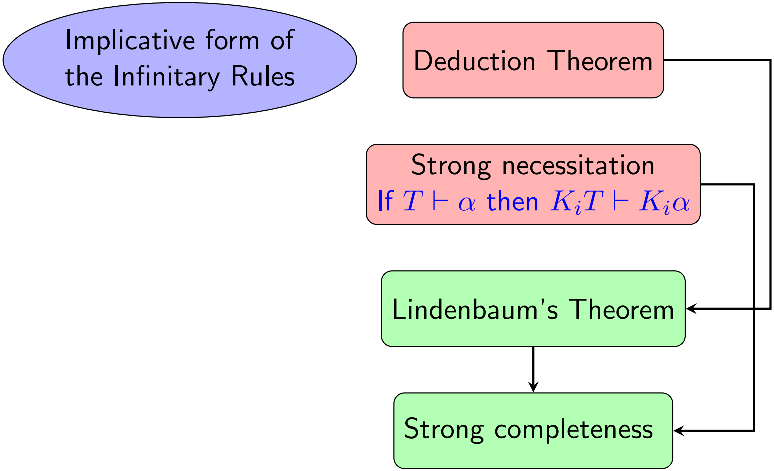

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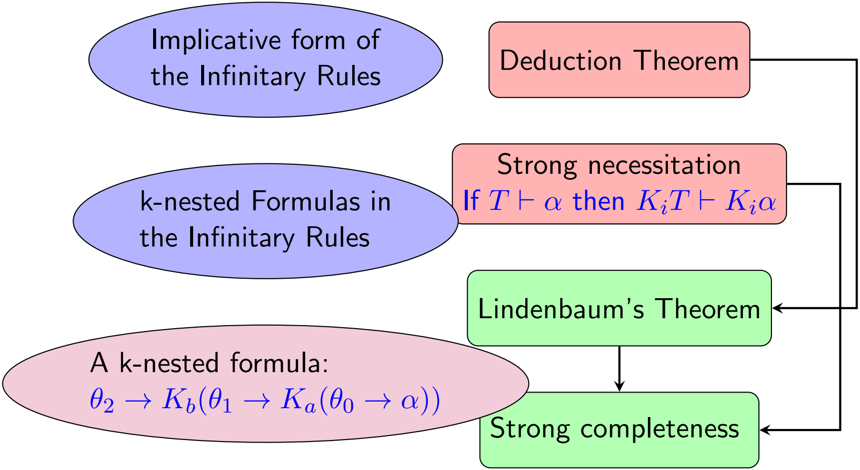

First-order extension FOCKL

$$(\forall x) w_i(\alpha(x), K_j(\beta(x))) \geq r$$

Language (FOCKL) = Language (CKL)+ Language (CFO)

FOCKL-Semantics

Definition

$M = (W, D, I, \mathcal{K}, Prob)$

- W is a nonempty set of worlds,
- D is nonempty domain,
- I is an interpretation for every world,
- $\mathcal{K} = \{\mathcal{K}_i \mid i \in \mathbf{A}\}$ is a set of binary equivalence relations on W
- $Prob(i, u) = (W_i(u), H_i(u), \mu_i(u))$ such that
  - $W_i(u)$ is a non-empty subset of W,
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Domain is fixed and the terms are rigid.

Axiomatization

- Axiomatization of CKL
- $(\forall x)(\alpha \to \beta) \to (\alpha \to \forall x \beta)$, where x is not free-variable in α
- $(\forall x)\alpha(x) \to \alpha(t/x)$, where $\alpha(t/x)$ is obtained by substituting all free occurrences of x in $\alpha(x)$ by the term t which is free for x in $\alpha(x)$
- $\forall x K_i \alpha(x) \to K_i \forall x \alpha(x)$ (Barcan formula)
- From α infer $(\forall x)\alpha$.

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Theorem (Soundness and completeness)

A theory T is consistent iff T is satisfiable.

Conclusion

In this work:

- Propositional logic of knowledge and conditional probability
- First-order extension

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No semantic relationship between the modalities for knowledge and probability.

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CKL

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The end

Thank you!